Reexamination of the Helfrich-Hurault effect in smectic-*A* liquid crystals

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The Helfrich-Hurault effect is a phase transition that occurs in samples of cholesteric or smectic liquid crystals subject to external electric or magnetic fields. In this paper we analyze the Helfrich-Hurault effect of smectic-*A* liquid crystals in an electrostatic field taking into account the complete electromechanical coupling. A comparison is made with the results already obtained for the partially coupled case where one takes into account only the effect of the field on the crystal configuration and considering that field unaffected.

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I. INTRODUCTION

The smectic-*A* liquid crystals [1] exhibit an intermediate order between that of a solid and a liquid; they possess a long-range orientational order and one-dimensional positional ordering. Their rodlike molecules organize themselves into layers. The average alignment is represented by a unit vector **n**, called the optical axis or the director, which is locally perpendicular to the layers. In the presence of an electric field a liquid crystal tends to align its molecules normal or along the direction of the field, depending on the dielectric behavior of the molecules [1]. In smectic-*A* phases, rotation of the molecules induces the distortion of the layers. Hence, electric or magnetic fields are able to induce a switch from an undistorted liquid crystal configuration to a distorted one. In cholesteric or smectic liquid crystals, this phase transition is called the Helfrich-Hurault effect [1–3].

Theoretical results for the Helfrich-Hurault transition in infinite samples of smectic-*A* liquid crystals can be found in [1]. More recently, Stewart [4] extended the theoretical results of Helfrich [2] and Hurault [3] for infinite samples of cholesteric liquid crystals under the influence of magnetic fields, to finite cells of smectic-*A* liquid crystals subject to a uniform pressure and magnetic or electric field. However, in all these approaches the electric field is considered partially coupled with the mechanical system in the sense that it can induce layer distortion, but it is not affected by the latter.

In the electro-optic devices, samples of liquid crystal are placed between two plates, which work as two capacitor electrodes. The optical properties can be altered by applying a voltage difference between the two electrodes. The optical axis distortion induces an alteration of the dielectric properties and, consequently, of the liquid crystal electric state. The so-modified electric field further affects the optical properties and so on. Deuling [5] first took into account the mutual interaction between the field and the material for nematic liquid crystals. Self *et al.* [6] looked at post-threshold effects of electrically driven Freedericks transition in a nematic cell confined between two infinite parallel plates. Nevertheless, in nematics, the fully electromechanical coupling does not modify the Freedericks critical electric field. More recently, one of us [7] has proposed a model for smectic-*A* liquid crystals which incorporates the Maxwell equations in order to describe the mutual interaction between the smectic and the electric field. This model is fully nonlinear and when one considers plane deformations of smectic-*A* infinite samples in a homeotropic texture that contain a very large number of layers, the linear analysis of the equations shows a modification of the classic Helfrich-Hurault threshold. In this paper we generalize these findings to the case of a finite sample of smectic-*A* in a homeotropic alignment between two parallel plates.

The equilibrium equations, namely the layers shape equation and the Maxwell equations, are obtained through variational arguments. We will show how the classical Helfrich-Hurault threshold can be modified by taking into account the fully electromechanical coupling.

This paper is organized as follows: in Sec. II we give the expression of the free energy which characterizes the material-field interaction and we also derive the related Euler-Lagrange equations. The critical threshold is determined by solving an eigenvalues problem; explicit solution is obtained in Sec. III; some conclusions are pointed out in Sec. IV.

II. EQUILIBRIUM EQUATIONS

Smectic-*A* liquid crystals can be viewed as a continuum of surfaces $\Sigma(\zeta)$ defined by the equation

$$\omega(\mathbf{x},\zeta)=0, \tag{1}$$

where **x** denotes the position in the space of a point on the surface characterized by the ζ parameter. Moreover, smectic-*A* are characterized by the property that the rodlike molecules, described by a vectorial field **n**, are orthogonal to the respective surfaces; this propriety is expressed by the relation

$$\mathbf{n}=\frac{\nabla\omega}{|\nabla\omega|}, \tag{2}$$

where ∇ denotes the spatial gradient operator; so the tilt of the molecules depends on the layers' deformation.

