Minimization of the energy flow in the synchronization of nonidentical chaotic systems

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We argue that maintaining a synchronized regime between different chaotic systems requires a net flow of energy between the guided system and an external energy source. This energy flow can be spontaneously reduced if the systems are flexible enough as to structurally approach each other through an adequate adaptive change in their parameter values. We infer that this reduction of energy can play a role in the synchronization of bursting neurons and other natural oscillators.

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I. INTRODUCTION

Two identical chaotic systems conveniently coupled can reach spontaneously a regime of complete synchronization that is often called identical synchronization [1,2]. If the systems are different complete synchronization does not spontaneously occur although different synchronization regimes can be forced through the establishment of an appropriate coupling device [3–9]. There is not much work on the derivation of measures to evaluate the interdependences of the synchronizing systems [10–12]. A physically detectable difference between the spontaneous synchronization of identical systems and the forced synchronization of different chaotic systems stems from the fact that in the former case the maintenance of the synchronized regime is costless while in the latter there is certainly an energy cost [12]. In both situations the maintenance of the synchronized regime requires a coupling device that cannot be removed. Nevertheless, in the case of identical systems there is no flow of energy through this coupling device while in the case of different systems it must be a continuous flow of energy with a net nonzero average value per unit time. On the other hand, a forced synchronized regime between two nonidentical systems creates conditions that facilitate their adaptation in the sense that the guided system can cause the values of its parameters to approach the ones in the guiding system [13].

In this paper we show that this adaptive flexibility makes it possible for nonidentical coupled oscillators to attain a synchronized regime at a lower cost than they would do without the adaptation ability. Consequently, with this result, we argue that in some real physical and biological environments, where maintaining a synchronized regime among some systems is an essential feature of the behavior, nonidentical systems of the same family can spontaneously evolve toward closer values of their parameters in order to maintain synchronization at a much lower value of the energy cost. Biological oscillators are structures particularly flexible in adapting their parameters. Currently, great effort is being devoted to the study of models showing the irregular spike bursting characteristic of some neurobiological systems [14–17], but no extensive work has been done on the energy implications of the synchronization process.

In this section we quickly review the aspects of synchronization energy [12] and adaptation [13] that we need to put together in this work. Section II presents the flows of energy that appear in the synchronization of chaotic oscillators with and without adaptation. Finally, in Sec. III we discus the results and present our conclusion.

A. Feedback synchronization energy

Consider an autonomous dynamical system $\dot{x}=f(x)$ where $x\in\mathbb{R}^n$ and $f:\mathbb{R}^n\to\mathbb{R}^n$ is a smooth function. We can express the velocity vector field $f(x)$ as sum of two vector fields $f(x)=f_c(x)+f_d(x)$, one of them $f_d(x)$ containing its divergence and the other $f_c(x)$ containing its full rotation [18]. In this decomposition f_c is the conservative component of the flow and f_d is the dissipative component. For the conservative component f_c there exists an energy-type function $H(x)$ that remains constant, that is, $\dot{H}(x)=0$. Thus, the equation $\nabla H^T f_c(x)=0$ defines for each dynamical system a partial differential equation from which its energy function $H(x)$ can be evaluated [12]. The energy is dissipated, passively or actively, due to the divergent component of the velocity vector field according to the equation $\dot{H}=\nabla H^T f_d(x)$. The existence of a function of the phase space variables that can be used to measure the energy of a particular state of a given chaotic system permits the evaluation of the energy exchange of the system with its environment when it moves along a particular trajectory. In what follows we are going to use the expression $\dot{H}=\nabla H^T f_d(x)$ to evaluate the energy balance that takes place when the system is forced to synchronize another guiding oscillator.

Let us force the chaotic oscillator $\dot{x}=f(x)$ to synchronize a different guiding chaotic system $\dot{y}=g(y)$ via feedback coupling according to the scheme

$$\dot{y}=g(y,p),$$

$$\dot{x}_k=f(x_k,q)+K(y-x_k), \tag{1}$$

where $x,y\in\mathbb{R}^n, f,g:\mathbb{R}^n\to\mathbb{R}^n$ are smooth functions, K is the $n\times n$ diagonal matrix with diagonal entries $k_i=k>0$, p and q

stand for the parameters of the oscillators, and $x_k(t)$ indicates the states of the guided system when the gain parameter is set to k. Notice that $K(y-x_k)$ is the coupling interface required in order to be physically able to implement the coupling of both systems $\dot{x}=f(x)$ and $\dot{y}=g(y)$.

