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Relativistic acceleration of charged particles in uniform and mutually perpendicular electric and magnetic fields as viewed in the laboratory frame

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We obtain exact solutions in the laboratory frame for the relativistic position, velocity, and energy of a charged particle moving in constant, uniform, and mutually perpendicular electromagnetic fields. We also obtain an exact equation relating the proper time and the laboratory time, and examine all these quantities in the large and small time limits. Working in the laboratory frame makes the accelerated motion that can occur when $|\mathbf{E}|\geqslant c|\mathbf{B}|$ easier to analyze and understand.

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I. INTRODUCTION

The motion of a charge in constant, uniform, and mutually perpendicular electric and magnetic fields is treated in a number of classic texts (e.g., Jackson [1], Landau and Lifshitz [2]). The standard approach begins by Lorentz transforming from the laboratory frame to an inertial frame in which the equations of motion can be solved more easily. When $|\mathbf{E}|<c|\mathbf{B}|$ it is straightforward to transform the solution back to the laboratory frame and show that a laboratory observer sees the well known $\mathbf{E}\times\mathbf{B}$ drift, with no acceleration in the $\mathbf{E}\times\mathbf{B}$ direction. If $|\mathbf{E}|\geqslant c|\mathbf{B}|$ then a charge can accelerate in this direction, and none of the standard treatments transform the solutions back to the laboratory frame. Consequently, exact solutions in the laboratory frame for the position, velocity, and energy of an accelerating charge in these particular electromagnetic fields have been lacking, as has an analysis and discussion of the resulting motion.

In a recent paper Takeuchi [3] obtained an exact solution in the laboratory frame for the position of a charge accelerating when $|\mathbf{E}|\geqslant c|\mathbf{B}|$. Takeuchi then found the rate at which the energy increased in the limit of large times.

In this paper we complete the work presented in Ref. [3] in several ways. First, we derive an *exact* expression for the energy as a function of both the laboratory and proper times. Second, we derive exact solutions for the proper and laboratory velocities, and for the relation between the laboratory and proper times. We then use all these results to analyze the trajectories observed in the laboratory frame. As part of this analysis, we show that our results agree with those in Ref. [3] in the limit of large times. Taken together, the results of this paper and Ref. [3] provide a complete description in the laboratory frame of the motion of a charge moving in any $\mathbf{E}\times\mathbf{B}$ field.

Our approach uses a relativistic velocity variable called the *symmetric velocity*. The symmetric velocity has been used before to analyze Thomas precession and Wigner rotation geometrically, and in other situations involving noncollinear velocity addition and Lorentz boosts [4–7]. This paper presents another example in which using the symmetric velocity results in the exact solution of a relativistic problem, and makes certain important features of that problem easier to understand.

The paper is organized as follows. In the first section we define the s velocity $\mathbf{w}$ and establish its relation to the standard velocity $\mathbf{v}$. In the second section we derive the Lorentz force equation satisfied by the s velocity. In the third section we obtain exact solutions for the s velocity as functions of the proper time in each of the three cases $|\mathbf{E}|<c|\mathbf{B}|$, $|\mathbf{E}|=c|\mathbf{B}|$, and $|\mathbf{E}|>c|\mathbf{B}|$. In the fourth section we transform the solutions for the s velocity into exact solutions for the position and velocity in the laboratory frame, and examine their asymptotic form when $t\to 0$ and $t\to\infty$. In the fifth section we derive the rate at which the energy increases in the two cases $|\mathbf{E}|\geqslant c|\mathbf{B}|$. In the sixth section we mention some more formal properties of the s velocity and, in the final section, we summarize our results and discuss specific physical situations in which they can be applied.

II. THE *S* VELOCITY

We can associate with any velocity $\mathbf{v}$ a dynamic variable called the *symmetric velocity* $\mathbf{w}_s$. The symmetric velocity $\mathbf{w}_s$ and its corresponding velocity $\mathbf{v}$ are related by

$$\mathbf{v}=\frac{\mathbf{w}_s+\mathbf{w}_s}{1+(|\mathbf{w}_s|/c)(|\mathbf{w}_s|/c)}=\frac{2\mathbf{w}_s}{1+|\mathbf{w}_s|^2/c^2}. \tag{1}$$

Several different motivations for using the s velocity are given in Refs. [4,5] and will not be discussed here. Instead of $\mathbf{w}_s$, we shall find it more convenient to use the unit-free vector $\mathbf{w}=\mathbf{w}_s/c$, which we call the *s velocity*. The relation of a velocity $\mathbf{v}$ to its corresponding s velocity is

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$$\mathbf{v} = \Phi(\mathbf{w}) = \frac{2c\mathbf{w}}{1 + |\mathbf{w}|^2}, \tag{2}$$

where Φ denotes the function mapping the s velocity $\mathbf{w}$ to its corresponding velocity $\mathbf{v}$.

Using Eq. (2) and the identity $\gamma^2\beta^2=(\gamma-1)(\gamma+1)$, the s velocity $\mathbf{w}$ can be expressed in terms of $\mathbf{v}$ as

$$\mathbf{w} = \Phi^{-1}(\mathbf{v}) = \frac{\mathbf{v}/c}{1 + \sqrt{1 - |\mathbf{v}|^2/c^2}} \tag{3}$$

$$\Rightarrow \mathbf{w} = \frac{\gamma\boldsymbol{\beta}}{1 + \gamma}, \tag{4}$$

where $\gamma=(1-v^2/c^2)^{-1/2}$ and $\boldsymbol{\beta}=\mathbf{v}/c$. From this we see that $\mathbf{w}\to\boldsymbol{\beta}/2$ when $\beta\to 0$ and $\mathbf{w}\to\boldsymbol{\beta}$ as $\beta\to 1$.

Again using the identity $\gamma^2\beta^2=(\gamma-1)(\gamma+1)$ we see that the s velocity and γ are related by the equations

$$|\mathbf{w}|^2 = \frac{\gamma - 1}{\gamma + 1} \quad \text{and} \quad \gamma = \frac{1 + |\mathbf{w}|^2}{1 - |\mathbf{w}|^2}. \tag{5}$$

Finally, we can express the relativistic momentum $\gamma m\mathbf{v}$ (where m denotes the rest mass of the particle) in terms of the s velocity as

$$\gamma m\mathbf{v} = m\left(\frac{2c\mathbf{w}}{1 - |\mathbf{w}|^2}\right). \tag{6}$$

The s velocity has some interesting and useful mathematical properties. The set of all three-dimensional relativistically admissible s velocities forms a unit ball

$$D_s = \{\mathbf{w} \in \mathbb{R}^3 : |\mathbf{w}| < 1\}. \tag{7}$$

Corresponding to the Einstein velocity addition equation, we may define an addition of s velocities in D_s such that

$$\Phi(\mathbf{a}\oplus_s\mathbf{w}) = \Phi(\mathbf{a})\oplus_E\Phi(\mathbf{w}). \tag{8}$$

