PHYSICAL REVIEW E **72**, 045202(R) (2005)

Chaotic channel

M. S. Baptista and J. Kurths
Universität Potsdam, Institut für Physik, Am Neuen Palais 10, D-14469 Potsdam, Deutschland
(Received 25 February 2005; revised manuscript received 31 August 2005; published 17 October 2005)

This work combines the theory of chaotic synchronization with the theory of information in order to introduce the chaotic channel, an active medium formed by connected chaotic systems. This subset of a large chaotic net represents the path along which information flows. We show that the possible amount of information exchange between the transmitter, where information enters the net, and the receiver, the destination of the information, is proportional to the level of synchronization between these two special subsystems.

DOI: 10.1103/PhysRevE.72.045202 PACS number(s): 05.45.Vx, 05.45.Xt

A communication system, as defined by Shannon [1], is composed of an information source that produces a message, a transmitter that transforms the message into a signal suitable for transmission over a channel, such that the message can be retrieved in the receiver with a minimal amount of errors. The most outstanding result in Shannon's theory of communication is the formula that gives the channel capacity, i.e., the average upper bound for the mutual information exchange between the transmitter and the receiver, or in other words, the possible amount of information that can be transmitted in a physical medium.

In chaos-based communication systems, each step of the communication can be performed using a chaotic system. As shown in Ref. [2], chaotic systems can naturally possess the properties of a transmitter, since they can be controlled such that the information of the message is encoded in its chaotic trajectory. Moreover, a chaotic trajectory is suitable for transmission over noisy and frequency band-limited channels, in the sense that the receiver can recover the message with a small amount of errors [3–6].

A channel as defined by Shannon is a physical medium that enables information to pass throughout until it arrives to the receiver. Analogously, we define a chaotic channel as an active physical medium formed by at least two connected chaotic systems that enable information from a source to pass from the first one (the transmitter) to the last one (the receiver). A chaotic net, formed by many connected elements might possess only a few chaotic channels, in the sense that the channel is the path of connected systems along which information flows. We define a transmitter and a receiver in this net to be both elected subsystems of the whole chaotic net.

A first step to understand the chaotic channel goes back to the works [7–9] in which it is shown that two coupled chaotic systems can become completely synchronized (CS), i.e., the distance between their initially different trajectories tends to a small value as time tends to infinity [10]. This property was explored as a communication system, making the pair of coupled systems to work as an active medium that transports information from a driving system (the transmitter) to a slave system (the receiver) [8,9]. The condition under which CS takes place is given by the conditional exponents [8]. Basically, two coupled chaotic systems have two sets of conditional exponents. One set is associated with the synchronization manifold and the other one associated with the transversal manifold. The presence of positive transversal exponents usually indicates that CS does not exist.

A second step is given by Refs. [12,13]. A chaotic trajectory produces to an observer a certain amount of uncertainty that defines information, quantified by the Kolmogorov-Sinai entropy H_{KS} [12], which is the proper way of calculating the Shannon source entropy of a chaotic set. For systems with a measurable (the trajectory is bounded to a finite domain) and ergodic (average quantities can be calculated in space and time) invariant (with respect to time translations of the system and to smooth transformations) natural measure, that is smooth along the unstable manifold, H_{KS} equals the sum of the positive Lyapunov exponents [13]. So, as a source of information, the more chaotic a system is, the more information it produces.

A third step is given in Ref. [14], which showed that the conditional exponents like the Lyapunov exponents are relevant physical quantities to describe a network that is formed by coupled chaotic systems. In particular, in addition to Pesin's identity [13], it was suggested [14] that the summation of the positive conditional exponents λ^+ between two subsystems of a large network could be a measure of the apparent rate of information production in each pair of subsystems, as if they were detached from the whole group.

We show in this paper, by plausible physical reasoning, that the appropriate quantity to quantify the amount of information in the chaotic channel is

$$I_C(S_i,S_j) = \Sigma\lambda_{\parallel}^{+} - \Sigma\lambda_{\perp}^{+}, \tag{1}$$

where $I_C(S_i,S_j)$ represents the mutual information between the transmitter, S_i, and the receiver, S_j. The term $\Sigma\lambda_{\parallel}^{+}$ is the sum of the positive exponents associated to the synchronization manifold between S_i and S_j, and $\Sigma\lambda_{\perp}^{+}$ is the sum of the positive exponents associated to the transversal manifold between S_i and S_j. The term $\Sigma\lambda_{\parallel}^{+}$ represents the information (entropy production per time unit) produced by the synchronous trajectories, and it corresponds to the amount of information transmitted. The term $\Sigma\lambda_{\perp}^{+}$ represents the information produced by the nonsynchronous trajectories, and it corresponds to the information lost in the transmission, the information that is erroneously retrieved in the receiver.

