

Partially asymmetric zero-range process with quenched disorder

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We consider the one-dimensional partially asymmetric zero-range process where the hopping rates as well as the easy direction of hopping are random variables. For this type of disorder there is a condensation phenomenon in the thermodynamic limit: the particles typically occupy one single site and the fraction of particles outside the condensate is vanishing. We use extreme value statistics and an asymptotically exact strong disorder renormalization group method to explore the properties of the steady state. In a finite system of L sites the current vanishes as $J \sim L^{-z}$, where the dynamical exponent, z , is exactly calculated. For $0 < z < 1$ the transport is realized by $N_a \sim L^{1-z}$ active particles, which move with a constant velocity, whereas for $z > 1$ the transport is due to the anomalous diffusion of a single Brownian particle. Inactive particles are localized at a second special site and their number in rare realizations is macroscopic. The average density profile of inactive particles has a width of $\xi \sim \delta^{-2}$ in terms of the asymmetry parameter δ . In addition to this, we have investigated the approach to the steady state of the system through a coarsening process and found that the size of the condensate grows as $n_L \sim t^{1/(1+z)}$ for large times. For the unbiased model z is formally infinite and the coarsening is logarithmically slow.

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I. INTRODUCTION

The properties of interacting many-particle systems can be strongly affected by the presence of quenched disorder. This is particularly true for systems of self-driven particles [1–3], where even pointlike defects [4] are able to change the macroscopic properties of the system. Often these defects cause phase separated states, a phenomenon that is known from jam formation at bottlenecks [1,3].

Particularly interesting features arise if not only the amplitude of the hopping rates are quenched random variables, but the directional bias as well. Then the dynamics of the particles is governed by a complex landscape of energy barriers. As the escape time grows exponentially with the heights of the barriers, the largest barriers in the system determine the velocity of the particles. This property of the many-particle system is in agreement with a more classical problem of single particle diffusion in a disordered environment [5], which is rather well understood and serves for numerous applications, e.g., polymer translocation through a narrow pore [6] or the motion of molecular motors on heterogeneous tracks [7].

Since there is no general framework for studying nonequilibrium disordered systems, it is of interest to investigate specific simple models. Here, we consider the zero range process [8] (ZRP) with quenched disorder. The ZRP is particularly well suited for theoretical analysis because the stationary weights of a given configuration factorize and can be exactly calculated [9]. But the ZRP is not only conceptually

interesting, there are a number of important applications as well: it has been used in order to describe, e.g., the formation of traffic jams [10], the coalescence in granular systems [11,12], and the gelation in networks [13,14]. On translational invariant lattices in one dimension the ZRP can be exactly mapped (up to translations of configurations on the lattice) onto a one-dimensional asymmetric simple exclusion process [15–17] (ASEP) if sites are considered as particles and masses as hole clusters.

The effect of quenched disorder on the properties of the ZRP has been investigated in one dimension in the totally asymmetric version [18], i.e., when the particles hop in one direction with position-dependent rates. Lattice disorder in the ZRP is transformed into particle dependent hopping rates in the ASEP, which is generally referred to as particle-wise disorder. In this case a dynamical phase transition takes place from a low density phase, where one observes the condensation of holes, to a homogeneous high density state [19,20]. At low densities the average speed of the particles is related to the lower cutoff of the effective hopping rates. The properties of the phase transition are determined by the asymptotics of the effective hopping rate distribution at the lower cutoff and not by its mean or variance. If the ZRP is partially asymmetric and the easy direction of hopping is random as well, one observes a strong dependence of the average speed of the particles on the system size rather than on the density as in case of the totally asymmetric model [21]. Already this result illustrates the qualitative difference between the two different realizations of the disorder.

In case of the ASEP one can also realize lattice disorder, i.e., site-dependent hop rates. The ASEP both with particle-wise [19,20] and site-wise disorder [3,21–23] have been extensively studied, in particular for systems that are totally asymmetric. The partially asymmetric ASEP with particle-wise disorder has been recently studied by a strong disorder

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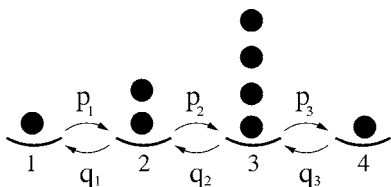


FIG. 1. The zero-range process on a one-dimensional lattice. The topmost particle at site i hops with rate p_i to site $i+1$ and with rate q_{i-1} to site $i-1$.

renormalization group (RG) method [21], which provides asymptotically exact results for large sizes. This RG method was originally introduced in order to study random quantum spin chains [24], but afterwards it was applied for a large variety of quantum [25,26] as well as classical [27] systems, both in and out of equilibrium [28]; for a review see Ref. [2].

In this paper we use this RG method to study the partially asymmetric ZRP with quenched disorder. With this type of disorder there is a condensation phenomenon: almost all particles occupy one single site, whereas the fraction of particles outside the condensate converges to zero in the thermodynamic limit. Here we address questions regarding the size dependence of different quantities, such as the stationary current, the density profile, the mean density in the bulk and the number of particles outside the condensate. We also investigate the approach to the steady state through a coarsening process and the time dependence of the mass of the condensate.

The paper is organized as follows. In the next section we will introduce the disordered ZRP and discuss its relation to the ASEP with particle disorder. In Sec. IV we introduce the RG method which is used to calculate the steady state current in the system. The density profile as well as finite-size behavior of the bulk density is calculated in Sec. V, whereas the coarsening behavior is analyzed in Sec. VI. We summarize and discuss our results in the final section. Some details of the calculations are given in the Appendixes.

II. DEFINITIONS AND PRELIMINARIES

In the ZRP particles hop from site to site on a lattice and the hop rates depend on the departure site and on the number of particles (mass) at that site. With these conditions the stationary weight of a configuration is given in a factorized form, which offers an opportunity to analyze the steady state properties exactly [8]. The type of ZRP we consider in the present work is defined on a one-dimensional periodic lattice with $l=1, 2, \dots, L$ sites and N particles. Particles are allowed to hop to nearest neighbor sites. The hopping rates are quenched random variables, where we denote the forward hopping rate from site l to $l+1$ by p_l and the backward hopping rate from site $l+1$ to l by q_l (see Fig. 1). Compared to the most general ZRP, we do not consider mass-dependent hopping rates, i.e., p_l and q_l are valid for all occupation numbers $n_l \geq 1$. A configuration of the system is characterized by the distribution of masses (number of particles) n_l , where $\sum_{l=1}^L n_l = N$.

The one-dimensional ZRP defined above is equivalent to an ASEP where $l=1, 2, \dots, L$ particles are placed on a ring

with $L+N$ sites. The l th particle hops to empty neighboring sites with a forward (backward) rate of q_l (p_l) and a configuration is defined by the number of empty sites, n_l , behind the particle (i.e., in front of the particle there are n_{l+1} holes).

