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Dynamical stabilization of solitons in cubic-quintic nonlinear Schrödinger model

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We consider the existence of a dynamically stable soliton in the one-dimensional cubic-quintic nonlinear Schrödinger model with strong cubic nonlinearity management for periodic and random modulations. We show that the predictions of the averaged cubic-quintic nonlinear Schrödinger (NLS) equation and modified variational approach for the arrest of collapse coincide. The analytical results are confirmed by numerical simulations of a one-dimensional cubic-quintic NLS equation with a rapidly and strongly varying cubic nonlinearity coefficient.

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Collapse phenomena are observed in many areas of physics: the self-focusing of intense laser beams, Langmuir waves in plasma, the collapse of the Bose-Einstein condensates (BECs) with attractive interactions, etc. The nonlinear Schrödinger equation (NLSE) with cubic nonlinearity used to describe these systems has stable solutions in the one-dimensional (1D) case, when the dispersion and nonlinearity effects can effectively balance each other. In two and three dimensions, the focusing nonlinearity overcomes the dispersion and the blow-up phenomenon occurs [1].

Few mechanisms for the arrest of collapse have been suggested. Among them can be mentioned the dispersion [2,3] and nonlinearity management methods [4–8]. The analysis based on the variational approach, method of moments, and numerical simulations showed that the nonlinearity management method is effective in suppressing collapse in the scalar and vector two-dimensional (2D) NLSE with focusing cubic nonlinearity. For the three-dimensional (3D) cubic NLSE with nonlinearity, management theoretical predictions and numerical results do not bring a clear and definitive picture [9,10], so more analytical and numerical work is necessary.

In this Rapid Communication we investigate the phenomenon of arrest of collapse by using the strong cubic nonlinearity management scheme in the 1D cubic-quintic (CQ) NLSE. The CQ NLSE with nonlinearity management presents practical interest since it appears in many branches of physics such as nonlinear optics and BEC. In nonlinear optics, it describes the propagation of pulses in double-doped optical fibers [11]; in BEC it models the condensate with two- and three-body interactions [12,13]. In optical fibers, periodic variation of the nonlinearity can be achieved by varying the type of dopants along the fiber. As pointed out in [11,14], the values and the signs of the cubic and quintic nonlinear parameters can be adjusted by properly choosing the characteristics of the two dopants, and variations of the cubic parameter can be obtained without modifying the quintic parameter. In BEC, the variation of the atomic scattering length by the Feshbach resonance technique leads to the oscillations of the mean-field cubic nonlinearity [15], but it also induces variations of the quintic nonlinearity because the three-body interaction is dependent on the scattering length [16]. The analysis of periodic variations of the cubic and quintic terms follows the same lines and shows that stabilization can also be obtained. The detailed analysis will be presented elsewhere.

The CQ NLSE when the cubic term is equal to zero is the critical quintic NLSE. The quintic Townes soliton is an unstable solution of the quintic NLSE [1]. In this paper we consider the configuration with the rapid and strong periodic modulation in time of cubic nonlinear interaction. This type of modulations corresponds to the management applied to the nonlinearity with a lower power. We first apply a variational approach to the averaged NLSE. The averaged equation for the 1D cubic NLSE in the case of strong nonlinearity management has been derived in [17,18], but the presence of the quintic term dramatically changes the picture, as this term would normally lead to collapse. We also propose a modified variational approach for the managed CQ NLSE where the ansatz is designed to take into account the fast self-phase modulation due to the cubic nonlinearity management. These two approaches predict the stabilization of the quintic Townes soliton.

We consider the CQ NLSE

$$iu_t + u_{xx} + \gamma(t)|u|^2u + \chi|u|^4u = 0, \tag{1}$$

with an attractive quintic nonlinearity $\chi > 0$. The time-varying cubic coefficient $\gamma(t)$ possesses an average value γ_0 and a fast varying part γ_1

$$\gamma(t) = \gamma_0 + \frac{1}{\varepsilon}\gamma_1\left(\frac{t}{\varepsilon}\right), \tag{2}$$

where $\varepsilon \ll 1$ corresponds to strong and rapid management. Here γ_1 can be either a periodic function or a stationary random function. We first address the case of a periodic management $\gamma_1(\tau+1)=\gamma_1(\tau)$ and $\int_0^1\gamma_1(\tau)d\tau=0$. We denote by Γ_1 the zero-mean antiderivative of γ_1. Following the same procedure as in [18], we can average the CQ NLSE over fast variations and show that the solution takes the form