In their natural configuration the layers are flat, and the smectic-*A* can be described as a bunch of parallel planes. Let

us denote with $\mathbf{n}^0$ the homogeneous field director for a cell of smectic in its natural configuration. Then, we consider a Cartesian coordinate system $O(x,y,z)$ with the third axis parallel to $\mathbf{n}^0$. We suppose that the smectic-A liquid crystal, in a homeotropic alignment, is confined to the volume $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, and $0 \leqslant z \leqslant d$. In an actual experiment the planes placed at $z=0$ and $z=d$ can be the electrodes of a capacitor. When an electric potential V_a is applied between the electrodes, it gives rise to an electric field $\mathbf{E}=-\nabla\phi$ inside the material, where ϕ is the electrostatic potential.

We consider the smectic-A to be a perfect insulator with dielectric anisotropy. The static dielectric constants $\varepsilon_\parallel$ and $\varepsilon_\perp$ are measured along and normal to the molecular axis, respectively; the quantity $\varepsilon_a=\varepsilon_\parallel-\varepsilon_\perp$ gives the dielectric anisotropy. In particular, if ε_a is negative, then the molecules tend to rotate their axis towards the plane orthogonal to the electric field. It is widely known [1] that there is a critical voltage V_{cr} above which the electric forces exceed the elastic strength; then the smectic switches from the undistorted state to a distorted one.

Let us consider an applied potential V_a slightly larger than the critical threshold V_{cr}, so we have

$$\phi(x,y,0) = 0, \tag{3a}$$

$$\phi(x,y,d) = V_a = V_{cr}(1+\epsilon^2), \quad \epsilon \ll 1 \tag{3b}$$

and, as a consequence, we suppose that the smectic layers undergo a slight distortion; by using Eq. (2) the correspondent distorted director is

$$\mathbf{n} = [-\epsilon u_{,x}, -\epsilon u_{,y}, 1-\epsilon^2(\nabla_s u)^2/2] + o(\epsilon^2), \tag{4}$$

where $\epsilon u(x,y,z)$ represents the layer displacement in $\mathbf{n}^0$ direction. ∇_s denotes the bi-dimensional gradient in the (x,y) plane, $\nabla_s=(\partial/\partial x, \partial/\partial y)$. Also, we assume that the molecules stick to the boundary with a free orientation. This means that ϵu vanishes on the boundary, while its derivative with respect to the boundary outward normal $\boldsymbol{\nu}$ is arbitrary. In particular, we have the boundary conditions

$$\epsilon u(x,y,0) = \epsilon u(x,y,d) = 0. \tag{5}$$

The free energy per volume unit related to the distortion and to the presence of the electric field is a sum of two contributions. The first is the elastic energy (see pp. 343 of [1]), including the nematic energy of distortion (namely, the splay and the saddle-splay terms) and the smectic energy of compression of the layers

$$w_A = \frac{K_1}{2}\epsilon^2[(\Delta_s u)^2 + 4(u^2_{,xy} - u_{,xx}u_{,yy})] + \frac{\bar{B}}{2}\epsilon^2 u^2_{,z}, \tag{6}$$

where K_1 and $\bar{B}$ are positive constants. The quantity $\lambda = \sqrt{K_1/\bar{B}}$ is the characteristic length of the material, of the order of the smectic layer thickness.

The second contribution is due to the interaction between the field director and the local electric field. According to [5,1], the interaction director-field energy density is, in Lorentz-Heaviside units [9],

$$w_E = -\tfrac{1}{2}\mathbf{D}\cdot\mathbf{E}, \tag{7}$$

where $\mathbf{D}=\varepsilon_\perp\mathbf{E}+\varepsilon_a(\mathbf{E}\cdot\mathbf{n})\mathbf{n}$ is the dielectric displacement. So, in terms of the electrostatic potential ϕ we have

$$w_E = -\tfrac{1}{2}[\varepsilon_\perp(\nabla\phi)^2 + \varepsilon_a(\mathbf{n}\cdot\nabla\phi)^2]. \tag{8}$$

The introduction of this term can modify the convexity of the total energy giving rise to the buckling of the layers under an imposed voltage difference.