The stationary weight of a configuration of the ZRP is given in a product form [8,9]:

$$P(\{n_l\}) = Z_{L,N}^{-1} \prod_{l=1}^L f_l(n_l), \quad (1)$$

where

$$Z_{L,N} = \sum_{\{n_l\}} \prod_{l=1}^L f_l(n_l) \delta\left(\sum_{l=1}^L n_l - N\right) \quad (2)$$

is the canonical partition sum. The factors are given by

$$f_l(n_l) = g_l^{n_l} \quad (3)$$

and the g_l satisfy the equations

$$g_l(p_l + q_{l-1}) = g_{l-1}p_{l-1} + g_{l+1}q_l. \quad (4)$$

These equations are identical to the stationary weights of a random walker with random hop rates: p_l ($l \rightarrow l+1$) and q_{l-1} ($l \rightarrow l-1$). The solution of Eq. (4) is given by

$$g_l = \frac{C}{p_l} \left[1 + \sum_{i=1}^{l-1} \prod_{j=1}^i \frac{q_{l+j-1}}{p_{l+j}} \right], \quad (5)$$

where C is a constant. From Eq. (4) one obtains

$$g_{l-1}p_{l-1} - g_lq_{l-1} = g_l p_l - g_{l+1}q_l = \text{const}, \quad (6)$$

independent of l . Taking the constant in Eq. (6) to be one we get

$$C = \left[1 - \prod_{l=1}^L \frac{q_l}{p_l} \right]^{-1}. \quad (7)$$

One can obtain a number of useful results for the stationary state [8]. For example, the occupation probability $p_l(n_l)$ that the l th site contains n_l particles is given by

$$p_l(n_l) = f_l(n_l) \frac{Z_{L-1, N-n_l}}{Z_{L,N}}, \quad (8)$$

and the particle current reads as

$$J_{L,N} = \langle p - q \rangle = \frac{Z_{L,N-1}}{Z_{L,N}}. \quad (9)$$

These expressions are simplified if the number of particles goes to infinity, $N \rightarrow \infty$, whereas there is no restriction on the value of L . In this limit we obtain for the canonical partition sum

$$Z_{L,N} = g_L^N \prod_{l=1}^{L-1} \frac{1}{1 - g_l/g_L}, \quad N \rightarrow \infty, \quad (10)$$

where we label the sites in a way that $g_L = \max(\{g_l\})$. Then the stationary current reads as

$$J_L = 1/g_L, \quad N \rightarrow \infty, \quad (11)$$

and the occupation probability follows a geometrical distribution:

$$p(n_i) = (1 - \alpha_i) \alpha_i^{n_i}, \quad \alpha_i = g_i/g_L, \quad N \rightarrow \infty, \quad (12)$$

so that

$$\langle n_i \rangle = \alpha_i / (1 - \alpha_i), \quad N \rightarrow \infty. \quad (13)$$

For large but finite N and L the sum of the occupation numbers should be equal to the total number of particles, which is evaluated in the saddle-point approximation in Ref. [8] as

$$\sum_{i=1}^L \langle n_i \rangle = \sum_{i=1}^L \frac{1}{1/(g_i J_{L,N}) - 1} = N. \quad (14)$$

In this work the hop rates are independent and identically distributed random variables taken from the distributions, $\rho(p)dp$ and $\pi(q)dq$, respectively, which will be specified later. Generally we allow for the existence of links with $p_l > q_l$ as well as links with $p_l < q_l$ with finite probability, i.e., the easy direction of hopping is a random variable, too.

We introduce a control-parameter, δ , which characterizes the average asymmetry between forward and backward rates

$$\delta = \frac{[\ln p]_{\text{av}} - [\ln q]_{\text{av}}}{\text{var}[\ln p] + \text{var}[\ln q]}, \quad (15)$$

such that for $\delta > 0$ ($\delta < 0$) the particles move on average to the right (left). Here and in the following $[\dots]_{\text{av}}$ denotes average over quenched disorder, whereas $\text{var}[x]$ stands for the variance of x .

One can show that for partially asymmetric processes with nonuniform bias at any nonzero density, $\rho = N/L > 0$, a Bose condensation occurs, in the sense that a finite fraction of the particles, $\langle n_L \rangle / N > 0$, are condensed at site L , and $\langle n_L \rangle / N$ tends to 1 in typical samples for $N \rightarrow \infty$. The condensation of particles can be understood by analyzing the distribution of g_i 's, as we will show in the next section (see for comparison Refs. [20,21]). We shall also show that the stationary current vanishes in the thermodynamic limit, i.e., $\lim_{L \rightarrow \infty} J_L = 0$. These results are in complete agreement with a recently introduced criterion [29], which predicts strong phase separation for vanishing stationary current.

The numerical calculations are carried out using two kinds of disorder distributions. First, a bimodal distribution, where $p_i q_i = r$ holds for all i and

$$\rho(p) = c \delta(p-1) + (1-c) \delta(p-r), \quad (16)$$

with $r < 1$ and $0 < c \leq 1/2$. Second, we use a uniform distribution defined by

$$\begin{aligned} \rho(p) &= p_0^{-1} \Theta(p) \Theta(p_0 - p), \\ \pi(q) &= \Theta(q) \Theta(1 - q), \end{aligned} \quad (17)$$

where $p_0 > 0$ and $\Theta(x)$ is the Heaviside function.

In the following we use the terminology that the model is asymmetric or biased for $\delta > 0$, and it is unbiased for $\delta = 0$.

This latter model is realized if the distribution of the hopping rates is symmetric. For random quantum spin chains, which show analogous low-energy properties, $\delta = 0$ corresponds to the critical point and (a part of) the $\delta > 0$ region is the so-called Griffiths phase [30]. Since the same type of mechanism takes place in the two systems we use the terminology of Griffiths phase for the random ZRP with $\delta > 0$, too.

III. ANALYSIS OF THE CURRENT

In Sec. II we pointed out that the stationary weights of the disordered ZRP are related to the statistical weights, g_i , of a random walker in a random environment. For further analysis it is important to notice that the random variables g defined by Eq. (5), are so-called Kesten variables [31], because there exist a number of rigorous mathematical results concerning their distribution $P_L(g)$. (For a short summary of these results see Appendix C of Ref. [2].) In the Griffiths phase, i.e., for $\delta > 0$ and in the thermodynamic limit, $L \rightarrow \infty$, the limit distribution presents an algebraic tail:

$$P_L(g) \sim g^{-1-1/z}, \quad L \rightarrow \infty, \quad (18)$$

where z is the positive root of the equation

$$\left[\left(\frac{q}{p} \right)^{1/z} \right]_{\text{av}} = 1. \quad (19)$$

In particular we obtain for small δ that

$$z \approx \frac{1}{2\delta}, \quad \delta \ll 1, \quad (20)$$

which is divergent and independent of the actual form of the distributions, $\rho(p)$ and $\pi(q)$.

If we now apply the results for Kesten variables given above to the disordered ZRP, two remarks are in order. First, the algebraic tail in Eq. (18) should be present in the large (but finite) L limit. Second, the g_i 's in Eq. (5) are not strictly independent. However, as will be shown later, g_l and g_k can be considered to be uncorrelated if $|l-k| > \xi$, where ξ is the finite correlation length of the problem. Therefore g_L has to be treated as the largest event of a distribution in Eq. (18) among $\sim L/\xi$ terms. According to extreme value statistics [32] the typical value of g_L follows from the relation $g_L^{-1/z} L = O(1)$, such that

$$g_L \sim L^z. \quad (21)$$

Another result, which we obtain from extreme value statistics, is the typical value of the i th largest occupation probability, $g(i)/g_L$:

$$\frac{g(i)}{g_L} \approx i^{-z}. \quad (22)$$

From these results and from Eq. (9) it follows that the typical value of the current is

$$|J_L| \sim L^{-z}, \quad \delta > 0, \quad (23)$$

thus vanishing algebraically in the thermodynamic limit. Our second result concerns the typical value of particles outside