As the trajectory $x_k(t)$ remains confined to an attractive region of phase space for every value of k, the net average energy variation corresponding to system $f(x_k)+K(y-x_k)$ will be zero. That is,

$$\langle[\nabla H^f(x_k)]^T[f_d(x_k)+K(y-x_k)]\rangle = 0,$$

where the angular brackets represent averaging on the attractor and H^f denotes the energy function of the system $\dot{x}=f(x)$.

Thus, the average energy per unit time $P(k) \equiv \langle[\nabla H^f(x_k)]^T K(y-x_k)\rangle$ that the coupling mechanism must provide the guided system with, in order to maintain the degree of synchronization attained with a coupling of gain parameter k and, consequently, force it to follow an unnatural trajectory $x_k(t)$, will be

$$P(k) = -\langle[\nabla H^f(x_k)]^T f_d(x_k)\rangle. \quad (2)$$

According to Eq. (2), the coupling device provides the flow of energy needed to compensate the energy exchange of system $f(x_k)$ with its environment. This energy can be considered as the cost of maintaining that particular level of synchronization.

The degree of synchronization reached, measured in terms of the error vector $e=x_k-y$, depends on the magnitude of the gain parameter k. The norm of the synchronization error can be made arbitrarily small as long as a sufficiently large gain k is implemented. To find the cost of maintaining a regime of complete synchronization we can substitute $y(t)$ for $x_k(t)$ in Eq. (2).

B. Adaptation

If the coupled systems are homochaotic, that is $g\equiv f$ but $p\neq q$, and the gain parameter k is large enough as to make the errors in the variables $e=x_k-y$ small, an operational law that adapts the parameters of the guided system to the ones of the guiding system is given by [13]

$$\dot{e}_i^p = -\left[\sum_{l=1}^{n}\left(\frac{\partial f_l(x_k,q)}{\partial q_i}\right)_{(y,p)} e_l\right], \quad (3)$$

where $e^p=q-p$ denotes the vector of parameter errors, which does not need to be small, and the summation is over every component of the vector field f. The above law is general and can be used to find specific adaptive laws to any kind of homochaotic systems provided they are coupled through a feedback scheme of large enough gain. One important characteristic of this law is that it can be implemented in a real environment even without a precise knowledge of the structure of the oscillators as long as the full state of the guiding system and the relevant parameters are accessible.

In Fig. 1 we show the results of the adaptation process that has been implemented to illustrate the main concern of this work, namely, studying the flow of energy that occurs when systems with adaptation capability are forced to synchronize. Details of the particular systems used will be given later on. Figures 1(a)–1(c) show adaptation when all parameters are available and Figs. 1(d)–1(f) when only two parameters are accessible for adaptation. In this last situation correct adaptation of the accessible parameters is also achieved. We emphasize that no particular knowledge of the structure of the systems has been required, what underlines the fact that adaption is a very general ability that can potentially be successful with many different families of chaotic oscillators. The very fact of two systems being forced to synchronize makes it possible that a general adaptation law becomes operative.

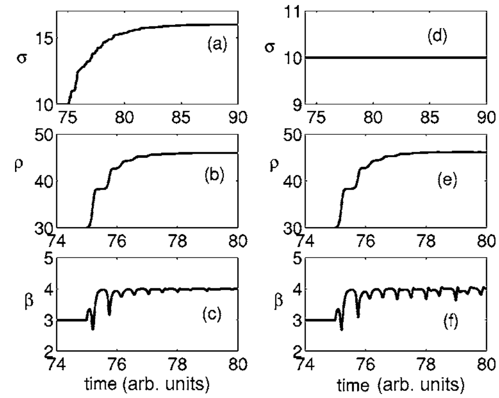

FIG. 1. Adapted parameters. In (a), (b), and (c) the full set of parameters is accessible for adaptation. In (d), (e), and (f) only two parameters are accessible. Time is in arbitrary units.

II. FLOW OF ENERGY

The average energy per unit time required to maintain a forced synchronized regime between coupled chaotic oscillators, Eq. (2), and the adaptation law of parameters, Eq. (3), will be used in this section to study the balance of energy in the synchronization of chaotic oscillators of the Lorenz family [19]. For this family, a particular solution of its energy equation is the nondefinite quadratic form $H=\frac{1}{2}[-(\rho/\sigma)/x^2+y^2+z^2]$ [12]. Using this energy function, the flows of energy that appear when a member of the Lorenz family is forced to synchronize to another oscillator can be evaluated.