Let us consider electrostatic potentials of the type

$$\phi = \frac{V_{cr}}{d}z + \epsilon\varphi(x,y,z); \tag{9}$$

notice that ϕ satisfies the boundary conditions (3) provided that the potential φ satisfies the homogeneous boundary conditions

$$\epsilon\varphi(x,y,0) = \epsilon\varphi(x,y,d) = 0. \tag{10}$$

We recover the classical theory by taking $\epsilon\varphi=0$ everywhere.

To make the problem more tractable, we assume

$$\epsilon u = u_0 \sin\left(\frac{\pi}{d}z\right) v(x,y), \tag{11a}$$

$$\epsilon\varphi = \varphi_0 \sin\left(\frac{\pi}{d}z\right) \psi(x,y), \tag{11b}$$

with u_0 and φ_0 small constants, allowing u and φ to satisfy the boundary conditions (5) and (10), respectively.

The total free energy is

$$G = \int_\Omega (w_A + w_E)\, d\Omega; \tag{12}$$

by inserting Eqs. (6), (9), and (11) into Eq. (12) gives

$$\begin{aligned} G = &-\frac{\varepsilon_\parallel}{2}E_{cr}^2\Omega + du_0^2\int_S \left[\frac{K_1}{2}(\Delta_s v)^2 + 2K_1(v^2_{,xy} - v_{,xx}v_{,yy})\right. \\ &\left. + \frac{\bar{B}}{2}\left(\frac{\pi}{2}\right)^2 v^2 + \frac{1}{2}\varepsilon_a E_{cr}^2(\nabla_s v)^2\right] dS \\ &- u_0\varphi_0\varepsilon_a d\int_S [(\nabla_s\psi)\cdot(\nabla_s v)E_{cr}]\, dS \\ &- \varphi_0^2 d\int_S \left[\frac{\varepsilon_\perp}{2}(\nabla_s\psi)^2 + \frac{\varepsilon_\parallel}{2}\left(\frac{\pi}{d}\right)^2\psi^2\right] dS, \end{aligned} \tag{13}$$

where S is the plane region $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, with boundary Γ. Δ_s is the usual two-dimensional Laplace operator and $E_{cr}=-V_{cr}/d$.

We now vary G in order to obtain the related Euler-Lagrange equations to the variables u and ψ. The variation of the first integral of (13) is carefully discussed by Stewart [4] and Landau and Lifshitz [8] in the case of simply supported boundary conditions. Let $\varsigma(x,y)$ be the variation of the displacement v. Then

$$\delta\int_S \left[\frac{K_1}{2}(\Delta_s v)^2 + 2K_1(v_{,xy}^2 - v_{,xx}v_{,yy}) + \frac{\bar{B}}{2}\left(\frac{\pi}{2}\right)^2 v^2 + \frac{1}{2}\varepsilon_a E_{cr}^2(\boldsymbol{\nabla}_s v)^2 \right] dS = \int_S \left[K_1\Delta_s^2 v + \bar{B}\left(\frac{\pi}{d}\right)v - \varepsilon_a E_{cr}^2 \Delta_s v \right] \varsigma \, dS \tag{14}$$

and, since $\varsigma=0$ and $\partial\varsigma/\partial\boldsymbol{\nu}$ is arbitrary on Γ, we obtain

$$v = 0, \quad \text{on } \Gamma \tag{15a}$$

$$v_{,xx} = 0, \quad \text{in } x = 0, \quad x = a, \tag{15b}$$

$$v_{,yy} = 0 \quad \text{in } y = 0, \quad y = b. \tag{15c}$$

In Eq. (14) Δ_s^2 denotes the two-dimensional biharmonic operator.