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$$u(t,x) = \left[w(t,x) + \varepsilon w_1\left(t,\frac{t}{\varepsilon},x\right) + \cdots \right] \times \exp\left[i\Gamma_1\left(\frac{t}{\varepsilon}\right)|w(t,x)|^2 \right],$$

where w is solution of the averaged CQ NLSE

$$iw_t + w_{xx} + \gamma_0|w|^2w + \chi|w|^4w + \sigma^2\{[(|w|^2)_x]^2 + 2|w|^2(|w|^2)_{xx}\}w = 0, \quad (3)$$

and $\sigma^2 = \int_0^1 \Gamma_1(s)^2 ds$. The averaged CQ NLSE has a Hamiltonian form, with the Hamiltonian

$$H = \int |w_x|^2 - \frac{\gamma_0}{2}|w|^4 - \frac{\chi}{3}|w|^6 + \sigma^2[(|w|^2)_x]^2|w|^2 dx. \quad (4)$$

We first prove that the solution of Eq. (3) cannot collapse because its supremum norm can be *a priori* bounded. This proof is essentially based on the Sobolev inequality $\|f\|_\infty^2 \leqslant C\|f\|_2\|f_x\|_2$, where we denote $\|f\|_p = [\int |f(x)|^p dx]^{1/p}$ for $p \in (1,\infty)$ and $\|f\|_\infty$ is the (essential) supremum of $|f|$. We first apply this inequality with $f=w$,

$$\|w\|_4^4 \leqslant \|w\|_\infty^2\|w\|_2^2 \leqslant C\|w\|_2^3\|w_x\|_2 \leqslant C\|w_x\|_2, \quad (5)$$

where C stands for a constant that may change from line to line and we have used the fact that $\|w\|_2$ is constant. Next we apply the Sobolev inequality with $f = v := |w|^3$,

$$\|w\|_6^6 \leqslant \|w\|_\infty^4\|w\|_2^2 = \|v\|_\infty^{4/3}\|w\|_2^2 \leqslant C\|v\|_2^{2/3}\|v_x\|_2^{2/3} \leqslant C\delta\|v\|_2^{4/3} + C\delta^{-1}\|v_x\|_2^{4/3},$$

where the last estimate holds true for any $\delta > 0$. By choosing $\delta = \|v\|_2^{2/3}/(2C)$ and noting that $\|v\|_2^2 = \|w\|_6^6$, we get

$$\|w\|_6^6 \leqslant C\|v_x\|_2. \quad (6)$$

Substituting (5) and (6) into (4) and using the fact that $\int [(|w|^2)_x]^2|w|^2 dx = (4/9)\|v_x\|_2^2$, we can finally write

$$H \geqslant \|w_x\|_2^2 - \gamma_0 C\|w_x\|_2 + (4/9)\sigma^2\|v_x\|_2^2 - \chi C\|v_x\|_2,$$

which shows that $\|w_x\|_2$ and $\|v_x\|_2$, and thus $\|w\|_\infty$ are uniformly bounded, which prevents the solution of the averaged CQ NLSE from collapsing. However, this result does not show that the solution of Eq. (3) does not collapse for fixed ε.

The next step consists of applying the variational approach to the averaged CQ NLSE [19,20]. The variational ansatz for the solution is chosen as the chirped function

$$w(t,x) = A(t)Q\left(\frac{x}{a(t)}\right)\exp[ib(t)x^2 + i\phi(t)], \quad (7)$$

with a given shape Q. Following the standard procedure, we substitute the ansatz into the Lagrangian density generating Eq. (3) and calculate the effective Lagrangian density in terms of A, a, b, ϕ, and their time derivatives. The evolution equations for the parameters of the ansatz are then derived from the effective Lagrangian by using the corresponding Euler-Lagrange equations. In particular, this approach yields a closed-form ordinary differential equation (ODE) for the width a

$$a_{tt} = -U'_{av}(a), \quad (8)$$

where the effective potential is

$$U_{av}(a) = \frac{6D_1 - 2D_3\chi N^2}{3a^2} - \frac{D_2\gamma_0 N}{a} + \frac{8D_4\sigma^2N^2}{a^4}, \quad (9)$$