The right-hand side of Eq. (8) is the equation for the relativistic addition of two noncollinear velocities, which can be written as

$$\mathbf{v}\oplus_E\mathbf{u} \equiv \frac{\mathbf{v} + \mathbf{u}_1 + \gamma^{-1}\mathbf{u}_2}{1 + c^{-2}\langle\mathbf{v}|\mathbf{u}_1\rangle}, \tag{9}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ denote the components of $\mathbf{u}$ parallel and perpendicular to $\mathbf{v}$. (See, for example, p. 25 in Ref. [7].) Using Eqs. (2) and (5), a straightforward but somewhat long calculation leads to the corresponding equation for s velocity addition:

$$\mathbf{a}\oplus_s\mathbf{w} = \frac{(1 + |\mathbf{w}|^2 + 2\langle\mathbf{a}|\mathbf{w}\rangle)\mathbf{a} + (1 - |\mathbf{a}|^2)\mathbf{w}}{1 + |\mathbf{a}|^2|\mathbf{w}|^2 + 2\langle\mathbf{a}|\mathbf{w}\rangle}. \tag{10}$$

(This calculation is carried out explicitly in Ref. [7] on p. 59.)

Equation (10) can be put into a more convenient form if, for any $\mathbf{a}\in D_s$, we define a map $\Psi_{\mathbf{a}}: D_s\to D_s$ by

$$\psi_{\mathbf{a}}(\mathbf{w}) \equiv \mathbf{a}\oplus_s\mathbf{w}. \tag{11}$$

This map is an extension to $D_s\in\mathbb{R}^n$ of the Möbius addition on the complex unit disk. It defines a conformal map on D_s[7,8].

The motion of a charge in $\mathbf{E}\times\mathbf{B}$ fields is two dimensional if the charge starts in the plane perpendicular to $\mathbf{B}$, and in this case Eq. (10) for s-velocity addition is somewhat simpler. We find the appropriate two-dimensional subspace of D_s as follows. Let Π be the two-dimensional plane generated by the s velocities $\mathbf{a}$ and $\mathbf{w}$. By introducing a complex structure on Π, the disk $\Delta=D_s\cap\Pi$ can be identified as a unit disk $|z|<1$ called the Poincaré disk [5]. In this case the s-velocity addition defined by Eq. (10) becomes

$$a\oplus_s w = \psi_a(w) = \frac{a + w}{1 + \bar{a}w}, \tag{12}$$

which is the well-known Möbius transformation of the unit disk.

III. THE LORENTZ EQUATION FOR THE S VELOCITY

The relativistic Lorentz force equation is

$$\frac{d}{dt}(\gamma m\mathbf{v}) = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}).$$

Using Eqs. (2) and (6), this equation can be rewritten as

$$\frac{d}{dt}\left[m\left(\frac{2c\mathbf{w}}{1 - |\mathbf{w}|^2}\right)\right] = q\left(\mathbf{E} + \frac{2c\mathbf{w}}{1 + |\mathbf{w}|^2}\times\mathbf{B}\right). \tag{13}$$

After differentiating, the left-hand side becomes

$$2mc\left(\frac{1}{1 - |\mathbf{w}|^2}\frac{d\mathbf{w}}{dt} + \frac{2\mathbf{w}}{(1 - |\mathbf{w}|^2)^2}\left\langle\mathbf{w}\Big|\frac{d\mathbf{w}}{dt}\right\rangle\right). \tag{14}$$

If we replace the left-hand side of Eq. (13) with Eq. (14) and then take the inner product of both sides with $\mathbf{w}$, we find that

$$\left\langle\mathbf{w}\Big|\frac{d\mathbf{w}}{dt}\right\rangle = \frac{q}{2mc}\langle\mathbf{w}|\mathbf{E}\rangle\frac{(1 - |\mathbf{w}|^2)^2}{1 + |\mathbf{w}|^2}. \tag{15}$$

Substituting this expression for $\langle\mathbf{w}|d\mathbf{w}/dt\rangle$ back into Eq. (14), the relativistic Lorentz force equation becomes

$$\frac{mc}{(1 - |\mathbf{w}|^2)}\frac{d\mathbf{w}}{dt} = \frac{\mathbf{E}}{2} + \frac{c\mathbf{w}}{1 + |\mathbf{w}|^2}\times\mathbf{B} - \frac{\mathbf{w}\langle\mathbf{w}|\mathbf{E}\rangle}{1 + |\mathbf{w}|^2}. \tag{16}$$

Multiplying both sides of Eq. (16) by $(1+|\mathbf{w}|^2)$ and using Eq. (5), we conclude that

$$\frac{\gamma mc}{q}\frac{d\mathbf{w}}{dt} = \left(\frac{1 + |\mathbf{w}|^2}{2}\right)\mathbf{E} + c\mathbf{w}\times\mathbf{B} - \mathbf{w}\langle\mathbf{w}|\mathbf{E}\rangle, \tag{17}$$

which is the Lorentz force equation for the s velocity $\mathbf{w}$ as a function of the laboratory time t.

Equation (17) can be simplified if we use the instantaneous proper time τ as the parameter of evolution. Since at any instant $d\tau=\gamma^{-1}dt$, Eq. (17) can be rewritten as

$$\frac{mc}{q}\frac{d\mathbf{w}}{d\tau} = \left(\frac{1+|\mathbf{w}|^2}{2}\right)\mathbf{E} + c\mathbf{w}\times\mathbf{B} - \mathbf{w}\langle\mathbf{w}|\mathbf{E}\rangle, \tag{18}$$

which is the relativistic Lorentz force equation for the s velocity $\mathbf{w}$ as a function of the proper time τ.

For any given electric or magnetic field, Eq. (18) determines the s velocity of a charged particle moving in those fields. Once the s velocity is known, the world line of the particle evolution, as a function of the natural relativistic parameter τ, can be found. The space component of the world line, $\mathbf{r}(\tau)$, is found by adding to the initial position $\mathbf{r}(0)$ the integral

$$\int_0^t \mathbf{v}\,dt' = \int_0^\tau \gamma\mathbf{v}\,d\tau' = \int_0^\tau \mathbf{u}\,d\tau',$$

where $\mathbf{u}(\tau)$ is the proper velocity. Using Eqs. (2) and (5), the proper and s velocities are related by the equation

$$\mathbf{u}(\tau) = \frac{d\mathbf{r}}{d\tau} = \gamma\frac{d\mathbf{r}}{dt} = \gamma\mathbf{v} = \frac{2c\mathbf{w}}{1-|\mathbf{w}|^2}, \tag{19}$$

so the space trajectory is found from the equation

$$\mathbf{r}(\tau) = \mathbf{r}(0) + 2c\int_0^\tau \frac{\mathbf{w}(\tau')}{1-|\mathbf{w}(\tau')|^2}d\tau'. \tag{20}$$

Since the charge is accelerating, γ is not constant and the time component of the world line $t(\tau)$ is

$$t(\tau) = \int_0^\tau \gamma(\tau')d\tau' = \int_0^\tau \frac{1+|\mathbf{w}(\tau')|^2}{1-|\mathbf{w}(\tau')|^2}d\tau'. \tag{21}$$

The Lorentz force equation for the s velocity in the laboratory frame [Eq. (18)] corresponds to Eqs. (12)–(14) in Ref. [3]. However, the solutions to Eq. (18) are easier to interpret than the solutions to the corresponding equations in Ref. [3] since they are related to the velocity $\mathbf{v}$ by only one equation, Eq. (2), rather than by the various transformations and substitutions needed to reach $\mathbf{v}$ in Ref. [3].