In order to vary the remaining terms we introduce the ψ variation $\zeta(x,y)$ such that $\partial\zeta/\partial\boldsymbol{\nu}=0$ on Γ. The variation of the second and third integral in the free-energy expression gives

$$\delta\int_S [(\boldsymbol{\nabla}_s\psi)\cdot(\boldsymbol{\nabla}_s v)E_{cr}]dS = -\int_S E_{cr}[(\Delta_s\psi)\varsigma + (\Delta_s v)\zeta]dS + \oint_\Gamma E_{cr}(\boldsymbol{\nu}\cdot\boldsymbol{\nabla}_s v)\zeta \, dl, \tag{16}$$

$$\delta\int_S \left[\frac{\varepsilon_\perp}{2}(\boldsymbol{\nabla}_s\psi)^2 + \frac{\varepsilon_\parallel}{2}\left(\frac{\pi}{d}\right)^2\psi^2 \right] dS = -\int_S \left[\varepsilon_\perp\Delta_s\psi + \varepsilon_\parallel\left(\frac{\pi}{d}\right)^2\psi \right]\zeta \, dS + \oint_\Gamma \varepsilon_\perp(\boldsymbol{\nu}\cdot\boldsymbol{\nabla}_s\psi)\zeta \, dl. \tag{17}$$

At equilibrium $\delta G=0$ and taking into account the arbitrariness of ς and ζ we obtain, in S,

$$\Delta_s^2 v + \left(\frac{\pi}{\lambda d}\right)^2 v - \frac{\varepsilon_a}{K_1}E_{cr}^2\Delta_s v + \frac{\varphi_0}{u_0 K_1}E_{cr}\varepsilon_a\Delta_s\psi = 0 \tag{18a}$$

$$\Delta_s\psi - \left(\frac{\varepsilon_a}{\varepsilon_\perp}+1\right)\left(\frac{\pi}{d}\right)^2\psi + E_{cr}\frac{\varepsilon_a}{\varepsilon_\perp}\frac{u_0}{\varphi_0}\Delta_s v = 0 \tag{18b}$$

and by arbitrariness of ζ on Γ it follows

$$\varepsilon_\perp\varphi_0\psi_{,x} + \varepsilon_a E_{cr}u_0 v_{,x} = 0, \quad \text{in } x = 0, \quad x = a \tag{19a}$$

$$\varepsilon_\perp\varphi_0\psi_{,y} + \varepsilon_a E_{cr}u_0 v_{,y} = 0, \quad \text{in } y = 0, \quad y = b. \tag{19b}$$

These conditions express the absence of polarization-induced charges on Γ.

Equation (18a) states that the layer's deformation v is determined by four terms: the first two come, respectively, from the nematic distortion and the smectic compression energy of the layer; the last two are due to the electrostatic interaction energy. On the other hand, Eq. (18b) looks like a Poisson equation for ψ with the charge density proportional to the curvature of the layer. Then the local electric field is the sum of the external field plus the "corrections" coming from (18b).

It is easy to see that the classical case is recovered in the limit of $\varepsilon_a/\varepsilon_\perp \to 0$. Then $\psi=0$ everywhere and it easily follows that the remaining equation (18a) is identical to the homogeneous version of the equation (3.1) of Stewart [4]. In nematic literature [10] this assumption is called *the magnetic approximation*.

III. DETERMINATION OF THE CRITICAL FIELD

For any future development we assume $\varepsilon_a<0$ in order to create competition between the electric field and the elastic strengths.

