Phase transition in the link weight structure of networks

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When transport in networks follows the shortest paths, the link weights are shown to play a crucial role. If the underlying topology with N nodes is not changed and if the link weights are independent from each other, then we show that, by tuning the link weights, a phase transition occurs around a critical extreme value index α_c of the link weight distribution. If the extreme value index of the link weight distribution $\alpha > \alpha_c$, transport in the network traverses many links whereas for $\alpha < \alpha_c$, all transport flows over a critical backbone consisting of $N-1$ links. For connected Erdös-Rényi random graphs $G_p(N)$ and square lattices, we have characterised the phase transition and found that $\alpha_c \simeq bN^{-\beta}$ with $\beta_{G_p(N)} \approx 0.63$ and $\beta_{\text{lattice}} \approx 0.62$.

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I. INTRODUCTION

Topological phase transitions in networks are known phenomena. The growth of the giant component in the Erdös-Rényi random graphs is perhaps the most well studied topological phase transition (see e.g. [1]). Beside the topology of a network, described by the adjacency matrix, to each link (i,j) we can assign a link weight w_{ij} being a real positive number (in some appropriate units) that quantifies a property of the link from node i to node j. For example, in a transportation network, a typical link weight is the physical distance between two points (nodes) in the network. The main interest of link weights is related to the purpose of the network and to an optimality criterion that forces transport along *shortest* paths. Thus, the link weight structure (the set of all link weights, $\{w_{ij}\}$) affects the transport in a network if paths are determined based on a shortest path criterion. In this paper, we report on a phase transition caused by changes in the link weight structure.

In the sequel, we will assume that transport between two nodes follows the shortest path that is the minimizer of the sum of the link weights of any path between those two nodes. In many large networks such as, e.g., the Internet, biological molecules and social relations, neither the topology nor the link weight structure is accurately known. This uncertainty about the precise structure leads us to consider both the underlying graph and each of the link weights as random variables. We assume that links are undirected, $w_{ij} = w_{ji}$ and that each link weight w_{ij} is independent from w_{kl} with link $(i,j) \neq (k,l)$. A second important confinement is that we assume that the graph (topology) and the link weight structure are orthogonal characteristics of a network, in the sense that the link weight structure can be changed independently from the underlying graph. In many biological networks, the link weight (or strength of a link) is coupled to the structure of the underlying topology such that both link weights and topology are not independent. However, in many man-made large infrastructures such as the Internet, the link weight structure can be chosen independently of the underlying topology. The latter assumption allows us to control or steer transport in the network. Finally, in contrast to evolving networks in which Bianconi and Barabási [2] have observed a Bose-Einstein condensation, the present setting is static: the underlying topology does not change, only the strength of the interconnections, i.e., the link weights, is tuned.

II. PROPERTIES OF α-TREES

The shortest path tree (SPT) rooted at some node is the union of the shortest paths from that node to all other nodes. Since the SPT is mainly sensitive to the smaller, non-negative link weights, the probability distribution of the link weights around zero will dominantly influence the properties of the resulting shortest path tree. A *regular* link weight distribution $F_w(x) = \Pr[w \leq x]$ has a Taylor series expansion around $x=0$,

$$F_w(x) = f_w(0)x + O(x^2)$$

since $F_w(0)=0$ and $F'_w(0)=f_w(0)$ exists. A regular link weight distribution is thus linear around zero. The factor $f_w(0)$ only scales all link weights, but it does not influence the shortest path. The simplest distribution of the link weight w with a distinct different behavior for small values is the polynomial distribution,

$$F_w(x) = x^\alpha 1_{x\in[0,1]} + 1_{x\in[1,\infty)}, \quad \alpha > 0, \tag{1}$$

where the indicator function 1_x is one if x is true else it is zero. The corresponding density is $f_w(x) = \alpha x^{\alpha-1}$, $0 < x < 1$. The exponent

$$\alpha = \lim_{x\downarrow 0} \frac{\log F_w(x)}{\log x}$$

is called the *extreme value index* of the probability distribution of w and $\alpha=1$ for regular distributions. By varying the exponent α over all non-negative real values, any extreme value index can be attained and a large class of corresponding SPTs, in short α-trees, can be generated.