We now use Eq. (18) to find the s velocity of a charge q in uniform, constant, and mutually perpendicular electric and magnetic fields. Since all of the terms on the right-hand side of Eq. (18) are perpendicular to the magnetic field vector whenever the initial s velocity is perpendicular to $\mathbf{B}$, $d\mathbf{w}/d\tau$ is in the plane perpendicular to $\mathbf{B}$. Consequently, if the initial s velocity is in the plane perpendicular to $\mathbf{B}$, $\mathbf{w}$ will remain in this plane and the motion will be two dimensional. Therefore, the simplest way to find the important features of the motion is look at cases in which the initial velocity of the charge is in the plane perpendicular to $\mathbf{B}$.

Working in Cartesian coordinates, we choose

$$\mathbf{E} = (0,E,0), \quad \mathbf{B} = (0,0,B), \quad \text{and} \quad \mathbf{w} = (w_1,w_2,0). \tag{22}$$

With these choices the equations for the components in Eq. (18) are

$$\frac{m_0c}{q}\frac{dw_1}{d\tau} = w_2Bc - w_1w_2E \tag{23}$$

and

$$\frac{m_0c}{q}\frac{dw_2}{d\tau} = \left(\frac{1+|\mathbf{w}|^2}{2}\right)E - w_1Bc - w_2^2E. \tag{24}$$

Equations (23) and (24) can be solved most easily using the notation $w=w_1+iw_2$. Multiplying Eq. (24) by i and adding it to Eq. (23) we find

$$\frac{m_0c}{q}\frac{dw}{d\tau} = -iBcw + \frac{iE}{2}[1+|w|^2+2iw_2w]. \tag{25}$$

However,

$$\begin{aligned}[1+|w|^2+2iw_2w] &= 1+w_1^2+w_2^2+2iw_2(w_1+iw_2)\\ &= 1+w_1^2+w_2^2+2iw_2w_1-2w_2^2\\ &= 1+w_1^2-w_2^2+2iw_2w_1\\ &= 1+(w_1+iw_2)(w_1+iw_2)\\ &= 1+w^2.\end{aligned}$$

Therefore, Eq. (25) can be written as

$$\frac{dw}{d\tau} = \frac{iqE}{2mc}\left[1-2c\frac{B}{E}w+w^2\right]. \tag{26}$$

We can simplify Eq. (26) by defining the real constants

$$\Omega \equiv \frac{qE}{2mc} \quad \text{and} \quad \tilde{B} \equiv \frac{cB}{E}. \tag{27}$$

With these definitions, the final form of the equation for the s velocity $w=w_1+iw_2$ of a charge q moving in the electric and magnetic fields specified in Eq. (22) is

$$\frac{dw}{d\tau} = i\Omega(w^2-2\tilde{B}w+1). \tag{28}$$

The solution of Eq. (28) is unique for a given initial condition

$$w(0) = w_0, \tag{29}$$

where the complex number w_0 represents the initial s velocity $\mathbf{w}_0=\Phi^{-1}(\mathbf{v}_0)$ of the charge.

Integrating Eq. (28) produces the equation

$$\int \frac{dw}{w^2-2\tilde{B}w+1} = i\Omega\tau + C, \tag{30}$$

where the constant C is determined from the initial condition (29). The way we evaluate this integral depends upon the sign of the discriminant $4\tilde{B}^2-4$ associated with the denominator of the integrand. If we define

$$\Delta \equiv \tilde{B}^2 - 1 = \frac{(cB)^2-E^2}{E^2}, \tag{31}$$

then the three cases $E<cB$, $E=cB$, and $E>cB$ correspond to the cases Δ greater than zero, equal to zero, and less than zero. In the first case we will find that the motion of q in the laboratory frame includes the well-known constant $\mathbf{E}\times\mathbf{B}$ drift velocity. In the second and third cases we will find that the charge q can accelerate indefinitely. We will derive the explicit form of the position and velocity of the charge in all three cases as measured in the laboratory frame. We also will derive the γ dependence of the total energy in the laboratory frame in the two cases $\mathbf{E}\geqslant\mathbf{B}$, and in this way determine the rate at which the total energy tends to infinity as measured by the laboratory observer.

IV. SOLVING FOR THE *s* VELOCITIES

A. Case 1

Consider first the case

$$\Delta=[(cB)^2-E^2/E^2]>0 \Leftrightarrow E<cB \quad \text{and} \quad \widetilde{B}>1.$$

The denominator of the integrand in Eq. (30) can be rewritten as

$$w^2-2\widetilde{B}w+1=(w-\alpha_1)(w-\alpha_2),$$

where α_1 and α_2 are the real, positive roots

$$\alpha_1=\widetilde{B}-\sqrt{\widetilde{B}^2-1} \quad \text{and} \quad \alpha_2=\widetilde{B}+\sqrt{\widetilde{B}^2-1}. \tag{32}$$

Decomposing the integrand into partial fractions, we find

$$\int\frac{dw}{w^2-2\widetilde{B}w+1}=\frac{-1}{\alpha_2-\alpha_1}\ln\left(\frac{w-\alpha_1}{w-\alpha_2}\right)+C.$$

Substituting this into Eq. (30), we get

$$\ln\left(\frac{w-\alpha_2}{w-\alpha_1}\right)=-i(2\sqrt{\Delta})\Omega\tau+C. \tag{33}$$

If we define the real, positive number

$$\nu\equiv(2\sqrt{\Delta})\Omega=\frac{q\sqrt{(cB)^2-E^2}}{mc}>0, \tag{34}$$

then we can rewrite Eq. (33) as

$$\frac{w-\alpha_1}{w-\alpha_2}=Ce^{-i\nu\tau}. \tag{35}$$

Using the fact that $\alpha_1\alpha_2=1$, Eq. (35) becomes

$$\frac{w-\alpha_1}{1-\alpha_1 w}=Ce^{-i\nu\tau}, \tag{36}$$

which can be rewritten as

$$\psi_{-\alpha_1}(w)=Ce^{-i\nu\tau}, \tag{37}$$

where ψ is defined by Eq. (12). Applying ψ_{α_1} to both sides of Eq. (37), and using the fact that $\psi_{-\alpha_1}(w)=\psi_{\alpha_1}^{-1}$, we get

$$w(\tau)=\frac{\alpha_1+Ce^{-i\nu\tau}}{1+\alpha_1Ce^{-i\nu\tau}}=\alpha_1\oplus_s Ce^{-i\nu\tau}. \tag{38}$$