A. Flow of energy without structural adaptation

We first evaluated the balance of energy in the synchronization of two Lorenz chaotic systems with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$ for the drive and $\sigma=10$, $\rho=30$, $\beta=3$ for the driven system, coupled via feedback coupling in the way described by Eq. (1). Figure 2 shows the energy per unit time dissipated by the driven system, $\dot{H}(x_k)=\nabla H^T f_d(x_k)$, throughout its motion on the attractor at four different values of the gain parameter k (the energy of the drive is not affected by k). This dissipated energy has been averaged over a convenient length of time in order to avoid large fluctuations. The temporal pattern of energy dissipation per unit time is similar

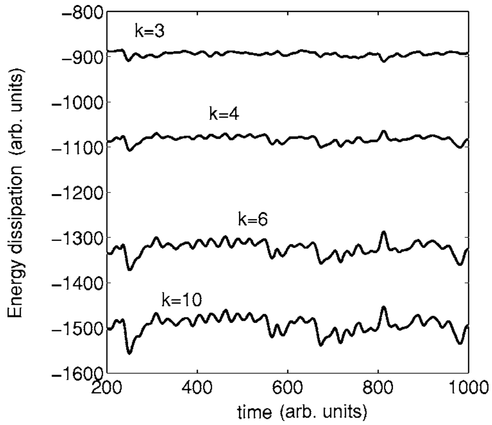

FIG. 2. A Lorenz system with parameters (16,45.92,4) guiding another different Lorenz system with parameters (10,30,3). Average energy dissipation over twenty units of time on the attractor of the guided system at four different values of the gain parameter k. Energy is in arbitrary units.

for the four values of k displayed but as the coupling forces the guided system to move away from its natural regions of the state space the average dissipated energy per unit time increases with the coupling strength k. For $k=0$, that is, with no guidance at all, the driven system moves on its natural region of the state space and its averaged dissipated energy on the attractor is zero (not shown in the figure to improve clarity).

For a more comprehensive understanding of the dependence of the average dissipated energy per unit time, given by $-P(k)$ in Eq. (2), on the strength of the coupling, we have studied its evolution for a continuous range of values of k.

First we synchronize two identical Lorenz systems with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$, coupled in the way described by Eq. (1). The gain parameter k has been varied smoothly ranging from $k=0$ to 2. For each value of k the degree of synchronization, measured as the norm of the synchronization error in the variables $\|x_k-y\|$, and the time derivative of the energy $\dot{H}$ have been averaged along a trajectory of the coupled system long enough to be considered averaged on the attractor. Figure 3(a) shows the progress of the synchronization regime. Identical synchronization appears at a value of the gain parameter of approximately $k=1.5$. Figure 3(b) shows how the average dissipation of energy changes as the coupling interface forces the guided system attractor to move through different regions of the phase space. As soon as the coupling strength is connected the average energy derivative of the guided system becomes negative, that is, it starts to dissipate on average an energy that the coupling device will have to provide in order to maintain the forced regime. The required energy increases with the gain parameter k until the onset of the dynamical changes that will produce the identical synchronization regime. At values of k in the neighborhood of $k=1.2$ the coupling device forces bifurcations in the quality of the attractors that quickly lead the guided system to reach identical synchronization, with no energy consumption at all. Note that the no consumption of energy in the identical synchronization regime is a long time average result but that there are still local variations of energy.

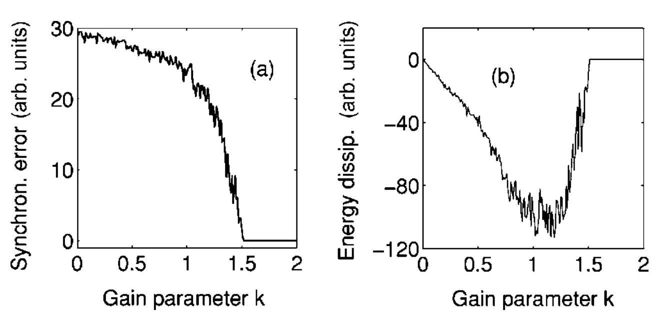

FIG. 3. A Lorenz system with parameters (16,45.92,4) guiding another identical Lorenz system at different values of the gain parameter k. (a) Synchronization error. (b) Average dissipated energy per unit time by the guided system.

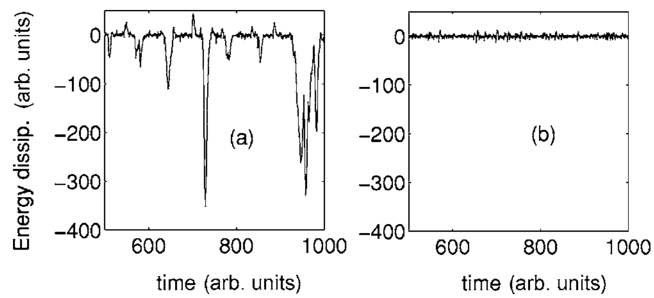

FIG. 4. Average over five units of time of the energy dissipated by a Lorenz system with parameters (16,45.92,4) guided by another identical Lorenz system. Coupling gain k= (a) 1.4 and (b) 1.5.