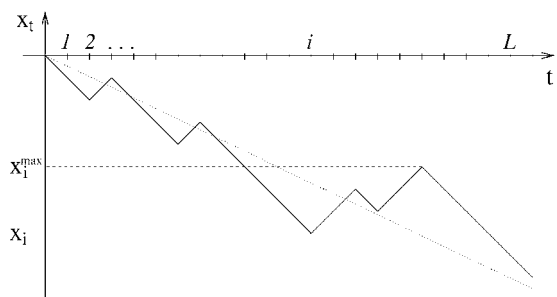


FIG. 2. Mapping the configurations to random walk paths.

the condensate, N_{out} , which follows from Eqs. (13) and (22) as

$$N_{out} = \sum_{l=1}^{L-1} \langle n_l \rangle \sim \sum_{i=1}^{L-1} i^{-z} \sim \begin{cases} L^{1-z}, & z < 1, \\ O(1), & z > 1. \end{cases} \quad (24)$$

Therefore, for any $z > 0$ the fraction of particles in the condensate goes to one in the limit of large system sizes $L \rightarrow \infty$ and constant density, $\rho = L/N$. The typical number of particles outside the condensate, however, behaves differently for $z < 1$, when it is divergent, and for $z > 1$, when it tends to a finite value. We shall discuss this issue in more detail in Sec. V.

Next, we turn to analyze the behavior of the g_i weights in the weakly asymmetric limit $\delta \ll 1$ and estimate the correlation length, ξ . For our analysis it is convenient to use a correspondence between the sequences of the hop rates and random walk paths. (A similar reasoning has been introduced for the random transverse-field Ising chain in Ref. [33].) We consider the situation where the walker starts at $i=0$, $x_0=0$ and takes in its l th step an oriented distance of $\delta x_l = \ln q_{l-1} - \ln p_l$, such that its position, x_l , is given by $x_l = \sum_{i=1}^l \delta x_i$; see Fig. 2. For $\delta > 0$ the motion of the walker is biased (due to the average slope of the energy landscape), but decorated by fluctuations, i.e., one observes local deviations from the average behavior. The typical time during which the walker makes a large excursion, ξ , against the bias follows from the Gaussian nature of the fluctuations and given by [33]

$$\xi \sim \delta^{-2}. \quad (25)$$

More precisely the probability of an excursion time, i , reads as $P(i) \sim \exp(-i/\xi)$. Similarly, from the Gaussian form of the fluctuations follows the typical transversal size of excursions of the walker, which scales as $\xi_{\perp} \sim \xi^{1/2} \sim \delta^{-1}$. Consequently the probability of a transversal size of excursions, Δ , is given by $P(\Delta) \sim \exp(-\Delta/\xi_{\perp})$.

ξ as defined in Eq. (25) is the correlation length of the random walk and at the same time it is the correlation length of the ZRP as discussed in the first part of this section. Indeed, the stationary weight, g_i , is connected to the part of the landscape which starts at position, i , and its value is dominated by the height of the largest excursion, $x_i^{max} - x_i$; see Fig. 2. Since the landscape becomes uncorrelated for distances (times), which are larger than ξ , the corresponding g_i weights are uncorrelated, too. The value of g_L , i.e., the largest weight, is related to the largest possible transversal fluctuation, Δ_L . In



FIG. 3. Renormalization scheme for the zero range process: Site i is eliminated and hopping from site $i-1$ to $i+1$ and vice versa occurs with rates \tilde{p} and \tilde{q} , respectively.

a chain of length, L , this extremal position can be chosen out of $\sim L$ positions, therefore the typical value of Δ_L follows from extreme value statistics as $\exp(-\Delta_L/\xi_{\perp})L = O(1)$, thus $\Delta_L \sim \ln L \delta^{-1}$. Keeping in mind the definition of δx_i and putting this result into Eq. (5) we obtain $\ln g_L \sim \delta^{-1} \ln L$, which is compatible with the exact result in Eq. (21) and the small δ expansion of z in Eq. (20).

In the random unbiased ZRP $\delta=0$ and the correlation length is divergent. In this case the transverse fluctuations of the walker in the thermodynamical limit are unbounded. In a finite system they typically behave as $\Delta_L \sim L^{1/2}$. From this follows that the fluctuations in the particle current in a typical sample are of the form

$$|\ln |J_L|| \sim L^{1/2}, \quad \delta=0. \quad (26)$$

We can thus conclude that in the random walk picture the largest local fluctuation of the energy landscape is responsible for the small value of the particle current. The particles are accumulated in front of this large barrier and build the condensate at a given site of the system. At other subleading barriers there are only a few $O(1)$ particles, the sum of which typically gives N_{out} , as estimated in Eq. (24). Later in Secs. V B 1 and V B 3 we shall use this random walk mapping to obtain the size of the cloud of particles around the condensate and the density profile.

IV. RENORMALIZATION AND THE PARTICLE FLOW

The results on the particle current presented in the previous section can be obtained, together with another results, by the application of an RG method. The use of the RG procedure has different advantages. (i) New results are obtained for the unbiased model. (ii) By this RG method the partially asymmetric model is mapped onto a totally asymmetric system and the approximations used in Sec. III are verified. (iii) It leads to an intuitive picture for the particle current and the condensation phenomena.

Here we first illustrate how single sites of the lattice can be decimated (see Fig. 3); afterwards the method is adopted to the random system and finally the properties of the fixed points, both for the unbiased and asymmetric (biased) models are discussed.

A. Exact decimation of a single site

The factorized form of the stationary distribution in Eq. (1) allows us to explicitly integrate out a typical site, say $i \neq L$. To do this we start with Eq. (6) and eliminate g_i :

$$\frac{p_{i-1}p_i}{p_i + q_{i-1}} g_{i-1} - \frac{q_{i-1}q_i}{p_i + q_{i-1}} g_{i+1} = 1, \quad (27)$$

so that the effective (two site) hop rates are identified as

$$\tilde{p} = \frac{p_{i-1}p_i}{p_i + q_{i-1}}, \quad \tilde{q} = \frac{q_{i-1}q_i}{p_i + q_{i-1}}. \quad (28)$$

At the same time the particle current is transformed as

$$\tilde{J} = \sum_{n_i=0}^{\infty} \frac{Z_{L-1, N-n_i-1}}{Z_{L-1, N-n_i}} p(n_i), \quad (29)$$

which can be obtained along the lines of Eq. (9). From a practical point of view it is of importance that the particle current remains invariant, which happens both (i) for $N \rightarrow \infty$ (even for finite L) and (ii) if $\langle n_i \rangle \rightarrow 0$. For a random system case (ii) is realized for a typical site, if $L \rightarrow \infty$, even if N is finite. Indeed, in this case $g_i = O(1)$ whereas $g_L = J_{L,N}^{-1} \rightarrow \infty$. Since a site outside the condensate generally contains a finite number of particles, these are all “typical” so that one expects to be able to repeat the transformation up to site L .

B. Renormalization of the random ZRP

In the following we shall apply the RG transformation in case (ii), i.e., for a random system in the infinite (or very large) lattice limit. The corresponding single particle problem of a random walker in a random potential is thoroughly studied in the literature and many exact results have been obtained by using the strong disorder RG method [2,27]. The principles of the RG procedure apply as well to the ZRP with many particles. There are, however, questions which are specific to the large N limit, such as the properties of the condensate, the profile of the particles, and the coarsening process towards the stationary state, which will be studied in the following sections.