the total mass is $N = \int |u|^2 dx$, and the effective parameters are

$$D_1 = \frac{\int Q'(x)^2 dx}{\int x^2Q^2(x)dx},$$

$$D_2 = \frac{\int Q(x)^4 dx}{\left(\int Q^2(x)dx\right)\left(\int x^2Q^2(x)dx\right)},$$

$$D_3 = \frac{\int Q(x)^6 dx}{\left(\int Q^2(x)dx\right)^2\left(\int x^2Q^2(x)dx\right)},$$

$$D_4 = \frac{\int Q(x)^4Q'(x)^2 dx}{\left(\int Q^2(x)dx\right)^2\left(\int x^2Q^2(x)dx\right)}.$$

Another approach is possible for the CQ NLSE with cubic nonlinearity management (1). It consists of applying a specific variational approach designed to capture the fast self-phase modulation induced by the fast nonlinearity management. The variational ansatz is sought in the form

$$u(t,x) = A(t)Q\left(\frac{x}{a(t)}\right)\exp[ib(t)x^2 + i\phi(t)] \times \exp\left[i\Gamma_1\left(\frac{t}{\varepsilon}\right)A^2(t)Q^2\left(\frac{x}{a(t)}\right)\right]. \quad (10)$$

Substituting this ansatz into the Lagrangian density generating Eq. (1) we get the system of ODEs

$$a_t = 4ab - \frac{D_2N}{a^2}\Gamma_1\left(\frac{t}{\varepsilon}\right), \quad (11)$$

$$b_t = \frac{D_1}{a^4} - 4b^2 - \frac{D_2\gamma_0N}{4a^3} - \frac{D_3\chi N^2}{3a^4} - \frac{D_2Nb}{a^3}\Gamma_1\left(\frac{t}{\varepsilon}\right) + \frac{8D_4N^2}{a^6}\Gamma_1^2\left(\frac{t}{\varepsilon}\right). \quad (12)$$

Next we perform an averaging of this system of ODEs, which yields exactly Eq. (8). This analysis shows that the two different approaches yield the same effective system for the soliton parameters, which strengthens its validity.

We now focus our attention to the particular case $\gamma_0 = 0$.

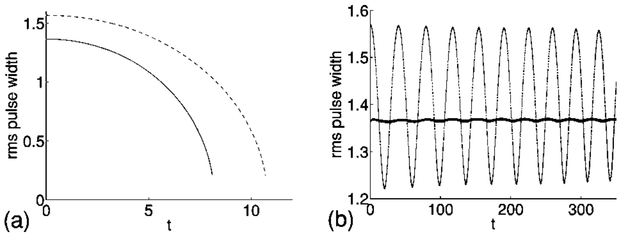

FIG. 1. Rms pulse width vs time. The initial pulse has a Townes shape with mass $N=1.02N_c \simeq 2.77$ and radius $a_0=a_c=0.88$ (corresponding to an initial rms pulse width $a_{rms,0}=1.38$, solid line) or $a_0=1.0$ ($a_{rms,0}=1.57$, dashed lines). (a) The collapse at finite time in the absence of nonlinearity management. (b) A periodic cubic nonlinearity management $\gamma(t)=10\cos(50t)$ is applied.

We first choose the shape function Q corresponding to the quintic Townes soliton $Q(x)=1/\sqrt{\cosh(x)}$, which gives the values $D_1=1/(2\pi^2)$, $D_2=8/\pi^4$, $D_3=2/\pi^4$, and $D_4=1/(8\pi^4)$. In absence of cubic nonlinearity management $\sigma=0$, we have $U'_{av}(a)=0$ for all a if the total mass is equal to the critical mass $N_c=\pi\sqrt{3}/(2\sqrt{\chi})$. This means that there exists an infinity of fixed points. However, as it can be checked from (8), the Townes soliton is not stable in the sense that it collapses if $N>N_c$ and it spreads out and vanishes if $N<N_c$, so this theoretical solution of the quintic NLSE with $N=N_c$ cannot be observed in practice [see Fig. 1(a)]. In presence of strong cubic management $\sigma>0$, we get the existence of a unique fixed point a_c if $N>N_c$, with

$$a_c = \frac{\sqrt{2}\sigma N}{\pi\sqrt{(N/N_c)^2-1}}. \quad (13)$$

The linear stability analysis of Eq. (8) shows that this fixed point is stable and that the oscillation period is

$$T_c = \frac{2\sigma^2 N^2}{[(N/N_c)^2-1]^{3/2}}. \quad (14)$$

We are especially interested in solutions whose masses are just above the critical mass, since our main goal is to prove that the Townes soliton can be stabilized by cubic nonlinearity management. In these conditions, the stable soliton width is rather large $\sim 1/(N-N_c)^{1/2}$, and the soliton oscillation period is very long $\sim 1/(N-N_c)^{3/2}$.