Figure 1 illustrates schematically the probability distribution of the link weights around zero $(0,\epsilon]$, where $\epsilon > 0$ is an arbitrarily small, positive real number. The larger link weights in the network will hardly appear in a shortest path provided the network possesses enough links. These larger link weights are drawn in Fig. 1 from the double dotted line to the right. The nice advantage that only small link weights dominantly influence the property of the resulting shortest

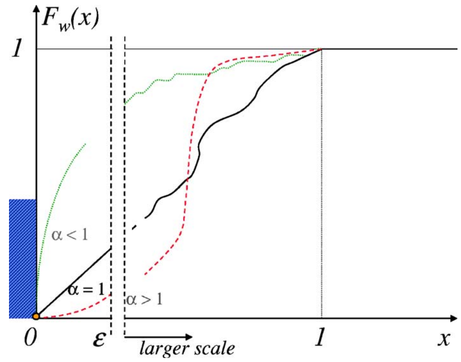

FIG. 1. (Color online) A schematic drawing of the distribution of the link weights for the three different α-regimes. The shortest path problem is mainly sensitive to the small region around zero. The scaling invariant property of the shortest path allows us to divide all link weights by the largest possible such that $F_w(1)=1$ for all link weight distributions.

path tree lies in that the remainder of the link weight distribution (denoted by the arrow with larger scale in Fig. 1) only plays a second order role.

Let us consider a connected graph $G(N,L)$ with N nodes and L links and with independent polynomial link weights specified by (1). We briefly present three special α-trees for $\alpha=\infty$, 1, and 0, respectively and then limit ourselves to the range $\alpha\in[0,1]$.

If $\alpha\rightarrow\infty$, it follows from (1) that $w=1$ almost surely for all links. Since all links have unit weight, the $\alpha\rightarrow\infty$ regime reduces to the computation of the SPT in the underlying graph. The $\alpha\rightarrow\infty$ regime is thus entirely determined by the topology of the graph because the link weight structure does not differentiate between links. Here, the $\alpha\rightarrow\infty$ regime is not further considered.

Link weights with $\alpha=1$ are, e.g., those that are uniformly or exponentially distributed. Earlier in [3], it was shown that the SPT in the complete graph with uniform (or exponential) link weights is precisely a Uniform Recursive Tree (URT). A URT is grown by sequentially attaching a new node uniformly to a node that is already in the URT. A URT is asymptotically the shortest path tree in the Erdös-Rényi random graph $G_p(N)$ [4] with link density p above the disconnectivity threshold $p_c\sim\ln N/N$. The interest of the URT is that analytic modeling is possible (see e.g. [5, Part III]). Moreover, in [6], the hop count is shown to be (at least asymptotically) *independent* of the link density provided $p>p_c$ in the underlying graph $G_p(N)$. For N large and fixed α around 1, the number of links in the shortest path, in short the hopcount H_N, is shown in [3] to satisfy

$$\mathbb{E}[H_N]\sim\frac{\ln N}{\alpha}, \tag{2}$$

$$\mathrm{Var}[H_N]\sim\frac{\ln N}{\alpha^2}. \tag{3}$$

If $\alpha\rightarrow 0$, the ratio $\sqrt{\mathrm{Var}[w]/\mathbb{E}[w]}\sim 1/\sqrt{\alpha}$ diverges which means that, in this limit, the link weights possess strong fluctuations. This observation inspired by Braunstein *et al.* [7] is crucial in the analysis of the behavior of the shortest path for small α. For $\alpha\rightarrow 0$, all SPTs coincide with the minimum spanning tree (MST) as reported by Dobrin and Duxbury [8]. Hence, for $\alpha\rightarrow 0$, all traffic in the graph routed along SPTs traverses precisely the same $N-1$ links that form the "critical backbone." Using arguments from the theory of critical phenomena and numerical simulations Braunstein *et al.* [7] showed that for Erdös-Rényi random graphs the hopcount scales like $\lim_{\alpha\rightarrow 0}\mathbb{E}[H_N(\alpha)]=O(N^{1/3})$. The $\alpha\rightarrow 0$ regime corresponds to a strong disorder regime and has been studied further in [9].

In summary, relatively small variations in the link weight structure cause large differences in the properties of the SPT. In particular, the average hopcount in a graph with N nodes follows a different scaling: $\mathbb{E}[H_N]=O(\ln N)$ for α around 1 while $\mathbb{E}[H_N]=O(N^{1/3})$ if $\alpha\rightarrow 0$. The logarithmic $O(\ln N)$-scaling corresponds to "small world" networks that are densely interconnected such that typical paths only possess a few hops. A well-known example of a small world network [10] is the graph whose nodes are persons and whose links are generated by the acquaintance relations between persons. The algebraic $O(N^{a/b})$-scaling corresponds to sparse networks where paths contain generally many hops. For example, the hopcount between two random points in a two-dimensional square lattice with N nodes scales as $O(N^{1/2})$.