In the following we adopt the RG rules in Eq. (28) for the random system. In the traditional application of the strong disorder RG method for a Brownian particle one renormalizes the energy landscape [27] in Fig. 2. Here we apply a somewhat different approach and also refer to a mapping to random quantum spin chains. The first step is to select the site (bond) to be decimated out. We choose the fastest rate, $\Omega = \max(\{p_i\}, \{q_i\})$, which means that processes which are faster than the time scale, $\tau = 1/\Omega$, are integrated out. Here for Ω we will refer to the rate or energy scale, which is given by the gap of the evolution matrix [17] of a block of sites. If the distribution of disorder is sufficiently broad, then the rate Ω is much larger than the neighboring rates and the expressions for the renormalized rates in Eq. (28) can be simplified. For example if $\Omega = p_i$ we have

$$\tilde{p} \approx p_{i-1}, \quad \tilde{q} \approx \frac{q_{i-1}q_i}{\Omega}, \quad (30)$$

and similarly for $\Omega = q_{i-1}$, by replacing q_i and p_i . It is evident from Eq. (30) that the generated new rate is smaller than the eliminated rate Ω . Repeating the decimation procedure the rate (energy) scale is gradually lowered and we monitor the distribution of the hop rates, $P(p, \Omega)$ and $R(q, \Omega)$, respectively. In particular we are interested in the scaling properties of the transformation at the fixed point, which is located at $\Omega^* = 0$. This latter statement is in accordance with the fact that the stationary current vanishes in the system.

Using the approximate decimation rules in Eq. (30) the RG equations can be formulated in the continuum limit as a set of integral-differential equations, that can be exactly solved at the fixed point, both at $\delta=0$ [25] and for $\delta \neq 0$ [34]. These calculations are identical to that of the random transverse-field Ising chain (RTFIC) and we borrow the results obtained for the latter model. Details about the mapping between the two problems as well as the fixed-point solutions are presented in Appendix A.

C. The unbiased ZRP

The unbiased ZRP with $\delta=0$ corresponds to the critical point of the RTFIC. At this point the distributions of forward and backward rates in Eq. (A5) are identical, having the same exponents: $r_0 = p_0 \sim 1/\ln(\Omega/\Omega_0)$. Here, Ω_0 is a reference rate (energy) scale. The appropriate scaling variable is given by $\eta = -(\ln \Omega - \ln p)/\ln \Omega = -(\ln \Omega - \ln q)/\ln \Omega$, having a distribution $\rho(\eta) = \exp(-\eta) d\eta$, $\eta > 0$. The length scale, L_Ω , which is the size of the effective cluster, and the rate (energy) scale are related as

$$L_\Omega \sim \left[\ln \frac{\Omega_0}{\Omega} \right]^2, \quad \delta = 0, \quad (31)$$

which shows unusual, activated scaling. In the ZRP in the course of renormalization huge particle clusters are created and the distance, X , they travel is the accumulated distance covered by the original particles that form the cluster $X = \sum_{k=1}^X x_k$. Finally, after eliminating all but the last site, we obtain the accumulated distance traveled by all the particles during time t . In the unbiased case $[\langle X \rangle]_{\text{av}} = 0$ holds due to symmetry. In this case the average mean square of the accumulated displacement, $[\langle X^2 \rangle]_{\text{av}}$, has the same scaling behavior as L_Ω^2 in Eq. (31) at a rate (energy) scale, $\Omega \sim t^{-1}$. Consequently we obtain

$$[\langle X^2 \rangle]_{\text{av}} \sim \ln^4 t, \quad (32)$$

in agreement with the diffusion of a Sinai walker [35]. Furthermore, an appropriate scaling combination between current (measured in a specific sample) and size is obtained as $|\ln |J_L|| L^{-1/2}$, which is compatible with the relation in Eq. (26).

D. The asymmetric ZRP

The ZRP with a global bias, $\delta > 0$ (or equivalently $\delta < 0$), corresponds to the so-called disordered Griffiths phase [30] of the RTFIC. In this case a time scale $\tau \sim \Omega_\xi^{-1}$ exists, which separates two characteristic areas of the renormalization process. In the initial part of the renormalization, for $\Omega > \Omega_\xi$, both backward and forward rates are decimated out, until effective clusters of typical size, ξ , are created, where the correlation length ξ is defined in Eq. (25) through Eq. (31). In the random walk picture in Fig. 2 this corresponds to eliminating the fluctuations of the landscape and obtaining a monotonically decreasing curve. Now continuing the decimations for $\Omega < \Omega_\xi$, almost exclusively forward rates are eliminated and the ratio of the typical backward and forward rates tends to zero. Consequently the system is renormalized

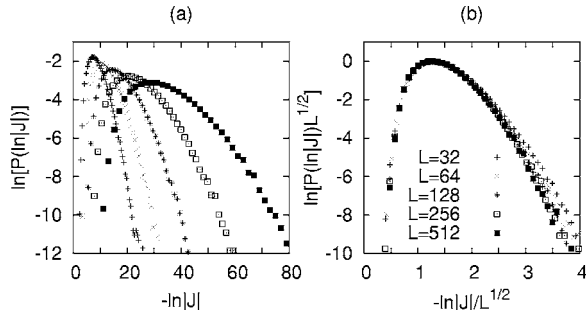


FIG. 4. Distribution of the steady state current in the random unbiased ZRP calculated from Eqs. (5) and (11). (a) Note that the distribution broadens with the size, which is a clear indication of an infinite disorder fixed point. (b) An appropriate scaling collapse is obtained using the scaling combination in Eq. (26).

to a totally asymmetric ZRP in which the distribution of the forward rates can be calculated [see Eq. (A5)], and follows a pure power law [see Eq. (A5)]:

$$P_0(p, \Omega) \approx \frac{1}{z\Omega} \left(\frac{\Omega}{p} \right)^{1-1/z}, \quad \Omega < \Omega_\xi, \quad (33)$$

where z is the dynamical exponent as defined in Eq. (19). For comparison with Eq. (18) one should note that for the totally asymmetric model the weights in Eq. (5) are $\sim 1/p_i$, consequently Eqs. (33) and (18) are of identical form. Furthermore the effective particles become indeed uncorrelated at a length scale of ξ .

If effective clusters of size L are created, the corresponding rate (energy) scale is lowered as

$$L \sim \left(\frac{\Omega}{\Omega_0} \right)^{-1/z}, \quad \delta > 0. \quad (34)$$

Consequently the smallest effective forward hop rate is given by $\bar{p}_L \sim L^{-z}$. This implies the same relation for the stationary current as given in Eq. (23).

The results on the dynamical behavior of the random ZRP presented in the previous subsections are expected to be asymptotically exact. Indeed during renormalization the distributions of the hop rates broaden without limits both at the critical point and in the Griffiths phase; consequently the decimation rule in (30) becomes exact at the fixed point. For static quantities, however, such as the density profile and static correlations the RG results are exact only in the vicinity of the critical point, where the correlation length is divergent. Here the RG results are correct in linear order of δ . We note that for the biased Sinai walker the RG approach has been generalized in order to calculate higher order corrections, too [36].

The scaling forms of the current in Eqs. (23) and (26) are tested by numerical calculations in Ref. [21] for the partially asymmetric ASEP with bimodal, i.e., discrete disorder in Eq. (16). Here we performed calculations on the ZRP using the uniform, i.e., continuous disorder in Eq. (17). The agreement is indeed satisfactory, both for the unbiased (Fig. 4) and for the biased (asymmetric) (Fig. 5) cases.