In the following numerical experiments, we take $\chi=1$, so the critical mass is $N_c \simeq 2.72$, and we apply the management $\gamma(t)=10\cos(50t)$, which gives $\sigma=1/(5\sqrt{2})$. We solve the CQ NLSE by a pseudospectral method starting from a Townes soliton shape with a mass $N>N_c$ and a radius a_0 close to a_c. For $N=1.02N_c$ [Fig. 1(b)], the theoretical fixed point and oscillation period are $a_c=0.88$ and $T_c=38$. If we choose $a_0=a_c$ for the input pulse width, then we observe in the numerical simulations that the mean-square radius of the pulse is almost constant, which shows that this solution is stable. If we choose $a_0=1>a_c$, then we observe slow oscillations around a_c with a period very close to T_c. These two observations demonstrate that the values a_c and T_c for the soliton parameters predicted by the variational analysis are very accurate.

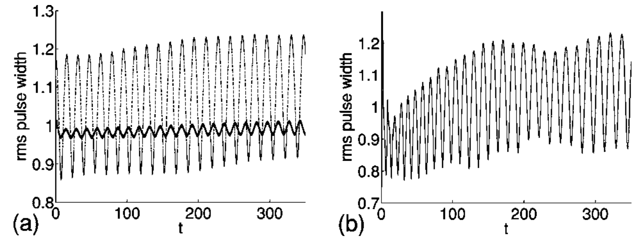

FIG. 2. Rms pulse width. (a) $N=1.04N_c$, $a_0=0.63$ ($a_{rms,0}=0.99$, solid line), $a_0=0.75$ ($a_{rms,0}=1.18$, dashed lines). (b) $N=1.08N_c$, $a_0=0.5$ ($a_{rms,0}=0.79$, solid line).

We have repeated the same experiments with different initial masses, to check the validity of formulas (13) and (14). For $N=1.04N_c$ (respectively, $N=1.08N_c$), the theoretical fixed points and oscillation periods are $a_c=0.63$ and $T_c=14$ (respectively, $a_c=0.42$ and $T_c=5.1$). As it can be seen in Fig. 2(b), when the initial mass is significantly larger than the critical mass, the pulse first experiences a strong distortion and emits radiation. The resulting pulse is still stable, its mass is still well above the critical mass, but the fixed point and oscillation period are not well predicted by the variational analysis. If the initial mass is even larger, then this first phase leads to collapse, which shows that the stabilizing effect of the cubic nonlinearity management $\gamma(t)=10\cos(50t)$ is only efficient when the initial mass is in the range $[N_c, 1.1N_c]$.

We now prove the robustness of the solution of the CQ NLSE driven by cubic nonlinearity management with respect to the initial pulse shape. More exactly, we show that a stable solution can be obtained with an initial pulse profile that is significantly different from the Townes profile. The important conditions that have to be satisfied by the initial pulse is that its mass should be just above the critical mass corresponding to the input pulse profile, and that its initial radius should be chosen in the vicinity of the fixed point. These conditions are imposed by the analysis of Eq. (8), and they are confirmed by numerical simulations. Let us consider an arbitrary pulse shape Q. The critical mass is then $N_c=\sqrt{(3D_1)/(D_3\chi)}$ and the fixed point in presence of nonlinearity management is $a_c=2\sqrt{2}\sqrt{D_4/D1}\sigma N/\sqrt{(N/N_c)^2-1}$, with the period $T_c=(4\sqrt{2}\pi D_4/D_1^{3/2})\sigma^2N^2/[(N/N_c)^2-1]^{3/2}$. A stable solution of the managed CQ NLS equation can be obtained by injecting a pulse with shape Q not too far from the Townes soliton, with a mass just above N_c and a radius close to a_c. In the case of the Gaussian ansatz $Q(x)=\exp(-x^2/2)$, we have $D_1=1$, $D_2=\sqrt{2/\pi}$, $D_3=2/(\sqrt{3}\pi)$, and $D_4=1/(3\sqrt{3}\pi)$. We report in Fig. 3 the results of numerical simulations carried out with $\chi=1$ so the critical mass is $N_c \simeq 2.86$. We use the same management as above. In a first phase ($t<20$), the pulse emits a small amount of radiation ($<1\%$) and its shape converges to the one of the Townes soliton. A similar phenomenon has been observed in numerical simulations carried out in [8] for the 2D cubic NLSE with cubic nonlinearity management. After this transition period, the soliton width experiences oscillations around the fixed point a_c with an oscillation period close to the theoretical value T_c.