Finally, the capacity as defined in Shannon's work, is the maximum of the mutual information. So, while

 ©2005 The American Physical Society

the capacity of a net that respects certain conditions [13] is given by $H_{KS}=\Sigma_{\lambda_k>0}\lambda_k$, with λ_k representing all the positive Lyapunov exponents of the net, the capacity of a chaotic channel between systems S_i and S_j is given by $C(S_i,S_j) = \max[I_C(S_i,S_j)]$, with the condition that

$$C(S_i,S_j) \leq H_{KS}, \tag{2}$$

where the maximum is taken over all possible coupling strengths. We implement this approach for a system of two coupled maps, and for a system of three coupled Rössler oscillators, showing that this approach is valid for both descriptions of dynamical systems, the discrete and the time continuous. Further we argue how to extend these results to large networks of coupled chaotic systems, as well.

The discrete channel—*a channel of communication formed by discrete chaotic elements:* We model a discrete channel by two coupled maps $x^{(1)}_{n+1}=(1-c)2x^{(1)}_n+2cx^{(2)}_n \times(\text{mod } 1)$ and $x^{(2)}_{n+1}=(1-c)2x^{(2)}_n+2cx^{(1)}_n$ (mod 1), with $c\leq 0.25$, representing the coupling strength. In here, the channel is completely described only by the transmitter, the subsystem of variable $x^{(1)}$, and the receiver, the subsystem of variable $x^{(2)}$. The Lyapunov exponents of these coupled systems are $\lambda_1=\log(2)+\log(1-c)$ and $\lambda_2=\log(2)+\log(1-2c) -\log(1-c)$. Therefore, $H_{KS}=2\log(2)+\log(1-2c)$. The synchronization manifold, $x_{\parallel}$, is defined by the following variable transformation: $x_{\parallel}=x^{(1)}+x^{(2)}$, and the transversal manifold is defined by $x_{\perp}=x^{(1)}-x^{(2)}$. The conditional exponents are $\lambda_{\parallel}=\log(2)$ and $\lambda_{\perp}=\log(2)+\log(1-2c)$. For no coupling ($c=0$), these two mappings work as independent sources of information, and the capacity for generating information of these two sources are given by the sum of the capacity of each one, which in this case is equal to $H_{KS}=2\log(2)$. The mutual information should vanish (note that for $c=0$, $\lambda_{\parallel}-\lambda_{\perp}=0$) with the errors produced by the nonsynchronous trajectories being maxima (note that $\lambda_{\perp}$ is maxima for $c=0$). This I_C function increases as the coupling c increases, once the larger is c, the larger is the synchronization level, and consequently the amount of information retrieved in the receiver. So, we see that it is reasonable to consider that $I_C=\lambda_{\parallel}-\lambda_{\perp}$. Note that Eq. (2) holds. We get equality for $c=0.25$ $[C=H_{KS}=\log(2)]$ when CS is reached between the transmitter and receiver. At this moment, the errors produced by the nonsynchronous trajectories should vanish. That is exactly what happens to $\lambda_{\perp}$. Therefore, we see again that it is reasonable to consider that $\lambda_{\perp}$ is related to the errors caused by the nonsynchronous trajectories in the decoding of the information in the receiver. So, when there is no CS, errors may occur in the transmission ($\lambda_{\perp}>0$), while when there is CS, errors may not occur and the channel transmits information in its full capacity.

The continuous channel—*a channel of communication formed by continuous chaotic elements:* A small chaotic network is modeled by the following system of three coupled Rössler oscillators: $\dot{x}_i=-\alpha_i y_i-z_i+A_{ji}(x_j-x_i)$, $\dot{y}_i=\alpha_i x_i+ay_i$, $\dot{z}_i=b+z_i(x_i-c)$, with $a=0.15$, $b=0.2$ and $c=10$, and i, $j=1,2,3$, with $i\neq j$. S_i represents the system of the variables (x_i,y_i,z_i) and S_j represents the system of the variables (x_j,y_j,z_j). A_{ji} indicates the coupling strength between S_j and S_i. The configuration of the net is set, for most of our examples, to have S_1 and S_2 bidirectionally coupled with $A_{12} =A_{21}$, and S_3 is unidirectionally coupled to S_2, that is, $A_{23} \geq 0$ and $A_{32}=0$. $\alpha_1=1$, $\alpha_2=1.0002$, and $\alpha_3=0.998$, and thus all the systems have different parameters.