[Equation (38) also can be derived directly from Eq. (36).] Equation (38) shows that in a system K' moving with s velocity α_1 relative to the laboratory, the s velocity of the charge corresponds to circular motion with initial s velocity

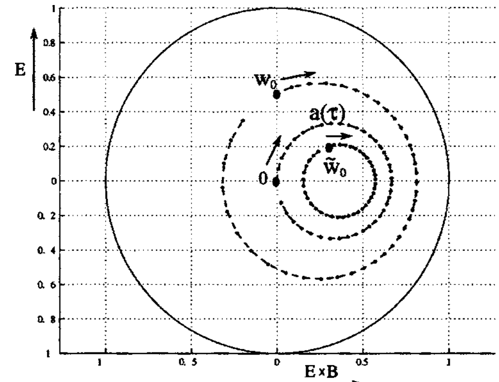

FIG. 1. The trajectories of the s velocity $w(\tau)$ of a charged particle with $q/m=10^7$ C/kg in constant uniform fields $E=1$ V/m and $cB=1.5$ V/m. The initial conditions are $w_0=-0.02+i0.5$ for the first trajectory and $\widetilde{w}_0=0.3+i0.2$ for the second. We also draw $a(\tau)$ corresponding to $w_0=0$. Note that all the trajectories are circles.

$$C=\psi_{-\alpha_1}(w_0). \tag{39}$$

The s velocity observed in K' is shown in Fig. 1.

From Eqs. (2) and (32) it follows that the laboratory velocity corresponding to s velocity α_1 is

$$\frac{2c\alpha_1}{1+|\alpha_1|^2}=(E/B)\mathbf{i}=\mathbf{v}_d=v_d\mathbf{i}, \tag{40}$$

which is the well-known $\mathbf{E}\times\mathbf{B}$ drift velocity. Applying the map Φ defined in Eq. (8) to both sides of Eq. (38), we get

$$\mathbf{v}(\tau)=\mathbf{v}_d\oplus_E e^{-i\nu\tau}\Phi(C). \tag{41}$$

Equation (41) says that the total velocity of the charge, as a function of the proper time, is the sum of a constant drift velocity $\mathbf{v}_d=(E/B)\mathbf{i}$ and circular motion, as expected.

If we let

$$\Phi(C)=|\Phi(C)|e^{i\nu\tau_0}=\widetilde{v}_0e^{i\nu\tau_0}, \tag{42}$$

then

$$\mathbf{v}(\tau)=\mathbf{v}_d\oplus_E\widetilde{v}_0e^{-i\nu(\tau-\tau_0)}. \tag{43}$$

Using the Einstein velocity addition formula given in Eq. (9), we find that

$$v_x(\tau)=\frac{v_d+\widetilde{v}_0\cos\nu(\tau-\tau_0)}{1+c^{-2}v_d\widetilde{v}_0\cos\nu(\tau-\tau_0)},$$

$$v_y(\tau)=\gamma_d^{-1}\frac{\widetilde{v}_0\sin\nu(\tau-\tau_0)}{1+c^{-2}v_d\widetilde{v}_0\cos\nu(\tau-\tau_0)}. \tag{44}$$

In these equations

$$\gamma_d\equiv\left(1-\frac{v_d^2}{c^2}\right)^{-1/2}. \tag{45}$$

Note that in this case

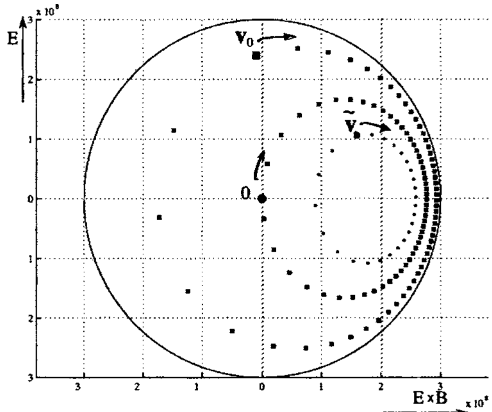

FIG. 2. The velocity trajectories $\mathbf{v}(t)$ on D_v of the test particle of Fig. 1 in the same electromagnetic field. The initial velocities are $\mathbf{v}_0=(-0.1,2.4,0)\times 10^8$ m/s, $\tilde{\mathbf{v}}_0=(1.59,1.06,0)\times 10^8$ m/s, and 0. The velocity of the particle is shown at time intervals $dt=10$ s.

$$\gamma(\tau)=\frac{1}{\sqrt{1-(v_x^2+v_y^2/c^2)}}=\gamma_d\gamma_0\left(1+\frac{v_d\tilde{v}_0^2}{c^2}\cos\nu(\tau-\tau_0)\right), \tag{46}$$

where $\gamma_0\equiv(1-\tilde{v}_0^2/c^2)^{-1/2}$. Consequently, the proper velocity is

$$\gamma\mathbf{v}=\gamma_0(\gamma_d[v_d+\tilde{v}_0\cos\nu(\tau-\tau_0)],\tilde{v}_0\sin\nu(\tau-\tau_0)). \tag{47}$$

The two equations in Eq. (44) show that relative to the laboratory frame, $\mathbf{v}(\tau)$ traces out an ellipse, as shown in Fig. 2.

B. Case 2

Next consider the case $\Delta=[(cB)^2-E^2]/E^2=0\Leftrightarrow E=cB$ and $\tilde{B}=1$. The denominator in the integrand of Eq. (30) is

$$w^2-2w+1=(w-1)^2$$

and the integral is

$$\int\frac{dw}{w^2-2\tilde{B}w+1}=-\frac{1}{w-1}+C.$$

Using this result in Eq. (30), we find that

$$-\frac{1}{w-1}=i\Omega\tau+C \tag{48}$$

or

$$w(\tau)=1-\frac{1}{i\Omega\tau+C} \tag{49}$$

with

$$C=-\frac{1}{w_0-1}. \tag{50}$$

This s velocity is graphed in Fig. 3.