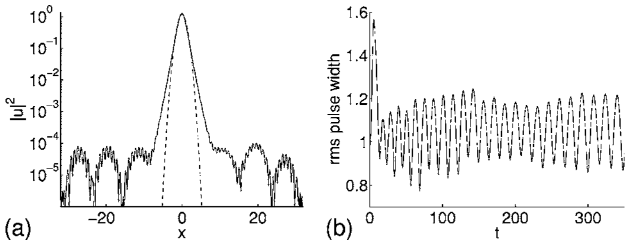

FIG. 3. Dynamics of a Gaussian pulse with initial mass $N=1.02N_c=2.92$ and radius $a_0=a_c=1.4$ ($a_{rms,0}=0.99$). (a) The pulse profile at time $t=350$ (solid line) compared to the input Gaussian profile (dashed lines). (b) The rms pulse width vs time.

We finally study the stabilization induced by random nonlinearity management. Accordingly we now assume that γ_1 is a zero-mean stationary random process. Considering Eqs. (11) and (12), it can be seen that the important process is actually $\Gamma_1(\tau)=\int_0^\tau \gamma_1(s)ds$, and it is critical that this process does not grow in a diffusive manner, which would mean that cubic nonlinearity accumulates. This condition is fulfilled if γ_1 is the derivative of a stationary random process, or if we apply a pinning scheme. This technique was presented by Chertkov *et al.* 21 to compensate for accumulated fiber dispersion. The periodic insertion of additional pieces of fiber with well-controlled lengths and dispersion values was found to prevent from pulse deterioration. The pinning method can be applied to compensate for accumulated cubic nonlinearity as well. In Fig. 4 we show that random cubic nonlinearity management stabilizes a Townes soliton in a manner similar to a periodic management. However, we can detect a very slow spreading out, whose origin could be explained by the imperfect compensation of the accumulated nonlinearity by the pinning scheme.

In conclusion we have analyzed the stabilizing role of the strong cubic nonlinearity management in the 1D CQ NLSE.

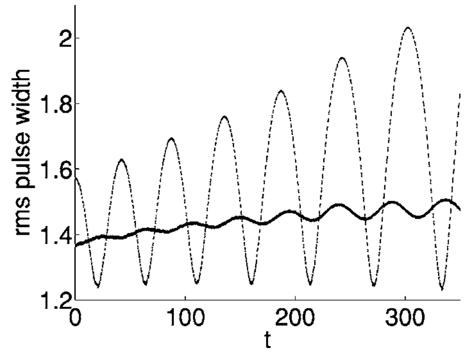

FIG. 4. Rms pulse width vs time. The initial pulse has a Townes shape with mass $N=1.02N_c\simeq 2.77$ and radius $a_0=a_c=0.88$ ($a_{rms,0}=1.38$, solid line) or $a_0=1.0$ ($a_{rms,0}=1.57$, dashed lines). We apply a random cubic nonlinearity management $\gamma(t)=10m(50t)$ where m is zero-mean stationary random process with a unit coherence time compensated by a pinning scheme.

We have proved that the averaged CQ NLSE, in a dramatic distinction from the nonmodulated system, supports stable solutions beyond the critical mass. In particular the quintic Townes soliton, which is unstable in the nonmodulated system, becomes stable in presence of strong nonlinearity management if its width lies in the vicinity of a fixed point. This fixed point and the associated oscillation frequency are well predicted by the variational approach applied to the averaged CQ NLSE. We have also developed and applied a modified variational approach to the CQ NLSE with strong management, which gives the same values for the stabilized soliton parameters. We have checked that a random management is also stabilizing, if we take care to select a random process whose cumulative value is well controlled. These results can be applied for the search of nonlinearity managed solitons in double-doped optical fibers and BECs with three-body interactions.

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