Figure 2 visualizes the different structure of a typical MST (a) and a typical URT (b) of the same size $N=100$.

III. PHASE TRANSITION IN α-TREES

The existence of a critical extreme value index $\alpha_c>0$ separating both the $\alpha\rightarrow 0$ and $\alpha\rightarrow 1$ regimes has been proved in [9]. Here, we report on a fascinating phase transition between both regimes around a critical α_c as illustrated in Fig. 3. We first consider the complete graph K_N as the underlying topology to which polynomial link weights are assigned. The results are obtained by simulations only and improve the best order estimate $\alpha_c=O((N\log N)^{-2})$, analytically derived in [9], considerably. In our simulations, we created K_N with $N=25\cdot 2^n$, $0\leqslant n\leqslant 3$. The SPT rooted at each node in K_N to all other nodes is computed with Dijkstra's algorithm from which the union $G_{\cup_{\mathrm{spt}}}$ is constructed. For every pair of N and α, we generate 10 000 different link weight structures on K_N. The fraction of the graphs $G_{\cup_{\mathrm{spt}(\alpha)}}$ that are a tree approximates $\Pr[G_{\cup_{\mathrm{spt}(\alpha)}}=\mathrm{MST}]$. The simulations were performed in high precision as explained in Appendix A.

Figure 3 shows the probability $F_T(\alpha)=\Pr[G_{\cup_{\mathrm{spt}(\alpha)}}=\mathrm{MST}]$ that the union of all shortest paths $G_{\cup_{\mathrm{spt}(\alpha)}}$ between all node pairs in the complete graph with polynomial link weights is a minimum spanning tree. In real networks where almost all flows follow shortest paths through the network, the union of all shortest paths is the observable part of a network. For example, the union of all trace routes (Internet paths) between all node pairs in the Internet, would represent the observable graph of the Internet. The real Internet is larger