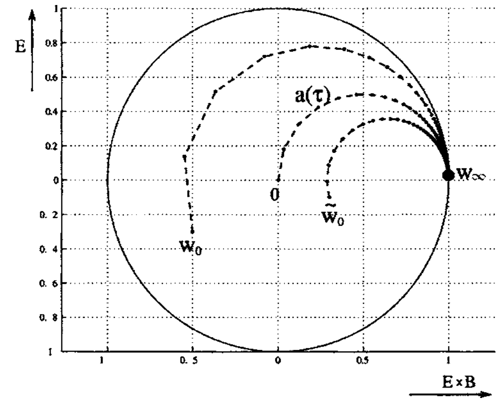

FIG. 3. The trajectories of the s velocity $w(\tau)$ of a charged particle with $q/m=10^7$ C/kg in constant, uniform fields $E=1$ V/m and $cB=1$ V/m. The initial conditions are $w_0=-0.5-i0.3$ and $\tilde{w}_0=0.3-i0.1$. Also shown is $a(\tau)$, corresponding to $w_0=0$. Note that each trajectory is a circular arc and that they all end at $w_\infty=1$.

C. Case 3

Finally, consider the case $\Delta=[(cB)^2-E^2]/E^2<0\Leftrightarrow E>cB$ or $\tilde{B}<1$.

Just as in case 1, we rewrite the denominator of the integrand in Eq. (30) as

$$w^2-2\tilde{B}w+1=(w-\alpha_1)(w-\alpha_2),$$

where

$$\alpha_1=\tilde{B}-i\delta\quad\text{and}\quad\alpha_2=\tilde{B}+i\delta=\bar{\alpha}_1 \tag{51}$$

and

$$\delta=\sqrt{1-\tilde{B}^2}>0. \tag{52}$$

As in case 1, we decompose the integrand in Eq. (30) into partial fractions and, repeating all the steps shown there, we get

$$\ln\left(\frac{w-\alpha_1}{w-\alpha_2}\right)=\sqrt{E^2-(cB)^2}\left(\frac{q}{mc}\right)\tau=\nu'\tau \tag{53}$$

$$\Rightarrow\frac{w-\alpha_1}{w-\alpha_2}=\frac{w-\alpha_1}{w-\bar{\alpha}_1}=Ce^{\nu'\tau}, \tag{54}$$

where ν' is defined as

$$\nu'=\left(\frac{q}{mc}\right)\sqrt{E^2-(cB)^2}. \tag{55}$$

It follows from Eq. (54) that as $\tau\to\infty$ the s velocity $w(\tau)$ tends to $W_\infty=\bar{\alpha}_1=\tilde{B}+i\delta$, as seen in Fig. 4.

Equation (54) implies that the initial condition can be expressed as

$$\frac{w_0-\alpha_1}{w_0-\bar{\alpha}_1}=C. \tag{56}$$

Using Eqs. (54) and (56) in the identity

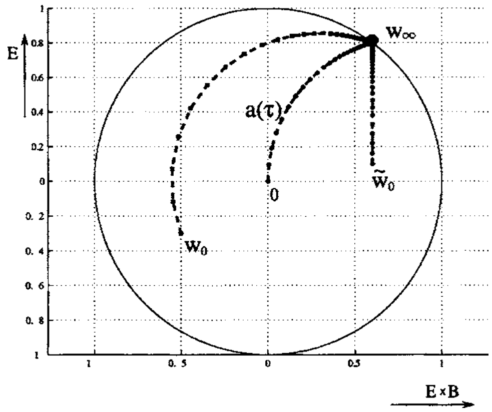

FIG. 4. The trajectories of the s velocity $w(\tau)$ of a charged particle with $q/m=10^7$ C/kg in constant, uniform fields $E=1$ V/m and $cB=0.6$ V/m. The initial conditions are $w_0=-0.5-i0.3$ and $\tilde{w}_0=0.6+i0.1$. Also shown is $a(\tau)$, corresponding to $w_0=0$. Note that the trajectories all end at $w_\infty=0.6+i0.8$.

$$\tanh(\nu'\tau)=\frac{Ce^{2\nu'\tau}-C}{Ce^{2\nu'\tau}+C},$$

we find that

$$\tanh(\nu''\tau)=\frac{\dfrac{w-\alpha_1}{w-\bar{\alpha}_1}-\dfrac{w_0-\alpha_1}{w_0-\bar{\alpha}_1}}{\dfrac{w-\alpha_1}{w-\bar{\alpha}_1}+\dfrac{w_0-\alpha_1}{w_0-\bar{\alpha}_1}},$$

where $\nu''=\nu'/2$. If we use Eq. (51) in this equation then, after some algebraic simplification, we get

$$i\tanh(\nu''\tau)=\frac{\delta(w-w_0)}{1-\tilde{B}w_0+(w_0-\tilde{B})w}.$$

Consequently, the solution for $w(\tau)$ is

$$w(\tau)=\frac{\delta w_0+i(1-\tilde{B}w_0)\tanh(\nu''\tau)}{\delta-i(w_0-\tilde{B})\tanh(\nu''\tau)}. \qquad (57)$$

This s velocity is graphed in Fig. 4.

To understand this solution, let us introduce the constant s velocity $w_d\equiv\tilde{B}/(1+\delta)$. A direct calculation shows that

$$\psi_{-w_d}(w(\tau))=i\tanh(\nu''\tau)\oplus_s\tilde{w}_0. \qquad (58)$$

In this equation ψ is defined by Eq. (12), $w(\tau)$ is defined by Eq. (57), and

$$\tilde{w}_0=\frac{-\tilde{B}+(1+\delta)w_0}{(1+\delta)-\tilde{B}w_0}=\psi_{-w_d}(w_0), \qquad (59)$$

which is the initial s velocity in the frame moving with constant s velocity w_d. Applying ψ_{w_d} to both sides of this equation and using the fact that $\psi_{-w_d}(w)=\psi_{w_d}^{-1}$, we get

$$w(\tau)=w_d\oplus_s[i\tanh(\nu'\tau)\oplus_s\tilde{w}_0]. \qquad (60)$$

Equation (60) shows that in a system K' moving with s velocity w_d relative to the laboratory, the s velocity of the charge corresponds to uniformly accelerated motion in the y direction (which is the direction of $\mathbf{E}$) with initial s velocity $\tilde{w}_0$. Consequently, we can identify w_d as the drift velocity of the frame in which the trajectory of the charge is most easily described. Also, just as in case 1, this drift velocity has the same constant value and direction for both positive and negative charges.

Using Eqs. (2), (27), and (52), the laboratory velocity corresponding to w_d is

$$\frac{2cw_d}{1+|w_d|^2}=\frac{2c\tilde{B}/(1+\delta)}{1+\tilde{B}^2/(1+\delta)^2}=\frac{2c\tilde{B}(1+\delta)}{1+2\delta+\delta^2+\tilde{B}^2}=\frac{2c\tilde{B}(1+\delta)}{2+2\delta}$$
$$=c\tilde{B}=(c^2B/E)\mathbf{i}=\mathbf{v}_d=v_d\mathbf{i}, \qquad (61)$$

as expected.