Assuming $\vec{X}_i$ to describe the state variables of subsystem i, then the synchronization manifold between subsystem S_i and S_j is given by $\vec{X}^{\parallel}_{ij}=\vec{X}_i+\vec{X}_j$, which yields the ordinary differential equation (ODEs) that describe this manifold. $\dot{x}^{\parallel}_{ij}=[(\alpha_j-\alpha_i)y^{\perp}_{ij}-(\alpha_i+\alpha_j)y^{\parallel}_{ij}]/2-z^{\parallel}_{ij}+G^{\parallel}_{ij}$, $\dot{y}^{\parallel}_{ij}=[(\alpha_i+\alpha_j)x^{\parallel}_{ij} +(\alpha_i-\alpha_j)x^{\perp}_{ij}]/2+ay^{\parallel}_{ij}$, $\dot{z}^{\parallel}_{ij}=2b+(0.5x^{\parallel}_{ij}-c)z^{\parallel}_{ij}+0.5x^{\perp}_{ij}z^{\perp}_{ij}$. The transversal manifold is defined as $\vec{X}^{\perp}_{ij}=\vec{X}_i-\vec{X}_j$, which give us $\dot{x}^{\perp}_{ij}=[(\alpha_j-\alpha_i)y^{\parallel}_{ij}-(\alpha_i+\alpha_j)y^{\perp}_{ij}]/2-z^{\perp}_{ij}+G^{\perp}_{ij}$, $\dot{y}^{\perp}_{ij}=[(\alpha_i+\alpha_j)x^{\perp}_{ij} +(\alpha_i-\alpha_j)x^{\parallel}_{ij}]/2+ay^{\perp}_{ij}$, $\dot{z}^{\perp}_{ij}=0.5x^{\perp}_{ij}z^{\parallel}_{ij}+(0.5x^{\parallel}_{ij}-c)z^{\perp}_{ij}$, with the terms $G^{\perp}_{ij}$ and $G^{\parallel}_{ij}$ expressing the coupling between the transmitter and the receiver, with other elements in the network.

To calculate the conditional exponents associated to the communication channel between S_i and S_j, we use in the method of Ref. [15] the following 6×6 Jacobian

$$\begin{matrix} \dfrac{\partial\dot{\vec{X}}^{\perp}_{ij}}{\partial\vec{X}^{\perp}_{ij}} & \dfrac{\partial\dot{\vec{X}}^{\perp}_{ij}}{\partial\vec{X}^{\parallel}_{ij}} \\ \dfrac{\partial\dot{\vec{X}}^{\parallel}_{ij}}{\partial\vec{X}^{\perp}_{ij}} & \dfrac{\partial\dot{\vec{X}}^{\parallel}_{ij}}{\partial\vec{X}^{\parallel}_{ij}} \end{matrix} \tag{3}$$

which is equal to

$$\begin{matrix} -(A_{ji}+A_{ij}) & -(\alpha_i+\alpha_j)/2-1 & 0 & (\alpha_j-\alpha_i)/2 & 0 & \\ (\alpha_i+\alpha_j)/2 & a & 0 & (\alpha_i-\alpha_j)/2 & 0 & 0 \\ 0.5z^{\parallel}_{ij} & 0 & 0.5x^{\parallel}_{ij}-c & 0.5z^{\perp}_{ij} & 0 & 0.5x^{\perp}_{ij} \\ (A_{ij}-A_{ji}) & (\alpha_j-\alpha_i)/2 & 0 & 0 & -(\alpha_1+\alpha_2)/2 & -1 \\ (\alpha_i-\alpha_j)/2 & 0 & 0 & (\alpha_i+\alpha_j)/2 & a & 0 \\ 0.5z^{\perp}_{ij} & 0 & 0.5x^{\perp}_{ij} & 0.5z^{\parallel}_{ij} & 0 & 0.5x^{\parallel}_{ij}-c \end{matrix}$$

For each communication channel, we obtain two positive conditional exponents. Since $(\alpha_i-\alpha_j)/2\neq 0$, and not always $A_{ij}-A_{ji}=0$, the Jacobian in (3) cannot be separated into smaller diagonal blocks, where either only transversal or only parallel variables are present. Therefore, to discern which exponent is associated to which manifold, one uses the fact that the exponent with large magnitude is the one associated to the synchronization manifold, and the other is associated with the transversal manifold. This is a requirement needed to have the mutual information always positive or null, between a pair of elements in the network.