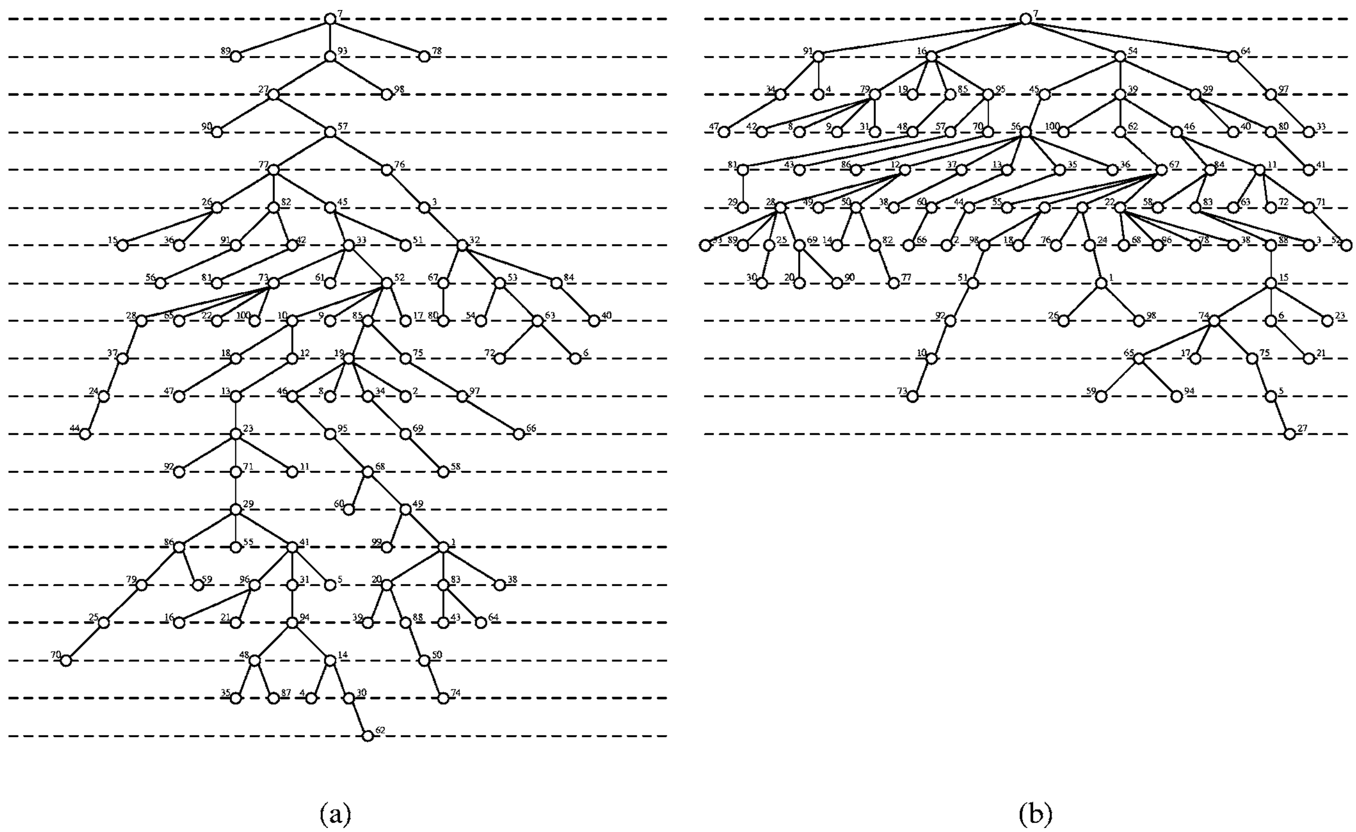

FIG. 2. An example in the graph with $N=100$ nodes of (a) the MST which is the SPT for $\alpha=0$ and (b) the URT which is the SPT for $\alpha=1$. Both trees are structured per level sets where each level shows the number of nodes at different hopcount from the root (here node with label 7).

because it also contains dark links for back-up paths needed in case of failures. We observe in Fig. 3 a phase transition around $\alpha/\alpha_c=1$, where α_c is defined as $F_T(\alpha_c)=\frac{1}{2}$. For $\alpha<\alpha_c$, most graphs $G_{\cup_{\text{spt}(\alpha)}}$ are trees with high probability while for $\alpha>\alpha_c$ hardly any graph $G_{\cup_{\text{spt}(\alpha)}}$ is a tree. Moreover, Fig. 3 shows that the phase transition obeys a same function in the normalized α/α_c for any size of the network N. The insert in Fig. 3 illustrates that the numerically computed derivative $f_T(\alpha)=dF_T(\alpha)/d\alpha\approx\text{Weibull}(2/3,2;\alpha/\alpha_c)$, where the Weibull probability density function is $\text{Weibull}(a,b;x)=abx^{b-1}\exp(-ax^b)$. Hence, $F_T(\alpha)\approx\exp(-(2/3)(\alpha/\alpha_c)^2)$ or, slightly better since by definition $F_T(\alpha_c)=1/2$,

$$F_T(\alpha)\approx 2^{-(\alpha/\alpha_c)^2}. \tag{4}$$

The phase transition is thus not symmetric around α_c. Due to the large degree of dependence (i.e., overlap of paths) in the union $G_{\cup_{\text{spt}(\alpha)}}$, we are unable to compute the probability $F_T(\alpha)$ analytically. If we ignore dependence, the fit with a Weibull distribution is, however, suggested by the fact (see Appendix B) that the minimum of a set of m independent polynomial random variables and the minimum of m sums of k independent polynomial random variables tends, for large m, to a Weibull distribution. A Weibull distribution is one of the three types of extremal distributions (see e.g. [5, Chap. 6]) for independent random variables. The remarkably good fit with a Weibull is surprising because the assumption of independence clearly does not hold for paths in $G_{\cup_{\text{spt}(\alpha)}}$.

The width of the phase transition is defined by $2\Delta\alpha=\alpha_h-\alpha_l$, where $F_T(\alpha_h)=\epsilon$ and $F_T(\alpha_l)=1-\epsilon$, e.g., $\epsilon=0.05$. It follows from (4) that

$$\Delta\alpha=(\alpha_c/2)(\sqrt{-\log_2 F_T(\alpha_l)}-\sqrt{-\log_2 F_T(\alpha_h)})$$

or $\Delta\alpha=k\alpha_c$ for some constant $k(\epsilon)$. Simulations accurate within a relative error of 1% indicate as shown in Fig. 4 that the critical extreme value index α_c (and thus also the width $\Delta\alpha$ of the phase transition) seems to obey a curious $O(N^{-\beta})$-scaling law with β around 0.625 which is much larger than the previously estimated worst case bound of $\alpha_c=O((N\log N)^{-2})$. Critical exponents β of this magnitude seems frequently appearing in phase transitions in physics [11]. At present, we do not have precise arguments to explain the observed $O(N^{-\beta})$-scaling law for the critical extreme value index α_c.