Developing v and ψ as

$$v(x,y) = \sum_{n,m=1}^{\infty} v_{n,m}\frac{2}{\sqrt{ab}}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right), \tag{20a}$$

$$\psi(x,y) = \sum_{n,m=1}^{\infty} \psi_{n,m}\frac{2}{\sqrt{ab}}\sin\left(\frac{n\pi}{a}x\right)\sin\left(\frac{m\pi}{b}y\right), \tag{20b}$$

which already satisfy the boundary conditions expressed in Eqs. (15). By substituting (20) into (18) we obtain in a straightforward way an infinite homogeneous linear system for the unknowns $v_{n,m}$ and $\psi_{n,m}$. After a closer inspection of this system we notice that it is factorized into an infinite set of 2×2 linear homogeneous systems, each of which can be written as

$$\begin{pmatrix} A_{n,m} & B_{n,m} \\ C_{n,m} & D_{n,m} \end{pmatrix}\begin{pmatrix} v_{n,m} \\ \psi_{n,m} \end{pmatrix} = 0, \tag{21}$$

where

$$A_{n,m} = \eta_{n,m}^2 + \left(\frac{\pi}{\lambda d}\right)^2 - \frac{\varepsilon_a}{K_1}(E_{cr})^2\eta_{n,m}, \tag{22a}$$

$$B_{n,m} = \frac{\varphi_0}{u_0 K_1}E_{cr}\varepsilon_a\eta_{n,m}, \tag{22b}$$

$$C_{n,m} = \frac{\varepsilon_a}{\varepsilon_\perp}E_{cr}\frac{u_0}{\varphi_0}\eta_{n,m}, \tag{22c}$$

$$D_{n,m} = \eta_{n,m} - \left(\frac{\varepsilon_a}{\varepsilon_\perp}+1\right)\left(\frac{\pi}{d}\right)^2, \tag{22d}$$

and $\eta_{n,m}=-\pi^2(n^2/a^2+m^2/b^2)$ are the eigenvalues of the laplacian operator. This simple structure is a consequence of the fact that in (18) only the laplacian operator and its square appear, which do not give rise to "off-diagonal" terms.

To have a nontrivial solution of (21) the determinant must be zero,

$$A_{n,m}D_{n,m} - C_{n,m}B_{n,m} = 0. \tag{23}$$

This equation can be solved for $(E_{cr})_{n,m}$ to obtain the critical field for each mode.

Before developing the calculations any further let us discuss in our formalism the magnetic approximation defined above.

A. The magnetic approximation

The magnetic approximation is obtained in the $\varepsilon_a/\varepsilon_\perp \to 0$ limit. After some simple algebra we obtain

$$(E_{cr})^2_{n,m} = \frac{K_1}{\varepsilon_a} \frac{\eta^2_{n,m} + \left(\frac{\pi}{\lambda d}\right)^2}{\eta_{n,m}}. \tag{24}$$

As usual the minimum value can be found by considering $\eta_{n,m}$ as a continuous variable. Then using standard techniques one finds

$$(E_{cr})_{min} = E_{magn} = \sqrt{-\frac{2K_1}{\varepsilon_a}\frac{\pi}{\lambda d}} \tag{25}$$

in correspondence to $\eta_{n,m} = \eta_{magn} = -\pi/(\lambda d)$, which is the classical Helfrich-Hurault critical field for smectic-*A* liquid crystals [1,4]. Moreover, the amplitudes $\psi_{n,m}$ are zero because $A_{n,m} \equiv 0$, in correspondence to (24).

B. The coupled case

After some tedious but straightforward algebra an expression is found for the critical field. Introducing the parameter α defined as

$$\alpha = \frac{\lambda}{d}\pi, \quad 0 \leqslant \alpha \leqslant \pi \tag{26}$$

and passing to the continuous using the dimensionless variable ξ defined as

$$\eta_{n,m} \to \left(\frac{\pi}{d}\right)^2 \frac{1}{\alpha}\xi < 0 \tag{27}$$

the result can be expressed as

$$(E_{cr})^2 = K_1\left(\frac{\pi}{\lambda d}\right)\frac{\varepsilon_\perp}{\varepsilon_a \varepsilon_\parallel}\frac{(\xi^2+1)(\xi-\alpha\beta)}{\xi(\xi-\alpha)}, \tag{28}$$

where $\beta = \varepsilon_\parallel/\varepsilon_\perp$. The stationary points of (28) are the roots of the following polynomial:

$$\xi^4 - 2\alpha\xi^3 + (\alpha^2\beta - 1)\xi^2 + 2\alpha\beta\xi - \alpha^2\beta = 0 \tag{29}$$

and can be written in a closed form. However, the expressions are so involved that we prefer to discuss in detail the special case $\alpha \ll 1$. Otherwise we give some numerical results.