If we had a fully connected network with equal coupling strengths, i.e, $A_{ij}=A$ for all i and j, and the chaotic elements had equal parameters, thus, the Jacobian in (3) could be easily separated in two diagonal blocks. The upper left diagonal block $\partial\dot{\vec{X}}^{\perp}_{ij}/\partial\vec{X}^{\perp}_{ij}$ would produce one positive exponent associated with the transversal manifold, and the lower right diagonal block $\partial\dot{\vec{X}}^{\parallel}_{ij}/\partial\vec{X}^{\parallel}_{ij}$ would produce another positive exponent associated with the synchronization manifold. For a general network formed by N systems, each system having dimension D, the conditional exponents associated with each communication channel between a pair of systems would be

RAPID COMMUNICATIONS

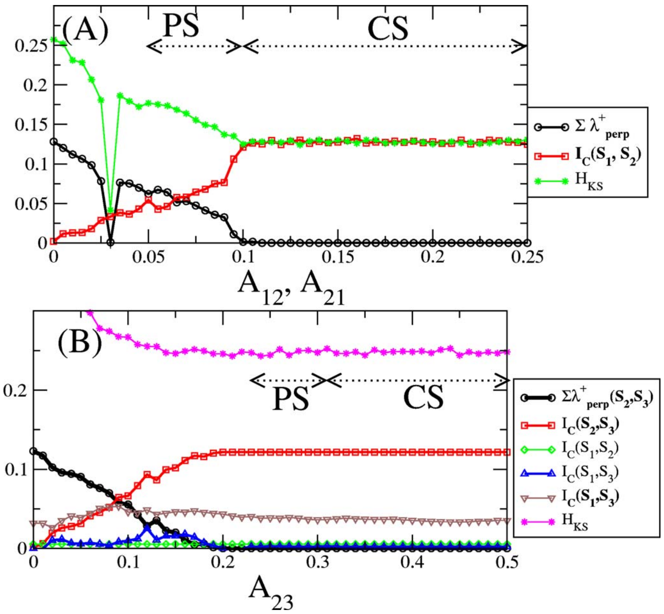

FIG. 1. (Color online) $\Sigma\lambda^+_{\mathrm{perp}}$ in the figure legend stands for the quantity $\Sigma\lambda^+_{\perp}$, in Eq. (1). $I_C(S_i,S_j)$ is the mutual information between S_i and S_j calculated from Eq. (1), and H_{KS} is the Kolmogorov-Sinai entropy, that gives the capacity of the network. In (a), the transmitter is S_1 and the receiver is S_2. For $A_{12}=A_{21}=0.03$, before PS is achieved, the Rössler type chaotic attractor turn into a three-band-type chaotic attractor, and for that reason $\Sigma\lambda^+_{\perp}\rightarrow 0$. PS is only achieved for $A_{12}=A_{21}=0.05$, when the attractor is, as usual, of the Rössler-type, and CS is achieved for $A_{12}=A_{21}>0.1$. In (b), we can consider as transmitters (receivers) S_1 or S_2 (S_2 or S_3), and the coupling strength between S_1 and S_2 (and vice-versa) is $A_{12}=A_{21}=0.05$. PS (and CS) between S_2 and S_3 is achieved for $A_{23}=0.23(A_{23}\geqslant 0.31)$. The units are in bits per time unit.

calculated from a Jacobian with dimension $2D\times 2D$. In principle, this approach can be used for any network in which the Jacobian in the transversal and synchronization variables produce exponents that can be properly associated with each manifold. For systems whose Jacobian produces more than two positive conditional exponents, it is hoped that there is a threshold, such that exponents with magnitude below it are associated to the transversal manifold, and above it are associated to the synchronization manifold. If that is possible, Eq. (1) can still be used.