A. Influence of the substrate

Instead of the complete graph K_N as underlying topology, we have performed the same set of simulations on the Erdös-Rényi random graph $G_p(N)$, where p is the probability that there is a link between two arbitrary nodes. One might expect that $\alpha_c(p)>\alpha_c(1)=\alpha_c$ because less links in the underly-

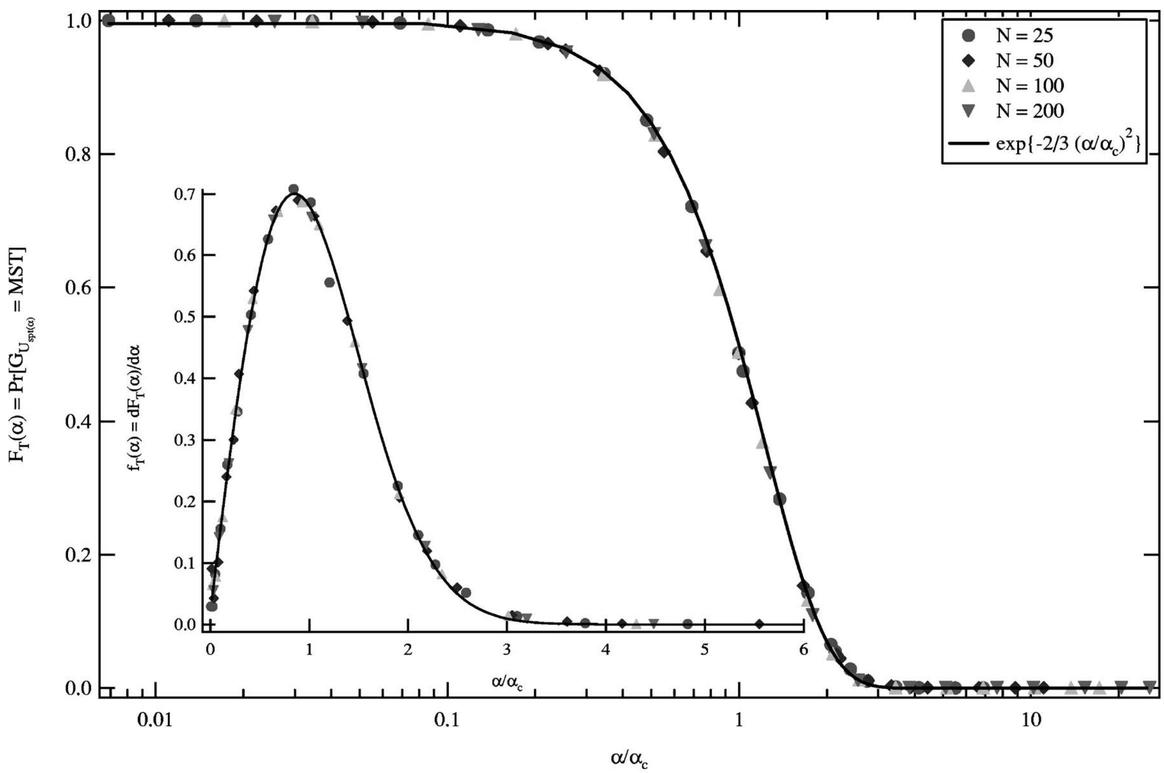

FIG. 3. The probability distribution $F_T(\alpha)$ as a function of the normalized α/α_c. The insert shows the probability density function $f_T(a)=dF_T(\alpha)/d\alpha$ together with the Weibull $\left(\frac{2}{3},2;\alpha\right)$ function.

ing graph only increase the probability that $G_{\cup_{\mathrm{spt}(\alpha)}}$ is a tree as the total number of links that can be included in the SPT is reduced. Or, alternatively, the sparser the substrate graph G, the more treelike it already is and, hence, the higher the probability that $G_{\cup_{\mathrm{spt}(\alpha)}}\subset G$ is a tree. However, the simulations even on small graphs ($N=25$) did not show any noticeable difference between $p=1$ (complete graph) and $p=p_c$. This means that the same link weight phase transition occurs in all connected Erdös-Rényi random graphs $G_p(N)$! The intuitive explanation similar to the arguments used in [6] is that Erdös-Rényi random graphs $G_p(N)$ for $p>p_c$ are still sufficiently dense in that the smallest link weights that determine the shortest path are not confined by the underlying topology.