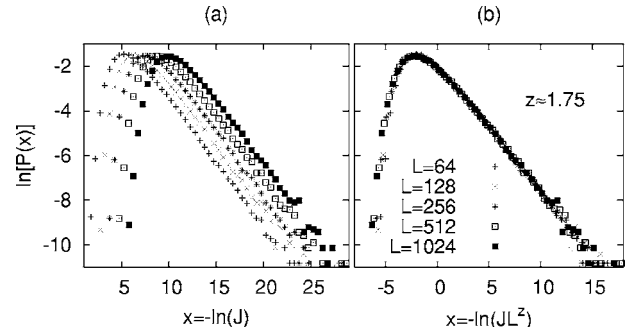


FIG. 5. Distribution of the steady state current in the random asymmetric ZRP, i.e., in the Griffiths phase calculated from Eqs. (5) and (11). (a) In a log-log plot the distribution is shifted to the left with increasing sizes, the asymptotic slope of the curves is $1/z$ and can be used to measure the dynamical exponent. (b) An appropriate scaling collapse is obtained using the scaling combination in Eq. (23) and the exact result for z in Eq. (19).

V. DISTRIBUTION OF PARTICLES

In order to characterize the microscopic states of the disordered ZRP it is instructive to distinguish between the $\langle n_L \rangle$ particles in the condensate and the N_{out} particles outside the condensate. These latter particles can be of two different kinds. The current in the system is carried by N_a active particles, which behave as single random walkers. There are also N_{ia} inactive particles in the system that are localized either around the condensate or at a remote specific site. In the following we study in details the distribution of active and inactive particles.

A. Active particles

Active particles carry the current in the system, which is given by $J_L = vN_a/L$. Here v is the average velocity of a Brownian particle in this random environment and given by [5] $v = O(1)$ for $z < 1$ and $v \sim L^{-z+1}$ for $z > 1$. In the latter case we speak about anomalous diffusion. Comparing this result for the current with the estimate in Eq. (23) one can obtain the number of active particles, N_a , which behaves differently for $z > 1$ and $z < 1$, respectively.

1. Single particle transport $z > 1$

For $z > 1$ the current is so small that it is produced by a finite number of active particles, i.e., $N_a = O(1)$. In this case the accumulated distance traveled by all particles is simply given by

$$X \sim t^{1/z}, \quad z > 1. \quad (35)$$

2. Many particle transport $z < 1$

For $z < 1$, however, the current in the ZRP is produced by $N_a \sim L^{1-z}$ active particles. Now the active particles have a constant velocity, thus during time t they travel a distance of $\sim t$.

Note that the number of active particles, N_a , is of the same order of magnitude as the typical number of particles

outside the condensate, N_{out} , see Eq. (24). This is due to the fact that in a typical sample there are only finite numbers of inactive particles. This issue is discussed in detail in the following subsection.

B. Inactive particles

Inactive particles in the random ZRP are of two kinds. The first kind of inactive particles is found in a “cloud,” which is localized next to the condensate. The formation of this cloud is due to an attractive property of the energy landscape: particles that left the condensate and did not travel farther than a distance ξ will typically turn back. The second sort of inactive particles are found to be localized at the subleading extrema of the energy landscape, i.e. they are at a site \tilde{l} , where $g_{\tilde{l}}$ in Eq. (5) takes the next-to-leading value. As shown in Eq. (22) the typical value of $\langle n_{\tilde{l}} \rangle$ is of the order of one. Its average value, however, is divergent and will be determined below.

1. Typical behavior of the cloud

Here we consider a typical sample in which the cloud is due to the attraction of the condensate and well separated from the subleading extrema of the landscape. In this situation the density of inactive particles is expected to decay exponentially, $n_l \sim \exp(-l/l_w)$, where l measures the distance from the condensate and l_w is the typical width of the cloud. We estimate l_w by the following consideration. The correlated cluster at the condensate has the largest size among the clusters and its value, ξ_L , follows from extreme value statistics. Using the distribution function of cluster sizes below Eq. (25) we obtain for its typical value $\xi_L \sim \xi \ln L$. At this distance, $l = \xi_L$, the typical value of the weight is $g_l = O(1)$, and with Eqs. (13), (12), and (21) we obtain $n_l \sim L^{-z} \sim \exp(-\xi_L/l_w)$. Consequently

$$l_w \sim \frac{\xi}{z} = \xi_{typ} \sim \delta^{-1}, \quad (36)$$

where we have made use of the scaling relation [2,37]: $\xi/\xi_{typ} = z$, where ξ_{typ} is the typical correlation length. The small δ behavior in the last equation follows directly from Eqs. (25) and (20). Thus we have obtained the result that the typical width of the cloud of inactive particles is measured by the typical correlation length of the biased Sinai walker.

The typical density profile of inactive particles has the same scaling behavior as the typical and the average value of the occupation number, α_i , as defined in Eq. (12). For this latter quantity we have checked numerically the scaling form in Eq. (36) for different values of δ and satisfactory agreement is found; see Fig. 6.

2. Average number of inactive particles

The typical value of inactive particles is finite. However, in rare realizations it is possible to find even a macroscopic number of inactive particles. This is due to the fact that the leading value of the weights in Eq. (5), g_L , and the subleading value, $g_{\tilde{l}} \equiv g$, can be arbitrarily close to each other. The average value of $[\langle n_{\tilde{l}} \rangle]_{av} \approx N_{ia}$ is dominated by such rare re-

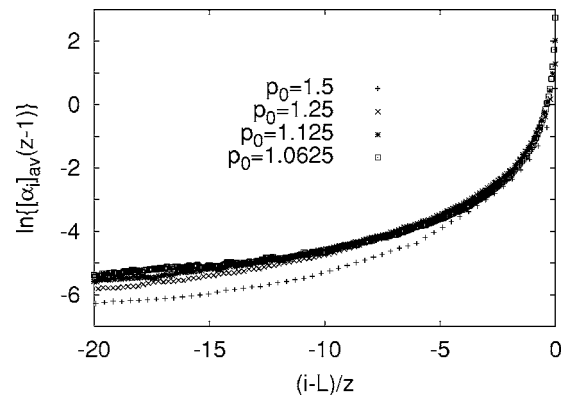


FIG. 6. Average of the occupation probability, α_i , which corresponds to the typical density of the cloud. The data were obtained by solving Eq. (14) numerically for $L=N=2048$ and the disorder average was performed over 500 000 samples. We used the uniform distribution in Eq. (17) for different values of the asymmetry parameter, δ , and thus for the dynamical exponent, $z > 1$. The data are rescaled according to (B4) and using the length scale in (36) with (19).