1. Many smectic layers

In this case the layer thickness λ is much smaller than the length d of the sample that is $\alpha \ll 1$. This is a good approximation in the case of many smectic layers contained in the sample.

To obtain the solution of (29) as a perturbative expansion in α, notice that when $\alpha = 0$ the solution of interest is $\xi_{min,0} = -1$ giving

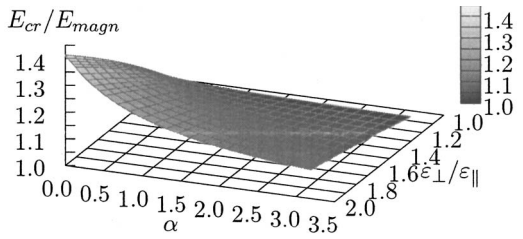

FIG. 1. Ratio between the calculated electric field E_{cr} and E_{magn} keeping constant $\varepsilon_a = -1$.

$$(E_{cr})_{min} = \sqrt{\frac{-2K_1}{\varepsilon_a}\left(\frac{\pi}{\lambda d}\right)\frac{\varepsilon_\perp}{\varepsilon_\parallel}}. \tag{30}$$

A consistent perturbative solution can be found at all the orders in α by simply putting $\xi_{min,p} = -1 + \Sigma_{k=1}^{p} a_k \alpha^k$ into (29) obtaining something like $\Sigma_{k=1}^{4p} b_k \alpha^k = 0$ and solving with respect to the a_k. For instance, the first terms are

$$a_1 = 1 - \beta,$$

$$a_2 = -\tfrac{1}{2}(1-\beta)(1+3\beta),$$

$$a_3 = -\beta(1-\beta)(1-4\beta). \tag{31}$$

The substitution of these expressions into (28) gives us the asymptotic expansion of the critical field

$$(E_{cr})_{min} = \sqrt{\frac{-2K_1}{\varepsilon_a}\left(\frac{\pi}{\lambda d}\right)\frac{\varepsilon_\perp}{\varepsilon_\parallel}} \times [1 - (1-\beta)\alpha - \beta^2(1-\beta)\alpha^2 + \ldots]. \tag{32}$$

Notice that we find two kinds of corrections to the well-known expression for the Helfrich-Hurault critical field. The first one is the overall factor $\sqrt{\varepsilon_\perp/\varepsilon_\parallel} > 1$, while the second, given by the asymptotic expansion (32), is related to the small but finite thickness of the layers, that is $0 < \alpha \ll 1$. So we find that for very small layer thickness the critical threshold is higher than the classical value of Helfrich-Hurault.

2. Numerical results

In situations where α is not a small quantity, a numerical calculation of the roots of (29) is needed and can be easily done with standard methods.

We are interested in the case of negative electric anisotropy, so we choose $\varepsilon_a = -1$ as a typical value. Figure 1 shows the ratio of E_{cr}/E_{magn}, so we can compare our results directly with the magnetic approximation.

We have also constructed the corresponding pictures for values of ε_a in the range $[-10^2, -10^{-3}]$ and we have found a quite similar behavior. As we can see from Fig. 1 in the region $\alpha \ll 1$, the critical field is greater than the magnetic value by the factor $\sqrt{\varepsilon_\perp/\varepsilon_\parallel}$ according to (32). When α increases, that is the number of layers decreases, there is a reduction in the critical threshold that becomes comparable with E_{magn}.