More surprisingly, we found that the square two-dimensional lattice, in which the number of nodes N is a square, also features a phase transition for which $F_T(\alpha)$ is precisely the same as for the complete graph and given by (4). As shown in Fig. 4, only the scaling law $\alpha_c=bN^{-\beta}$ has a slightly different prefactor b, but almost the same β (within the accuracy of the simulation). The fact that $b_{\mathrm{lattice}}<b_{G_p(N)}$ is intuitively expected since the diameter(lattice) $>$diameter($G_p(N)$) and the hopcount $H_{\mathrm{lattice}}>H_{G_p(N)}$. Hence, more links are used in the union $G_{\cup_{\mathrm{spt}(\alpha)}}$ and it takes longer (smaller α) to obtain a tree. Since the class of square lattices and that of connected Erdös-Rényi random graphs possess many opposite properties (regular versus irregular, large diameter versus a small world graph, small hopcount versus large hopcount, etc.), the simulations seem to suggest that $F_T(\alpha)$ may be the same for a much larger class of connected graphs containing cycles [14] as well as the scaling law $\alpha_c=bN^{-\beta}$, where b and β depend on the substrate topology. If this claim is correct, which we cannot prove rigorously, there always exists for that larger class a link weight structure with sufficiently small extreme value index that allows to steer transport in the underlying network in a similar way specified by (4). Also, it shows the nearly perfect orthogonality of such a link weight structure and the graph's topology.

B. Implications

Although eccentric at first glance, a number of interesting conclusions can be drawn. From a topological view, the trees in large networks such as the Internet indeed seem to consist of a critical bearer tree (corresponding to the $\alpha\to 0$-tree) overgrown with URT-like small trees (influence of $\alpha=1$). The latter cause that the hopcount in the Internet still scales logarithmically in N, rather than algebraically as for the $\alpha\to 0$-tree (MST). This effect is similar to the small world graphs: by adding a few links in a "large average hopcount graph," the hopcount may decrease dramatically [10].

While most of the phase transitions are natural phenomena, our finding illustrates that phase transitions can appear in large infrastructures when the link weights can be controlled independently from the underlying topology. In other words, if the link weight structure can be considered as a property of the network orthogonal to the graph's topology, we may switch the traffic in the network between two extreme transport profiles. Since also the width of the phase transition is narrow, from a control point of view, a network