To obtain the complete expression for $\mathbf{v}(\tau)$ we apply the map Φ defined in Eq. (8) to both sides of Eq. (60). To find the velocity corresponding to the s velocity $i\tanh(\nu''\tau)$, we use Eq. (2) and the formula of the hyperbolic tangent of a double angle to obtain

$$\Phi(\tanh(\nu''\tau))\mathbf{j}=\frac{2c\tanh(\nu''\tau)}{1+\tanh^2(\nu''\tau)}\mathbf{j}=c\tanh(\nu'\tau)\mathbf{j}. \qquad (62)$$

Thus, from Eq. (60) we get

$$\mathbf{v}(\tau)=\mathbf{v}_d\oplus_E[c\tanh(\nu'\tau)\mathbf{j}\oplus_E\tilde{\mathbf{v}}_0], \qquad (63)$$

where $\tilde{\mathbf{v}}_0$ is the initial velocity in the drift frame.

Equation (63) says that in order to calculate the velocity of the charge relative to the laboratory frame as a function of the proper time, you first add relativistically the initial velocity in the drift frame to a uniform acceleration in the direction of the $\mathbf{E}$ field and then add this to the drift velocity $\mathbf{v}_d=(c^2B/E)\mathbf{i}$.

To find the explicit equation of $\mathbf{v}(\tau)$, we decompose the initial velocity $\tilde{\mathbf{v}}_0$ in the drift frame as

$$\tilde{\mathbf{v}}_0=v_1^0\mathbf{i}+v_2^0\mathbf{j} \qquad (64)$$

with $v_2^0=c\tanh(\nu'\tau_0)$. Using the Einstein velocity addition formula given in Eq. (9), the formula for the hyperbolic tangent of the sum, and the identity $\gamma(c\tanh(\nu'\tau)=\cosh(\nu'\tau))$, we get

$$c\tanh(\nu'\tau)\mathbf{j}\oplus_E\tilde{\mathbf{v}}_0=\left(\frac{1/\cosh(\nu'\tau)}{1+\tanh(\nu'\tau)\tanh(\nu'\tau_0)}v_1^0,\; c\tanh[\nu'(\tau+\tau_0)]\right)$$
$$=\left(\frac{\cosh(\nu'\tau_0)}{\cosh[\nu'(\tau+\tau_0)]}v_1^0, c\tanh[\nu'(\tau+\tau_0)]\right).$$

Thus

$$v_x(\tau)=\frac{v_d\cosh[\nu'(\tau+\tau_0)]+\tilde{v}_1^0\cosh(\nu'\tau_0)}{\cosh[\nu'(\tau+\tau_0)]+c^{-2}v_d\tilde{v}_1^0\cosh(\nu'\tau_0)},$$

$$v_y(\tau) = \gamma_d^{-1}\frac{c\sinh[\nu'(\tau+\tau_0)]}{\cosh[\nu'(\tau+\tau_0)] + c^{-2}v_d\tilde{v}_1^0\cosh(\nu'\tau_0)}, \quad (65)$$

with $\gamma_d^{-1} = \sqrt{1-v_d^2/c^2}$.

Note that

$$\gamma(\tau) = \gamma_d\gamma_0'\left(\cosh[\nu'(\tau+\tau_0)] + \frac{v_d\tilde{v}_1^0}{c^2}\cosh(\nu'\tau_0)\right), \quad (66)$$

where

$$\gamma_0' \equiv \frac{1}{\sqrt{1-(\tilde{v}_0^2/c^2)\cosh(\nu'\tau_0)}}. \quad (67)$$

Consequently, the proper velocity is

$$\gamma\mathbf{v} = \gamma_0'(\gamma_d\{v_d\cosh[\nu'(\tau+\tau_0)] + \tilde{v}_1^0\cosh(\nu'\tau_0)\}, c\sinh[\nu'(\tau+\tau_0)]). \quad (68)$$

V. THE WORLD LINES OF THE EVOLUTION

Now that we have the s velocity for each of the three cases $E<cB$, $E=cB$, and $E>cB$, we can use Eqs. (20) and (21) to find the world line $t(\tau),\mathbf{r}(\tau)$ in each case. Although the exact motion will depend upon the initial conditions, the motion resulting from $\mathbf{v}=\mathbf{0}$ at $t=0$ will exhibit all of the features in which we are interested.

A. Case 1

First consider the case $\Delta=[(cB)^2-E^2]/E^2>0 \Rightarrow E<cB$ and $\tilde{B}>1$. In this case, from Eq. (43),

$$\mathbf{v}(0) = \mathbf{0} \Rightarrow \tilde{v}_0 = -v_d, \quad \tau_0 = 0,$$

and Eqs. (44) become

$$v_x(\tau) = \frac{v_d[1-\cos(\nu\tau)]}{1-c^{-2}v_d^2\cos\nu(\tau)},$$

$$v_y(\tau) = \gamma_d^{-1}\frac{-v_d\sin(\nu\tau)}{1-c^{-2}v_d^2\cos(\nu\tau)}. \quad (69)$$

By using Eq. (46) we obtain the expression for γ as a function of τ,

$$\gamma(\tau) = \gamma_d^2\left(1-\frac{v_d^2}{c^2}\cos\nu\tau\right)$$

and, using Eq. (47), the proper velocity is

$$\gamma\mathbf{v} = v_d(\gamma_d^2(1-\cos\nu\tau), -\gamma_d\sin\nu\tau), \quad (70)$$

with γ_d given by Eq. (45) and $\nu=qB/\gamma m$ given by Eq. (34).

The position of the charge as a function of the proper time is

$$\mathbf{r}(\tau) = \int_0^\tau \gamma\mathbf{v}(\tau')d\tau' = \frac{\gamma_d v_d}{\nu}(\gamma_d(\nu\tau-\sin\nu\tau), (\cos\nu\tau-1)). \quad (71)$$

Finally, the relation between the laboratory time t and the proper time τ during the evolution is

$$t(\tau) = \int_0^\tau \gamma(\tau')d\tau' = \frac{\gamma_d^2}{\nu}\left(\nu\tau - \frac{v_d^2}{c^2}\sin\nu\tau\right). \quad (72)$$

Equations (70)–(72) together give the complete solution for this case. To find the position and velocity of the charge at any time t, one would first use Eq. (72) to relate the laboratory time and the proper time, and then use the laboratory time in Eqs. (70) and (71).

There are several limiting cases that can be obtained from Eqs. (70)–(72) which provide useful information about the orbits when viewed in the laboratory frame as functions of the laboratory time t. The limits of interest are when the drift velocity approaches c and when the drift velocity approaches 0.