Figure 2 shows the ratio η_{cr}/η_{magn}. Notice that the value of $\eta_{cr} \equiv (\eta_{cr})_{min}$ gives us information about the modes that

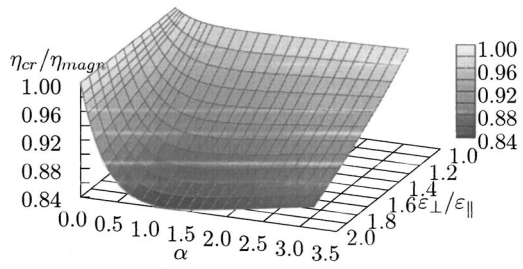

FIG. 2. Calculated ratio between η_{cr} and η_{magn} with $\varepsilon_a=-1$ constant.

start the Helfrich-Hurault transition. It is interesting to see that for small α the same modes are involved in the starting of the transition. Increasing α the modes involved lower and for roughly $\alpha=1$, there is a minimum corresponding to the layer thickness $d/3$. It can be seen by inspection that the lower the $\eta_{n,m}$, the smoother the undulation of the layer which starts the phase transition.

IV. CONCLUDING REMARKS

We have presented a model describing the interaction of an insulator smectic liquid crystal with an electrostatic field. Our model goes beyond the usual approach [1,4], that is, the so-called magnetic approximation in which the electrostatic field is not affected by the smectic. Instead we solved the usual shape equation completely coupled to the Maxwell equation for the electrostatic potential.

We have found that the electric field becomes nonuniform and it can be different from the classical one. In the case of a very large layer number, we have found an analytical expression for the critical field which shows that the classical Helfrich-Hurault field is an underestimation. This behavior is also confirmed numerically. However, by decreasing the number of layers the classical field becomes a better estimate.

We would like to point out a fundamental difference between the study of the critical threshold in nematic liquid crystals (the Freedericksz transition) and smectic-*A* liquid crystals. In the case of nematics [5,6], it is supposed that the problem is homogeneous on every plane parallel to the electrodes (i.e., not dependent on x and y variables); therefore every unknown field depends only on the z variable. The consequence is that the correction to the director field is of the first order, while that of the electric potential is of the second order. Therefore, in the eigenvalues problem [which involves just the $O(\epsilon)$ equations] the potential field does not influence the critical threshold. In other words, the linearized problem for nematics is still partially coupled. On the contrary, in the case of smectics the problem cannot be considered independent of the x-y variables due to the undulation of the layers. Both the correction of the displacement field and the correction of the electrostatic potential are $O(\epsilon)$. Thus, the complete electromechanical coupling already takes place in the linearized problem and therefore the eigenvalues problem to be solved involves two equations. Observe how, in accordance with our reasoning, the presence of the only source term of the linearized Maxwell equation [the last addendum in Eq. (18b)] follows by allowing inhomogeneous displacements (and therefore inhomogeneous director alignments) in the planes parallel to the delimiting electrodes. In light of this and our calculations, we conclude that the complete coupling is responsible for raising the Helfrich-Hurault critical threshold in smectic-*A* liquid crystals.

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[1] P. G. de Gennes and J. Prost, *The Physics of Liquid Crystals*, 2nd ed. (Clarendon Press, Oxford, 1993).
[2] W. Helfrich, Appl. Phys. Lett. **17**, 531 (1970).
[3] J. Hurault, J. Chem. Phys. **59**, 2086 (1973).
[4] I. W. Stewart, Phys. Rev. E **58**, 5926 (1998).
[5] H. Deuling, Mol. Cryst. Liq. Cryst. **19**, 123 (1972).
[6] R. H. Self, C. P. Please, and T. J. Sluckin, Eur. J. Appl. Math. **13**, 1 (2002).
[7] G. Napoli, IMA J. Appl. Math. (to be published).
[8] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity* (Pergamon Press, New York, 1986).
[9] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York, 1975).
[10] E. G. Virga, *Variational Theories for Liquid Crystals* (Chapman and Hall, London, 1994).