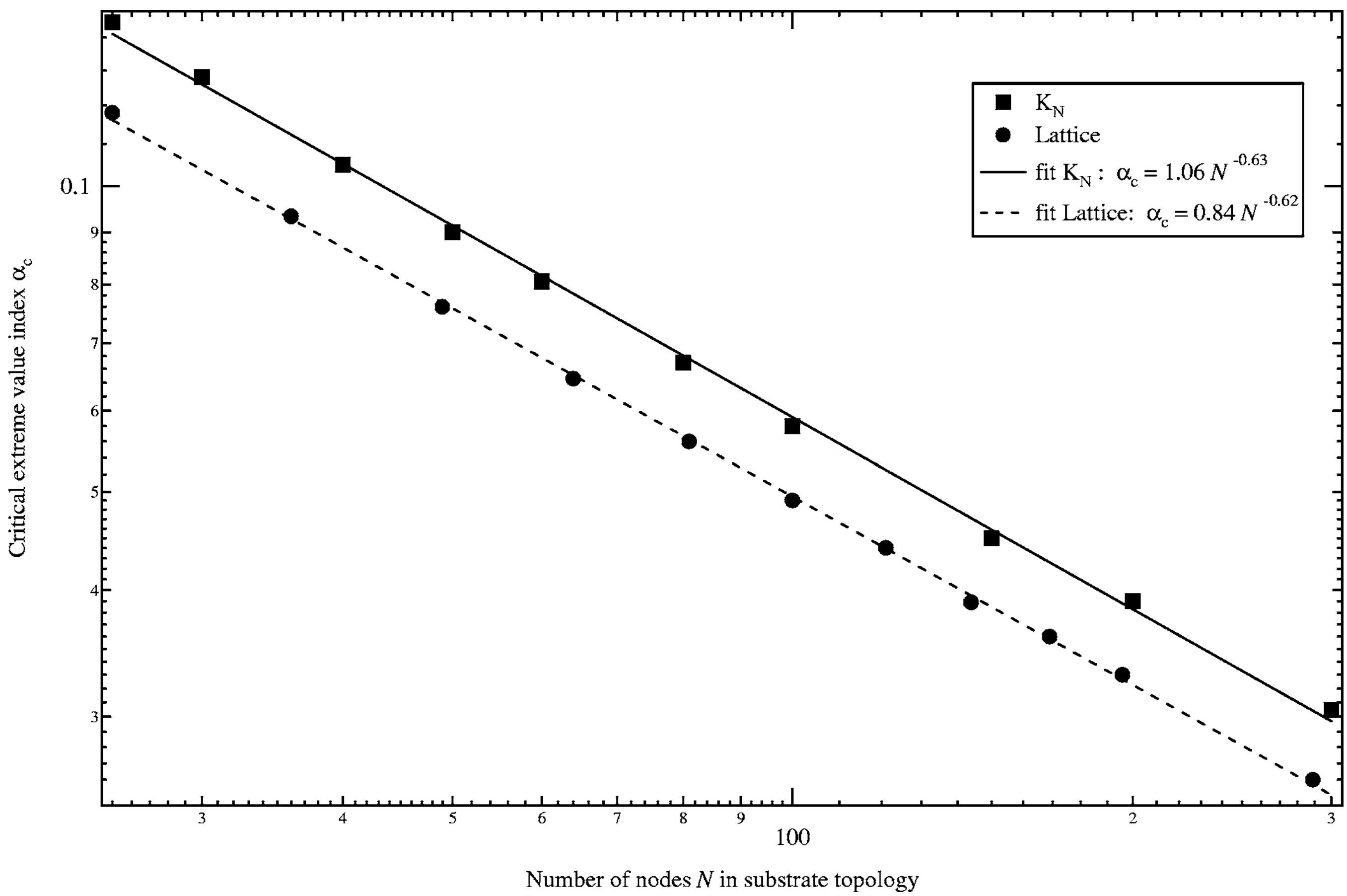

FIG. 4. The critical extreme value index α_c vs N for two substrate topologies, the complete graph and the square lattice.

operator may choose an independent link weight structure with extremal value index near to α_c: by changing the link weight's extreme value index with $\Delta\alpha$, he can switch traffic over two entirely different patterns.

From a robustness point of view, choosing α around 1 will lead to the use of more paths and, hence, a more balanced overall network load. Possible failures of a small set of nodes or links are unlikely to effect the global transport in the network. In the $\alpha \to 0$ regime, all flows are transported over the minimum possible fraction of links in the network. Any failure in a node or link disconnects the MST into two parts and may result in obstruction of transport in the network. In summary, from the view point of robustness, the $\alpha \to 0$ regime may constitute a weak regime although it is highly efficient: only $N-1$ links are used which means that a minimum of links need to be controlled and/or secured.

The link weight phase transition is similar to electrical conductivity in superconductive solids. Above a critical temperature T_c, the normal conduction consists of the ensemble of electrons that travel over different paths while below T_c, a superconducting state is formed in which electrical current flows as a kind of super-wave through the solid with nonmeasurable resistance. The analogy with nature shows that above α_c, a collection of small and seemingly unordered and local flows traverse the network, while below α_c, transport is transformed into a large and global network phenomenon, comparable with a macroscopic quantum effect (such as laser light and superconductivity).

Finally, we mention the related work of Porto *et al.* [12] who studied the crossover of optimal paths in disordered media from self-similar (roughly corresponding to $\alpha < \alpha_c$) to self-affine ($\alpha > \alpha_c$) behavior. Their results differ from ours because of their confinement in the link weight structure (the energy distribution associated with bonds), especially around $w=0$. We have shown that it is necessary for the orthogonality between link weight structure and topology that the link weight density $f_w(0) \to \infty$. Furthermore, the union of all possible shortest paths (the observable part of a network) seems to tend more rapidly to a universal behavior than the characteristics of a single arbitrary shortest path.

IV. SUMMARY

By extensive simulations, we have found that $F_T(\alpha) = \Pr[G_{\cup_{\mathrm{spt}(\alpha)}} = \mathrm{MST}]$ seems to obey the same equation (4) for the connected Erdös-Rényi random graphs and for square lattices, though with a slightly different scaling law $\alpha_c = bN^{-\beta}$ because $b_{\mathrm{lattice}} < b_{G_p(N)}$ while $\beta_{\mathrm{lattice}} \approx \beta_{G_p(N)}$. These observations suggest that $F_T(\alpha)$ may be the same for many connected graphs containing cycles. The prefactor b and the

TABLE I. Simulations that do not produce any rounding off error.

α	Number of digits	α	Number of digits
0.05	180	0.001	6750
0.02	360	0.0005	13500
0.01	720	0.0002	33300
0.005	1440	0.0001	67500
0.002	3600	0.00005	135000

exponent β in $\alpha_c = bN^{-\beta}$ are graph specific. It would be very interesting to have an analytic model that explains these observed phase transitions by changing the extreme value index of the link weight structure of a network.