Consider first the case when the drift velocity approaches the speed of light. In this case, $\nu=qB/\gamma m$ becomes quite small and

$$t \to \frac{\gamma_d^2}{\nu}\left(\nu\tau - \frac{v_d^2}{c^2}\nu\tau\right) = \tau. \quad (73)$$

Using this in the equations for the position and velocity, we find

$$\mathbf{r}(t) \to \frac{mv_d}{qB}(\gamma_d^3(\nu t-\sin\nu t), \gamma_d^2(\cos\nu t-1)) \quad (74)$$

and

$$\mathbf{v}(t) \to v_d(\gamma_d^2(1-\cos\nu t), -\gamma_d\sin\nu t). \quad (75)$$

The corresponding nonrelativistic equations are

$$\mathbf{r}(t) = \frac{mv_d}{qB}(\nu t-\sin\nu t, \cos\nu t-1) \quad (76)$$

and

$$\mathbf{v}(t) = v_d(1-\cos\nu t, -\sin\nu t). \quad (77)$$

Comparing the relativistic and nonrelativistic equations, we see that the relativistic position is elongated by factors of γ^3 in the x direction and γ^2 in the y direction. Naively, one might have expected a length contraction in the x direction and no change in the y direction but, in fact, this is not what happens because of the forces exerted by the **E** and **B** fields.

The relativistic frequency $\nu=qB/\gamma m$ is reduced by a factor of γ when compared with the nonrelativistic frequency. Another way to say this is that the period of the relativistic charge is dilated relative to the period of the nonrelativistic charge.

Next, consider the nonrelativistic limit $v_d \ll c$. In this case $t \to \tau$ so

$$\mathbf{v}(t) \to v_d(1-\cos\nu t, -\sin\nu t) \quad (78)$$

and

$$\mathbf{r}(t) \to \frac{mv_d}{qB}(\nu t - \sin \nu t, \cos \nu t - 1), \tag{79}$$

which are just the equations for the nonrelativistic motion given above, as expected.

B. Case 2

Consider next the case $\Delta=[(cB)^2-E^2]/E^2=0 \Leftrightarrow E=cB$ and $\tilde{B}=1$. In the last section we showed that under these conditions

$$w = 1 - \frac{1}{i\Omega\tau + C}, \tag{80}$$

with

$$C = -\frac{1}{w_0 - 1}. \tag{81}$$

When the initial velocity is zero $C=1$ and

$$w = w_1 + iw_2 = \frac{(\Omega\tau)(\Omega\tau + i)}{1 + (\Omega\tau)^2}. \tag{82}$$

Consequently, the proper velocity of the charge as a function of the proper time is

$$\gamma\mathbf{v} = \frac{2c\mathbf{w}}{1 - |\mathbf{w}|^2} = 2c(\Omega^2\tau^2, \Omega\tau), \tag{83}$$

and the position of the charge as a function of the proper time is

$$\mathbf{r}(\tau) = 2c\left(\frac{\Omega^2\tau^3}{3}, \frac{\Omega\tau^2}{2}\right). \tag{84}$$

Using Eq. (5),

$$\gamma(\tau) = 1 + 2(\Omega\tau)^2 \tag{85}$$

and, from Eq. (21),

$$t(\tau) = \int_0^\tau \gamma \, d\tau = \tau + \frac{2\Omega^2}{3}\tau^3. \tag{86}$$

Equations (83), (84), and (86) together give the complete solution for this case, an example of which is show in Fig. 5. To find the position and velocity of the charge at any time t, one would first use Eq. (86) to relate the laboratory time and the proper time, and then use the laboratory time in Eqs. (83) and (84).

For large times, $\tau \to (3t/2\Omega^2)^{1/3}$. Thus,

$$\mathbf{r}(t) \to \left(ct, c\left(\frac{3}{2\Omega^2}\right)^{2/3} t^{2/3}\right) \tag{87}$$

and

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} \to \left(c, \frac{2c}{3}\left(\frac{3}{2\Omega^2}\right)^{2/3} t^{-1/3}\right). \tag{88}$$

So when t is large, the x coordinate approaches ct and the y coordinate increases as $t^{2/3}$. Similarly, $v_x \to c$ and $v_y \to 0$ as $t^{-1/3}$.

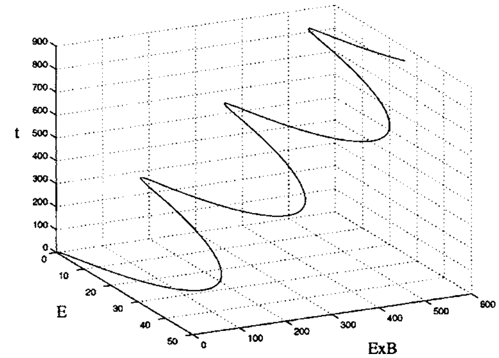

FIG. 5. The world line $\mathbf{r}(\tau), t(\tau)$ of the test particle of Fig. 1 in the same electromagnetic field. The initial velocity $\mathbf{v}_0=(0,0,0)$.

For small times, $t \approx \tau$ and

$$\mathbf{v}(t) \to \left(\left(\frac{qEt}{m}\right)\left(\frac{qBt}{2m}\right), \left(\frac{qEt}{m}\right)\right). \tag{89}$$

Therefore just after $t=0$, the acceleration of the charge in the y direction is qE/m, which is just the acceleration of a charge in a constant electric field. Once the charge acquires a v_y as a result of being accelerated by the electric field, v_x begins to grow because now there is a magnetic force on the charge in the x direction. Since $v_y=qEt/m$, Eq. (89) shows this explicitly when it is rewritten as

$$\mathbf{v}(t) \to \left(\frac{1}{2}\left(\frac{qv_yBt}{m}\right), \left(\frac{qE}{m}\right)t\right). \tag{90}$$

C. Case 3

Consider next the case $\Delta=[(cB)^2-E^2]/E^2<0 \Leftrightarrow E>cB$ and $\tilde{B}<1$.

It follows from Eq. (64) that $t=\tau_0=0 \Rightarrow \tilde{v}_1^0=-v_d$ and, from Eq. (67), that $\gamma_0'=\gamma_d$. Thus, Eqs. (65) become

$$v_x(\tau) = \frac{v_d[\cosh(\nu'\tau) - 1]}{\cosh(\nu'\tau) - c^{-2}v_d^2},$$

$$v_y(\tau) = \gamma_d^{-1}\frac{c \sinh(\nu'\tau)}{\cosh(\nu'\tau) - c^{-2}v_d^2}. \tag{91}$$

Using Eq. (66) we get

$$\gamma(\tau) = \gamma_d^2\left(\cosh(\nu'\tau) - \frac{v_d^2}{c^2}\right) \tag{92}$$

and

$$\mathbf{v}(\tau) = \frac{\gamma_d}{\gamma(\tau)}(\gamma_d v_d[\cosh(\nu'\tau) - 1], c \sinh(\nu'\tau)). \tag{93}$$