alizations, in which both $\langle n_L \rangle$ and $\langle n_{\tilde{l}} \rangle$ are macroscopic, i.e., they are of $O(N)$. However, as we checked numerically, the contribution to the average value of such samples in which more than two sites have macroscopic occupation is negligible. Therefore in the subset of rare realizations we should consider only two active sites, L and \tilde{l} , and the unnormalized weight that the subleading site contains $n_{\tilde{l}} = n$ particles is in leading order proportional to $g^n g_L^{N-n} = \alpha^n g_L^N$. The distribution of n is thus approximately $P_\alpha(n) = (1-\alpha)/(1-\alpha^{N+1})\alpha^n$ for $\alpha < 1$ and $P_\alpha(n) = 1/(N+1)$, for $\alpha = 1$ [38]. Thus we obtain for the expectation value

$$\langle n \rangle(\alpha) = \frac{\alpha}{1-\alpha} - (N+1) \frac{\alpha^{N+1}}{1-\alpha^{N+1}}, \quad \alpha < 1, \quad (37)$$

and $\langle n \rangle = N/2$ for $\alpha = 1$. Using the distribution function, $\rho(\alpha)$, we can average over the rare realizations, $[\langle n_{\tilde{l}} \rangle]_{av} = \int_0^1 \langle n \rangle(\alpha) \rho(\alpha) d\alpha$, which is dominated by the contribution as $\alpha \rightarrow 1$. Keeping in mind that the maximal value of $\langle n \rangle$ is $N/2$, we can write

$$N_{ia} \approx [\langle n_{\tilde{l}} \rangle]_{av} \approx \int_0^{1-2/N} \frac{\alpha}{1-\alpha} \rho(\alpha) d\alpha \sim \rho(1) \ln N, \quad (38)$$

provided that the distribution $\rho(\alpha)$ has a finite limiting value at $\alpha \rightarrow 1$. As we have checked numerically this is indeed the case, both for the unbiased and the biased (asymmetric) models. Consequently the average number of inactive particles is logarithmically divergent, although its typical value is finite. Furthermore the ratio between the average numbers of the active and inactive particles, N_a/N_{ia} , tends to zero for $z > 1$ and tends to infinity for $z < 1$.

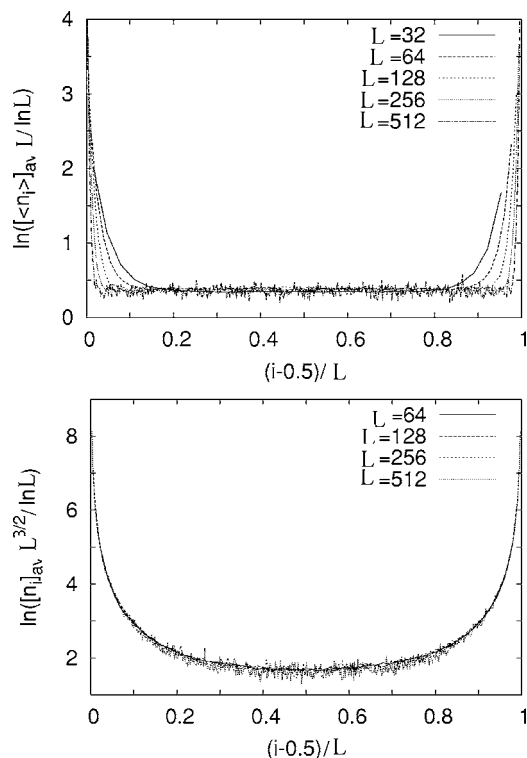


FIG. 7. Average density profiles of inactive particles for different system sizes, obtained by solving Eq. (14) numerically. The density was $\rho=1$ and average was performed over 500 000 samples. Up: asymmetric model with the uniform distribution, $p_0=3$. The profile is constant outside the renormalized cluster of the condensate, the size of which is finite. Down: unbiased model, $p_0=1$, rescaled according to (39).

3. Average density profile of inactive particles

The average density profile of inactive particles, $[\langle n_i \rangle]_{\text{av}}$, is proportional to $P(l)$, which is the probability density, that the subleading, almost degenerate $g \approx g_L$ is located at $l = \tilde{l}$. In the asymmetric model \tilde{l} can be either at the renormalized cluster of the condensate or at any other cluster. In the second case $P(l)$ is a constant, which happens for $l > \xi$. This behavior is illustrated in the upper part of Fig. 7.

In the unbiased model, as can be seen from Eq. (25) the width of the cluster diverges. Therefore \tilde{l} and L are always in the same cluster, and as a consequence the probability distribution is scale-free and a function of l/L . The form of the probability distribution for $l \ll L$ can be obtained from the random walk picture in Fig. 2. The rare event for this process is represented by a landscape that is drawn by an unbiased ($\delta=0$) random walker, which starts at one minimum and after l steps arrives to a degenerate second minimum. This means that the random walker is surviving (since it does not cross its starting position) and returns to its origin. The fraction of such random walks is given by [33] $P(l) \sim l^{-3/2}$, $l \ll L$, consequently $P(l) = L^{-3/2} \tilde{P}(l/L)$, where $\tilde{P}(x)$ is a smooth scaling function that behaves for small x as $\tilde{P}(x) \sim x^{-3/2}$. With this prerequisite we obtain for the average density profile of inactive particles for a given density $\rho = N/L > 0$:

$$[\langle n_i \rangle]_{\text{av}}(L) = \ln(L) L^{-3/2} \tilde{P}(l/L). \quad (39)$$

Results of numerical calculations are presented in the right part of Fig. 7, which are in excellent agreement with the theoretical predictions.

VI. COARSENING

In the stationary state of the process typically almost all particles occupy a single site. If initially the particles are uniformly distributed on the lattice, the system undergoes a coarsening process, meaning that the number of particles in the condensate and the typical size of empty regions is growing as time elapses. This process has already been studied for the nonrandom ZRP [39]; here we study the disordered case. To be specific we consider finite L and N and define the length scale as [19] $l(t) \equiv \sigma^2(t) = [(1/L) \sum_{i=1}^L \langle n_i^2 \rangle(t)]_{\text{av}}$, which characterizes the typical number of particles at sites with a nonmicroscopic n_i , or equivalently the typical distance between these sites. In the RG procedure the latter corresponds to the typical cluster sizes (defined as the sum of the original links) at the rate (energy) scale Ω . The growth rate of the typical distance, dl/dt , is proportional to the typical current at that rate (energy) scale. This leads to the differential equation:

$$\frac{dl}{dt} \sim J_l. \quad (40)$$

At the Griffiths phase we have in (23), $J_l \sim l^{-z}$, thus the solution of Eq. (40) is given by

$$l \sim t^{1/\zeta}, \quad \zeta = z + 1. \quad (41)$$

Here ζ is the dynamical coarsening exponent. Note that ζ is exactly known through Eq. (19) and it is a continuous function of the parameters appearing in the distributions of hop rates. Close to the critical point ζ becomes universal, depending only on the control parameter: $\zeta \approx 1/(2\delta)$; see Eq. (20). We note that the same result can be obtained by referring to the result of the RG procedure. As shown in Sec. IV D for $\Omega < \Omega_\xi$ the ZRP is transformed to a totally asymmetric ASEP with a rate distribution in Eq. (33). The coarsening of the totally asymmetric ASEP has been analyzed in Refs. [18,19,40] with the result given in Eq. (41).

At the critical point ζ diverges and the coarsening is ultraslow and strictly at $\delta=0$ the length scale grows anomalously (logarithmically) with t . This type of growth is the so-called *anomalous coarsening* [41], which can be observed, e.g., in spin glasses.