APPENDIX A: SIMULATIONS IN HIGH PRECISION

For small α, the simulations need to be performed in high precision due to the large relative fluctuation of the link weights close to zero (interval $[10^{-a}, 10^{-b}]$ where $a > b$ can be large). Calculations in double precision floating point numbers on 32-bit processors are found to be inaccurate for $\alpha < 0.1$. For, if $w_1 = 10^{-3} + 10^{-20}$ and $w_2 = 10^{-3} + 10^{-22}$, then for $w_1 - w_2$ the value 0 is returned implying that no distinction between w_1 and w_2 can be made. To prevent rounding-off errors, a high number of digits for very small α values is needed. In Table I, we present the number of digits used in the simulations that produces no rounding-off error.

The Dijkstra shortest path algorithm [13] has been modified to operate with link weights $w = w_{\text{mantissa}} 10^{w_{\text{exponent}}}$, where w_{mantissa} is an array of 32-bit integers and a w_{exponent} is one 32-bit integer. Each 32-bit integer in the w_{mantissa} array accommodates 9 digits, and the length of w_{mantissa} in digits can be adjusted by changing the size of the array. As observed from the table above, the number of w_{mantissa} digits will always be a multiple of 9.

APPENDIX B: THE WEIBULL DISTRIBUTION

We compute the minimum of a set of m i.i.d. polynomial random variables $\{w_k\}_{1 \le k \le m}$ in $[0, 1]$ with $F_w(x) = x^\alpha$ for $0 \le x \le 1$. By choosing an appropriate sequence $\{x_m\}$ such that $\lim_{m\to\infty} m(F(x_m)) = \zeta$ is finite (and preferably nonzero), a scaling law [5, Chapt. 6] for the minimum is

$$\lim_{m\to\infty} \Pr[\min_{1\le k\le m} w_k > x_m] = e^{-\zeta}. \quad \text{(B1)}$$

Applied to $F(x) = x^\alpha$, we have with $m x_m^\alpha = \zeta$ or $x_m = \left(\frac{\zeta}{m}\right)^{1/\alpha}$ that

$$\lim_{m\to\infty} \Pr\left[\min_{1\le k\le m} w_k > \left(\frac{\zeta}{m}\right)^{1/\alpha}\right] = e^{-\zeta}.$$

Let $x = \zeta^{1/\alpha}$, then

$$\lim_{m\to\infty} \Pr[m^{1/\alpha} \min_{1\le k\le m} w_k > x] = e^{-x^\alpha}$$

which shows that the scaled random variable $m^{1/\alpha} \min_{1\le k\le m} w_k$ tends to a Weibull distribution because $\int_x^\infty \text{Weibull}(a, b; u) du = \exp(-a x^b)$.

In addition, the probability that a sum of k independent random variables each with distribution function F_w is at most z and is given by the k-fold convolution,

$$F_w^{k*}(z) = \int_0^z F_w^{(k-1)*}(z-y) f_w(y) dy, \quad k \ge 2,$$

and where $F_w^{1*} = F_w$. By induction it readily follows from (1), that around $z \downarrow 0$,

$$F_w^{k*}(z) \sim \frac{z^{\alpha k} (\alpha \Gamma(\alpha))^k}{\Gamma(\alpha k + 1)}.$$

Suppose independence between the set of m of these k-hop paths, each with weight $w_j(P_k)$ for $1 \le j \le m$ and with same distribution function $F_w^{k*}(z)$, then the appropriate sequence $\{x_m\}$ satisfies

$$m F_w^{k*}(x_m) = m \frac{x_m^{\alpha k} (\alpha \Gamma(\alpha))^k}{\Gamma(\alpha k + 1)} = \zeta$$

or

$$x_m = \zeta^{1/\alpha k} \left(\frac{\Gamma(\alpha k + 1)}{m (\Gamma(\alpha + 1))^k} \right)^{1/\alpha k}$$

such that

$$\lim_{m\to\infty} \Pr[m^{1/\alpha k} \min_{1\le j\le m} w_j(P_k) > x] = \exp\left(-x^{\alpha k} \frac{(\Gamma(\alpha+1))^k}{\Gamma(\alpha k + 1)} \right)$$

which again shows the tendency towards a Weibull distribution with $a = (\Gamma(\alpha+1))^k / \Gamma(\alpha k + 1)$ and $b = \alpha k$.

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