From this it follows that

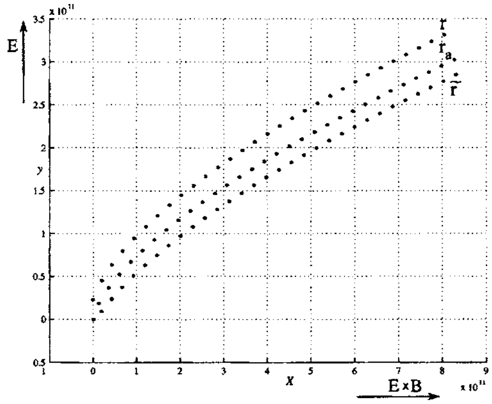

FIG. 6. The space trajectories $\mathbf{r}(t)$ of the test particles of Fig. 3 during 3000 s. The position of each particle is shown at fixed time intervals $dt=100$ s.

$$\mathbf{r}(\tau) = \int_0^\tau \gamma \mathbf{v}(\tau')d\tau' = \frac{\gamma_d}{\nu'}(\gamma_d v_d[\sinh(\nu'\tau) - \nu'\tau], c[\cosh(\nu'\tau) - 1]) \quad (94)$$

. The relation between the laboratory time t and the proper time τ is

$$t(\tau) = \int_0^\tau \gamma(\tau')d\tau' = \gamma_d^2\left(\frac{\sinh(\nu'\tau)}{\nu'} - \frac{v_d^2}{c^2}\tau\right). \quad (95)$$

Equations (93)–(95) together give the complete solution for this case, and example of which is shown in Fig. Fig. 6. To find the position and velocity of the charge at any time t, one would first use Eq. (95) to relate the laboratory time and the proper time, and then use the laboratory time in Eqs. (93) and (94).

When $t \to 0$, we see from Eq. (95) that

$$t \to \gamma_d^2\left(1 - \frac{v_d^2}{c^2}\right)\tau = \tau, \quad \gamma(\tau) \to 1,$$

and that the limit of v_y in Eq. (93) is

$$v_y \to \left(\frac{qE}{m}\right)t, \quad (96)$$

as expected. Similarly,

$$v_x \to \frac{1}{2}\left(\frac{qv_yB}{m}\right)t. \quad (97)$$

These are the same equations we found when we considered this limit in case 2. Thus, just after $t=0$, charges moving in the fields of cases 2 and 3 have exactly the same motion.

When $t \to \infty$ Eq. (95) becomes

$$t \to \gamma_d^2\frac{\sinh(\nu\tau)}{\nu} \approx \gamma_d^2\frac{\cosh(\nu\tau)}{\nu}.$$

Using this in Eq. (92), we see that $\gamma(\tau) \to \nu t$ and, from Eq. (93),

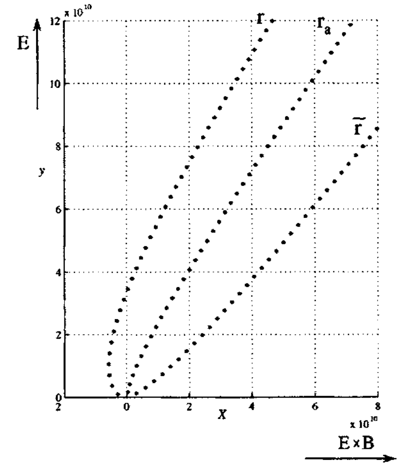

FIG. 7. The space trajectories $\mathbf{r}(t)$ of the test particles of Fig. 4 in the same electromagnetic field during 500 s. The position of each particle is shown at fixed time intervals $dt=10$ s.

$$\mathbf{v}(t) \to (v_d, c\gamma_d^{-1}) = \left(\frac{c^2B}{E}, c\sqrt{1 - \left(\frac{cB}{E}\right)^2}\right) \quad (98)$$

Consequently, in the limit $t \to \infty$, the velocity approaches the speed of light in the direction of the unit vector $(cB/E, \sqrt{1-(cB/E)^2})$, as can be seen in Figs. 4 and 7. Note that in this case a charge approaches its limiting velocity much more quickly than in case 2. Also note that as $cB \to E$, Eq. (98) → Eq. (88).

VI. ENERGY GAINS

The main goal of the paper by Takeuchi [3] was to derive the energy gain, in the laboratory frame, of a charged particle moving in the fields of cases 2 and 3. We now show that when we use the equations derived in the previous sections, our results agree with those found in Ref. [3].

First consider our case 2. When $E=cB$, Eqs. (85) and (86) show that when γ is very large, its time dependence increases as $t^{2/3}$. This is exactly the dependence derived in Ref. [3] and shown in Eq. (35) in that paper.

Next consider case 3. Using Eq. (95), we see that for large times $t \to \tau$. Putting this result into Eq. (92) we see that the increase in γ is proportional to the time t as t gets large. This conclusion agrees with Ref. [3], as expressed in Eq. (34) in that paper.

VII. MORE FORMAL MATHEMATICAL PROPERTIES OF THE *s* VELOCITY

The *s* velocity was discussed as a two-dimensional quantity in Ref. [5] and as a three dimensional quantity in Ref. [4,6,7]. The two-dimensional *s* velocity $\mathbf{w}$ was shown to be an element of a relativistic velocity space called *rapidity space* [5], whose metric is invariant under Lorentz transformations. The metric is hyperbolic, and rapidity space is a Poincaré disk. The coordinates of rapidity space are

$$\omega_x = \frac{\gamma\beta_x}{\gamma + 1} \quad \text{and} \quad \omega_y = \frac{\gamma\beta_y}{\gamma + 1}, \tag{99}$$

so it is a simple generalization of the usual nonrelativistic velocity space. Representing relativistic velocities as elements in rapidity space allows them to be analyzed geometrically, just as working with four-vectors in spacetime allows various phenomena (length contraction, time dilation, etc.) to be understood geometrically.

The three-dimensional *s* velocity has been shown to be an element of a three-dimensional ball that is a bounded symmetric domain [4,7]. All of the results presented here can be derived in a more general fashion by studying the group of all conformal automorphisms of the ball and the infinitesimal generators of that group. As mentioned in [5], *s* velocities are related to spinors, quaternions and Clifford algebras. Baylis [13] and Jancewich [14] have used Clifford algebras to derive some of the results presented here.

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VIII. SUMMARY AND CONCLUSIONS

In this paper we have extended and completed the work presented in Ref. [3] by deriving exact expressions in the laboratory frame for the relativistic velocity and energy of a charge moving in uniform, mutually perpendicular electric and magnetic fields. We obtained these results by using a relativistic velocity variable called the *s* velocity, which made the exact solutions easier to obtain and many aspects of them easier to understand.

The motion of a charge in electric and magnetic fields like the ones discussed here is important not only in plasma physics, but also in the study of the energy spectrum of ultrahigh-energy cosmic rays [9–11]. It also is fundamental to the study of hot galactic halo gases, and in the grain plasma theory of the evolution of galaxies [11,12].

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