The asymptotic time dependence of l can be obtained by inserting the scaling form of the typical current through an empty region of size l into the differential equation in Eq. (40). The scaling form is accordingly (26) given by $J_l \sim e^{-cl^{1/2}}$. Equation (40) has the asymptotic (large t) solution $l^{1/2} e^{cl^{1/2}} \sim t$, thus the growth of the length scale is logarithmically slow

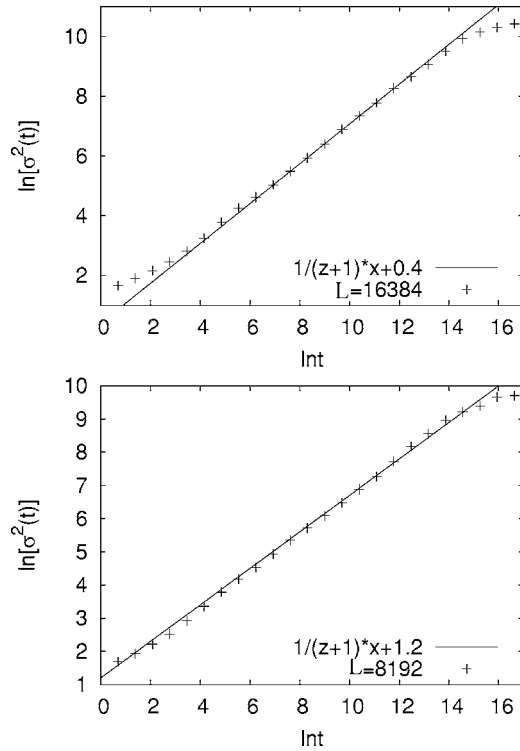


FIG. 8. Time dependence of the coarsening length scale, l , in the Griffiths phase in a log-log plot. We used the bimodal distribution (16) with $c=0.2$, $r=0.5$ where $z=0.5$ (left) and $c=0.3$, $r=0.5$ where $z \approx 0.818$ (right), having a density $\rho=3$, and disorder average was performed over a few hundred samples. The slope of the straight line is the theoretical result in Eq. (41): $1/(1+z)$.

$$l \sim \left[\ln \left(\frac{t}{\ln t} \right) \right]^2. \quad (42)$$

Results of numerical simulations are presented in Fig. 8 for the Griffiths phase (biased model) and in Fig. 9 for the unbiased model. The agreement with the theoretical results is satisfactory.

At the boundary of the Griffiths phase, where $z=0$, the coarsening is controlled by the behavior of the forward rate distribution at $p \rightarrow 0$. To see this, let us consider an arbitrary forward rate distribution for which $\rho(p) \sim p^{-1+1/z_0}$ ($z_0 > 0$) as $p \rightarrow 0$, and an arbitrary backward rate distribution, such that $\delta > 0$. Analyzing (19) it is easy to show that $\zeta > z_0 + 1$ in the whole Griffiths phase $\delta > 0$, and at the boundary

$$\lim_{\delta \rightarrow \infty} \zeta(\delta) = z_0 + 1, \quad (43)$$

which is the coarsening exponent of a totally asymmetric process obtained from the original one by setting the backward rates to zero. Thus, as expected, the presence of non-zero backward rates slows down the coarsening in so far as the dynamical exponent is larger in the whole Griffiths phase than that of the totally asymmetric process with zero backward rates, and the latter value is recovered at the boundary of the Griffiths phase.

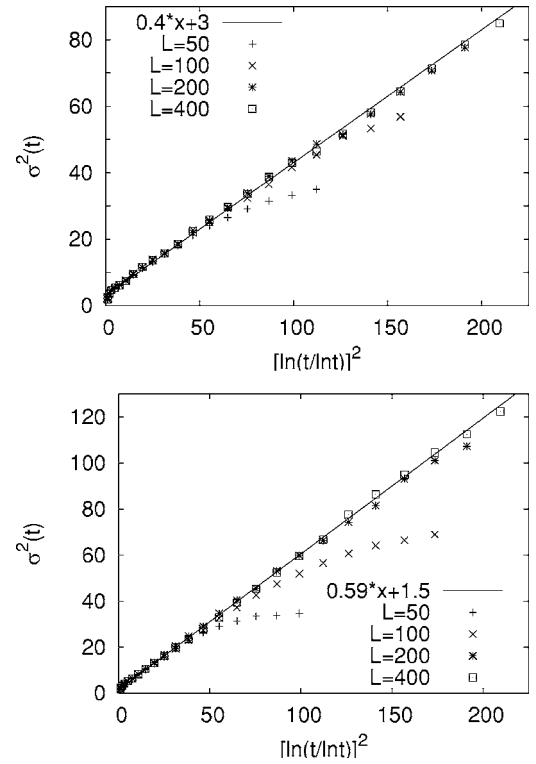


FIG. 9. Time dependence of the coarsening length scale, l , for the unbiased model using the theoretical combination in Eq. (42). We used the bimodal distribution (16) with $c=0.5$, $r=0.1$ (left) and $c=0.5$, $r=0.2$ (right), having a density $\rho=3$, and disorder average was performed over a few hundred samples.

VII. CONCLUSION

In this paper the prototype of a diffusive many-particle system, the zero range process, is studied in one dimension, in the presence of quenched disorder. This model system is of large importance since its stationary state is in a product form and therefore can be constructed analytically. At the same time there exists a wide range of different realizations of the ZRP, and therefore it can be applied to various problems of stochastic transport. We have analyzed the properties of the steady state of the disordered system and obtained many exact results. These results are expected to be generic for random, driven nonequilibrium systems and therefore can be useful to analyze more complicated, nonintegrable processes, too. It is also important to notice that our results are directly related to another fundamental model of nonequilibrium transport, i.e., the ASEP with particle-wise disorder.

A. Types of the transport

The transport in the system is related to the form of the hop rate distribution and characterized by the value of the dynamical exponent z defined in Eq. (19). We can distinguish four main types of transport in the random ZRP.

1. $z=0$

In this case the maximal value of the backward hop rates, q_{max} , is smaller than the minimal value of the forward hop

rates, $p_{min} > q_{max}$. This situation is equivalent to the totally asymmetric ZRP, which has been previously analyzed in the literature. In this case there is a finite stationary current in the system, such that the number of active particles (which carry the current) is $N_a \sim L$ and the particles have a finite velocity, $v = O(1)$. Condensation of particles at a particular site is not generic, but appears only for hop rate distributions that are vanishing fast enough at the lower cutoff $c > 0$ (see [1] for details).

2. $0 < z < 1$

In this case $q_{max} > p_{min}$ and the current is vanishing in the thermodynamic limit as $J \sim L^{-z}$. The transport in the system is effected by $N_a \sim L^{1-z}$ active particles, which move with a finite velocity, $v = O(1)$. Note that in the system there are infinitely many active particles, but their density is zero.

3. $1 < z < \infty$

The current in the system is vanishing according to $J \sim L^{-z}$, and the transport is effected by $N_a = O(1)$ active particles, which have a vanishing velocity. Thus the transport is realized by the anomalous diffusion of a few biased random walkers.

4. Unbiased ZRP: $z \rightarrow \infty$

The average current is zero and the fluctuations of the accumulated displacement are due to a few Sinai walkers and given in Eq. (32).

B. Problem with self-averaging

In the partially asymmetric random ZRP with $z > 0$ the number of active particles, N_a , and thus the transport is self-averaging, i.e., the typical and the average values have the same scaling behavior. The inactive particles, however, which are outside the condensate but do not contribute to the transport, have different properties. Their typical number is $O(1)$ whereas their average diverges as $\ln N$. The main contributions which lead to the logarithmic dependence on N are rare realizations of the disorder including a second site with macroscopic number of particles.

C. Relation with the totally asymmetric ZRP

The RG framework used in this paper has revealed a relation between the partially asymmetric ZRP with arbitrary type of initial disorder and the totally asymmetric ZRP with a hop rate distribution, which has a vanishing power-law tail. Indeed, during renormalization if the typical size of the renormalized new clusters becomes larger than the correlation length, the new effective particles perform a totally asymmetric motion with a power-law distribution of the hop rates given in Eq. (33). The exponent of this distribution is related to the form of the distribution in the original (i.e., partially asymmetric) model; see Eq. (19). In this way the problem, which has been thoroughly studied for the random totally asymmetric ZRP appears naturally as the fixed point problem of the partially asymmetric ZRP. The power-law

exponent then appears in a self-organized fashion. In the terminology of random systems both problems have the same type of *strong disorder fixed point*. We note that a similar relation is encountered between the biased Sinai walk and the directed trap model [2]. On the other hand, for the unbiased ZRP the singular behavior is governed by a so-called *infinite disorder fixed point*.