To illustrate our ideas, we calculate the conditional and the Lyapunov exponents by the method of Ref. [15]. For that, we integrate the network by a 4th-order Runge-Kutta integrator with time step of 0.02, for a time interval such that the system S_1 makes 2100 cycles, i.e., it crosses the plane $y=0$ (with $x>0$) 2100 times. We discard a transient of 100 cycles and the initial conditions are: $x_i=6.54$, $y_i=6.0$, and $z_i=0.1$. The results are shown in Fig. 1. In (a), we consider that there is no coupling between S_2 and S_3 ($A_{23}=0$), and therefore, $I_C(S_2,S_3)=0$, and the net can be thought to be formed by two coupled systems. As we increase the coupling between S_1 and S_2, the mutual information $I_C(S_1,S_2)$ increases from 0 to $I_C(S_1,S_2)\cong 0.127$ bits per time unit, the maximum value for the mutual information, that is the capacity of the chaotic channel, between S_1 and S_2. Two phenomena are important to characterize the chaotic channel: (i) First, the appearance of phase synchronization (PS) [10,16] between S_1 (transmitter) and S_2 (receiver). Whenever that happens $\Sigma\lambda^+_{\perp}<\delta$, where δ can be very small, and therefore the error in the retrieving of information in the receiver, caused by the nonsynchronous trajectories, can be small if δ is small, and consequently, there is a large chance that the message is completely recovered, with small probability of errors, as discussed in Ref. [6]; (ii) The appearance of CS, which for this particular network makes $\Sigma\lambda^+_{\perp}=0$, means that the message can be completely recovered, with small probability of errors, with the extra fact that the channel has its maximal capacity, i.e., $I_C(S_1,S_2)=C(S_1,S_2)$.

Then, we fix the coupling between S_1 and S_2 ($A_{12}=A_{21}=0.05$), to have PS between S_1 and S_2, and increase the coupling between S_2 and S_3. These three coupled systems can be treated as forming three communication channels, one from S_1 to S_2, another from S_2 to S_3, and finally one from S_1 to S_3. Assuming, S_1 to be the transmitter and S_2 the receiver, $I_C(S_1,S_2)\cong 0.002$. As we increase A_{23}, the channel formed by S_2 and S_3 has the same characteristics as shown in (a), that is, when S_2 and S_3 present PS, $\Sigma\lambda^+_{\perp}<\delta$, and when S_2 and S_3 are in CS (as happens for $A_{23}\geqslant 0.31$), $\Sigma\lambda^+_{\perp}=0$ (which is true for this particular network). With upper triangles, we show $I_C(S_1,S_3)$. Note that, as the coupling strength between S_2 and S_3 increases, no significant change in $I_C(S_1,S_3)$ is observed. As we introduce a coupling between S_1 and $S_3(A_{13}=0.04)$ the mutual information $I_C(S_1,S_3)$ increases considerably, as it can be seen by the down triangles, and thus, nonlocal couplings can enhance not only the synchronization level of the network but also the amount of information transmission.

Note that the capacity of the net, H_{KS} (represented by stars in Fig. 1) is always larger than or equal to the capacity of the channel, which agrees with Eq. (2). In the case of Fig. 1(a), as in the discrete channel, equality between H_{KS} and I_C happens when I_C is maximum, i.e., $I_C(S_1,S_2)=C(S_1,S_2)$, which is a consequence of the fact that CS exists between S_1 and S_2.

The noisy chaotic channel and the recovery of information: H_{KS} is a measure of uncertainty about the forward time evolution of the trajectory realized up to some precision, when a series of previous observations with the same precision had been already performed. It does not reflect the amount of information retrieved from one particular observation, realized with some specified precision. In order to understand how much information can be withdrawn from one single observation in a chaotic system, the accuracy with which this observation is realized determines this amount of information, which is a multiple of H_{KS} [4]. From Ref. [4], we have that each observation realized in a one-dimensional map provides $(g+1)H_{KS}$ bits. g is an integer number that is proportional to the accuracy of the observation and inversely proportional to the amount of noise in the chaotic trajectory. Using the deterministic property of chaotic systems, each observed trajectory point generates more g other trajectories points, that were not observed. For the continuous chaotic channel, each observation of the trajectory on a Poincaré plane can be used to reveal the other g nonobserved crossings of the trajectory in this same plane [17]. So, each observation gives R bits of information, with $R=(g$

$+1)I_C(S_i,S_j)\times\langle T(S_2)\rangle$ bits, and g being an integer number proportional to the accuracy of the observation (inversely proportional to the noise variance), with $\langle T(S_2)\rangle$ being defined in Ref. [17]. Note that the average time interval to obtain all this information is equal to $(g+1)\langle T(S_2)\rangle$, and therefore, the rate at which one recovers information in the receiver $[R/(g+1)\langle T(S_2)]$ is at most equal to the rate of information produced in the transmitter (I_C).