D. Condensation transition for limited number of particles

Throughout the analysis of the partially asymmetric model we considered exclusively the case of a finite density, $\rho = N/L > 0$. In this case there is always a condensate present, where one typically finds a finite fraction of the particles. However, if the number of particles scales as $N \sim L^\omega$, $0 < \omega < 1$, one expects that the condensate will disappear for sufficiently small values of ω . Indeed, the transport in the system can involve $N_a \sim L^{1-z}$ particles, for $0 < z < 1$, consequently for $\omega < 1 - z$ all the particles carry current and the condensate is absent in the system. At the borderline case, with $N = AL^{1-z}$ one expects that the fraction of particles in the condensate varies with A and that there is a condensation transition possibly at a finite value of $A = A_c = O(1)$. First numerical results support this scenario of the condensation transition, but a more detailed analysis of the phase transition, which is due to the slow dynamics of the process quite involved, is not yet completed. We are going to clarify this phenomenon in a separate work.

E. Higher dimensions

The ZRP process has the same type of factorized steady state for any type of lattices, thus also in higher dimensions. Also the stationary weights involve site-dependent quantities, g_i , which are the stationary weights of the random walk on the given lattice. The renormalization described in Sec. IV can be implemented numerically in this case. Similar calculations have already been performed for random quantum systems [2]. During the RG procedure one obtains large effective clusters having a complicated topology. One might ask the question whether the strong and infinite disorder nature of the fixed points as obtained in 1D remains also in higher dimensions. The answer to this question is negative. We know that the underlying random walk process has an upper critical dimension, $d_u = 2$, so that the Gaussian nature of the random walk remains [42] (in $d = 2$ with logarithmic corrections) even in the presence of quenched disorder. The same type of irrelevance of disorder is expected to hold for the ZRP, too.

ACKNOWLEDGMENTS

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APPENDIX A: MAPPING TO THE RTFIC, THE RG EQUATIONS, AND THEIR FIXED POINT SOLUTION

Renormalization of the one-dimensional ZRP with quenched disorder as described in Sec. IV is equivalent to the same procedure of a random transverse-field Ising chain (RTFIC) described by the Hamiltonian

$$H = - \sum_{l=1}^L J_l \sigma_l^x \sigma_{l+1}^x - \sum_{l=1}^L h_l \sigma_l^z, \quad (\text{A1})$$

where $\sigma_l^{x,z}$ are Pauli spin operators at site l , and the couplings, J_l , and the transverse fields, h_l , are quenched random variables. During renormalization the largest term in the Hamiltonian, $\Omega = \max(\{J_i\}, \{h_i\})$ is successively eliminated. For example, if the largest term is a coupling, $\Omega = J_i$, the connected two sites, $(i, i+1)$, flip coherently and form an effective two-site cluster. The renormalized value of the transverse field acting on the spin cluster is given by a second-order perturbation calculation as

$$\tilde{h} \approx \frac{h_i h_{i+1}}{\Omega}. \quad (\text{A2})$$

Decimating a strong transverse field, $\Omega = h_i$, will result in decimating out site i and generating a new coupling \tilde{J} between the remaining sites, $i-1$ and $i+1$. The value of \tilde{J} follows from self-duality; in Eq. (A2) one should replace h_i by J_i . Comparing the decimation rules for the ZRP in Eq. (30) to that for the RTFIC in Eq. (A2) one observes a complete agreement with the correspondences: $p_i \leftrightarrow J_i$ and $q_i \leftrightarrow h_i$. Now properties of the RG flow for the ZRP can be obtained from the equivalent expressions for the RTFIC.

For example the probability distribution of the backward, $R(q, \Omega)$, and forward, $P(p, \Omega)$, hop rates satisfy the set of integral-differential equations,

$$\begin{aligned} \frac{dR(q, \Omega)}{d\Omega} &= R(q, \Omega)[P(\Omega, \Omega) - R(\Omega, \Omega)] \\ &\quad - P(\Omega, \Omega) \int_q^\Omega dq' R(q', \Omega) R\left(\frac{q\Omega}{q'}, \Omega\right) \frac{\Omega}{q'}, \end{aligned} \quad (\text{A3})$$

$$\begin{aligned} \frac{dP(p, \Omega)}{d\Omega} &= P(p, \Omega)[R(\Omega, \Omega) - P(\Omega, \Omega)] \\ &\quad - R(\Omega, \Omega) \int_p^\Omega dp' P(p', \Omega) P\left(\frac{p\Omega}{p'}, \Omega\right) \frac{\Omega}{q'}, \end{aligned} \quad (\text{A4})$$

having a solution at the fixed point

$$\begin{aligned} P_0(p, \Omega) &= \frac{p_0(\Omega)}{\Omega} \left(\frac{\Omega}{p}\right)^{1-p_0(\Omega)}, \\ R_0(q, \Omega) &= \frac{r_0(\Omega)}{\Omega} \left(\frac{\Omega}{q}\right)^{1-r_0(\Omega)}, \end{aligned} \quad (\text{A5})$$

with $0 < p, q \leq \Omega$. The value of the exponents, $p_0(\Omega)$ and $r_0(\Omega)$, depend on original distributions and therefore on the value of the control parameter, δ . The specific values are given in Sec. IV.

APPENDIX B: AVERAGE OF THE OCCUPATION PROBABILITY

Here we start with the asymptotic distribution of the (uncorrelated) g_i weights in Eq. (18) with the condition that the largest value is fixed, $g_L = G$, so that

$$\rho(g|G)dg = \frac{1/z g^{-1/z-1}}{1 - G^{-1/z}} \Theta(G - g). \quad (\text{B1})$$

From this we obtain for the distribution of the occupation probabilities, $\alpha_i = g_i / g_L$, for large L as

$$\rho(\alpha)d\alpha \approx \frac{1}{z} \frac{1}{\alpha} \left[\frac{1}{L\alpha^{1/z}} - e^{-L\alpha^{1/z}} \left(1 + \frac{1}{L\alpha^{1/z}} \right) \right] d\alpha. \quad (\text{B2})$$

The average of α is therefore given by

$$[\alpha]_{\text{av}} = \int_0^1 \alpha \rho(\alpha) d\alpha \approx L^{-z} \int_0^L \left[\frac{1}{y} - e^{-y} \left(1 + \frac{1}{y} \right) \right] y^{-1+z} dy. \quad (\text{B3})$$

In leading order of L the second term in the bracket can be neglected and we arrive at

$$[\alpha]_{\text{av}} \approx L^{-z} \left(\text{const} + \frac{L^{z-1}}{z-1} \right) \sim L^{-z}, \quad z < 1, \quad (\text{B4})$$

$$[\alpha]_{\text{av}} \approx L^{-1} (\text{const} + \ln L) \sim L^{-1} \ln L, \quad z = 1,$$

$$[\alpha]_{\text{av}} \approx L^{-z} \left(\text{const} + \frac{L^{z-1}}{z-1} \right) \sim \frac{L^{-1}}{z-1}, \quad z > 1.$$

Thus, as far as $z > 1$ the occupation probability is non-self-averaging, and the decay exponent is becoming one independent of δ .

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