Concluding, we define the chaotic channel as a subset of a net of coupled chaotic systems along which information flows. We characterize this channel by showing how to calculate the amount of information exchanged between two important elements of the channel: the transmitter, which can be thought of as an entrance door of the information in the net, and the receiver, the ending point of the information. If phase synchronization exists between the elements of the channel, a transmitted message can be fully recovered at a rate smaller than if these elements are completely synchronized, a situation for which the channel achieves its capacity, this capacity being smaller than or equal the capacity of the network. This approach can be used whenever the conditional exponents can be associated to their appropriate manifold, i.e., either the synchronization or the transversal manifold. If that is possible, in principle, this approach could be used to understand information transmission in more complex systems, as natural chaotic nets, e.g., the human brain, which shows evidence of chaotic behavior [18], or other chaotic networks.

This research is partially financed by Alexander von Humboldt Foundation (M.S.B.) and SFB 555 (J.K.).

[1] C. E. Shannon and W. Weaver, Bell Syst. Tech. J. **27**, 379 (1948).

[2] S. Hayes, C. Grebogi, E. Ott, and A. Mark, Phys. Rev. Lett. **73**, 1781 (1994).

[3] M. S. Baptista, E. E. Macau, C. Grebogi, Y.-C. Lai, and E. Rosa, Phys. Rev. E **62**, 4835 (2000); M. Hasler and T. Schimming, Proc. IEEE **90**, 733 (2002); E. E. Macau and C. M. P. Marinho, Acta Astronaut. **53**, 465 (2003)

[4] M. S. Baptista and L. Lopez, Phys. Rev. E **65**, 055201(R) (2002).

[5] M. S. Baptista, E. E. Macau, and C. Grebogi, Chaos **13**, 145 (2003).

[6] L. Illing, J. Brocker, L. Kocarev, U. Parlitz, and H. D. I. Abarbanel, Phys. Rev. E **66**, 036229 (2002).

[7] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **29**, 1050 (1986).

[8] L. M. Pecora and T. L. Carrol, Phys. Rev. Lett. **64**, 821 (1990); L. M. Pecora and T. L. Carroll, Phys. Rev. A **44**, 2374 (1991).

[9] K. M. Cuomo and A. V. Oppenheim, Phys. Rev. Lett. **71**, 65 (1993).

[10] CS is considered to be achieved between two systems if the distances between their trajectories is smaller than ϵ, with ϵ being a small finite quantity. PS is achieved if the absolute phase difference between S_i and S_j is smaller than or equal to r, for a time corresponding to 200,000 cycles of the system S_i, where r is the average growing of the phase for one typical cycle in S_i. Each cycle is defined as the time the trajectory takes to cross twice the plane $y=0$, with $x>0$, and the phase of the chaotic attractor is calculated as defined in Ref. [11].

[11] M. S. Baptista, T. Pereira, and J. Kurths, e-print physics/05070901.

[12] A. N. Kolmogorov, Dokl. Akad. Nauk UzSSR **98**, 527 (1959).

[13] Ya. B. Pesin, Russ. Math. Surveys **32** 55 (1977).

[14] R. V. Mendes, Phys. Lett. A **248**, 167 (1998); Physica A **276**, 550 (2000).

[15] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, Physica D **16**, 285 (1985).

[16] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[17] I_C in Eq. (1) has the units of bits per time unit. It would be advantageous to calculate it in bits per cycle, that is, $I'_C=I_C\times\langle T(S_l)\rangle$, where $\langle T(S_l)\rangle$ is the average time of the returning of the system S_l, i.e., $\langle T\rangle=\Sigma_{k=1}^{R_l}\tau_k(S_l)/R_l$, with $\tau_K(S_l)$, a series of time intervals for the trajectory to leave and to return the Poincaré plane positioned at $y_l=0$. Recall that this can be done since the conditional exponents are time invariant, as well as the normal Lyapunov exponents.

[18] H. Korn and P. Faure, C R. Biol. **326**, 787 (2003).