To illustrate the ability of this energy approach to improve our understanding of the transition to synchrony, we present in detail the energy dissipated by the guided system at two different values of the gain parameter k, the one that just triggers synchrony, $k=1.5$, and another one, $k=1.4$, a little before synchrony is reached. Figure 4(a) shows an average over five units of time of the energy dissipated by the guided system at $k=1.4$. Synchrony is about to happen. It can be clearly appreciated how the dissipated energy remains most of the time oscillating inside the threshold of identical synchronization, which is shown in Fig. 4(b) for $k=1.5$ as reference, and jumps from time to time to higher values of dissipation corresponding to regions of the phase space where the basins of the attractors have not yet collapsed to the synchronization manifold.

Second, we synchronize two nonidentical Lorenz systems. The guiding system remains the same as before while the guided system has as parameter values $\sigma=10$, $\rho=30$, $\beta=3$. Figure 5(a) shows the degree of synchronization attained at different values of the gain parameter. Figure 5(b) shows the average energy dissipated per unit time by the guiding Lorenz system at different values of the gain parameter k. The average dissipation of energy per unit time has a minimum at a value of the gain parameter in the neighborhood of $k=10$. For larger values of the forcing of the coupling the average dissipated energy asymptotically increases toward its limit value. For each value of the gain parameter the corresponding synchronization regime can only be maintained if the

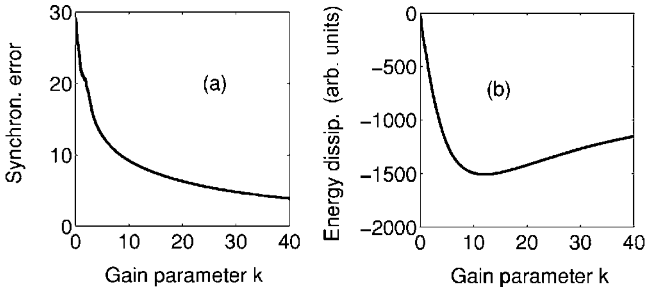

FIG. 5. A Lorenz system (16,45.92,4) guiding another different Lorenz system (10,30,3) at different values of the gain parameter k. (a) Synchronization error. (b) Average dissipated energy per unit time by the guided system.

coupling device provides the required energy to compensate the dissipation flow. Contrary to what happens with identical systems, in this case, identical synchronization does not spontaneously occur. The synchronization error has to be forced to remain small at the expense of some provision of energy.

B. Flow of energy with structural adaptation

As we have seen in the previously analyzed case of two different Lorenz systems, once they are coupled a continuous flow of energy is needed for them to maintain a synchronized regime. In this section we analyze the change in the balance of energy of the guided system when its parameters are free to adapt themselves to their nominal values in the guiding system. The adaptation law has been implemented following Eq. (3). In a first experiment we have assumed that all the three Lorenz parameters are available for adaptation. We assume that they are coupled with a coupling strength large enough to guarantee the convergence of the adaptation procedure. We have used $k=30$ for this experiment. The evolution of the values of the adapted parameters is shown in Figs. 1(a)–1(c). We started the adaptation procedure at $t=75$ and registered data between $t=60$ and 110 for proper observation of the evolution of the dissipated energy during the process. Figure 6(a) shows the degree of synchronization measured as the norm of the vector of errors in the variables. As soon as the adaptation procedure starts the average synchronization

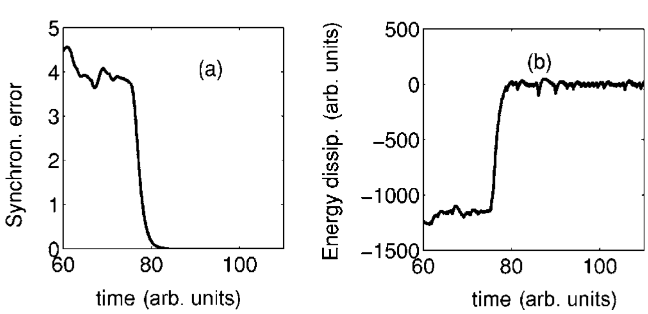

FIG. 6. Two coupled Lorenz systems with parameters (16,45.92,4) and (10,30,3) for the guiding and guided systems, respectively. Gain parameter $k=30$. Adaptation begins at $t=75$. (a) Average over two units of time of the synchronization error. (b) The same average for the dissipation of energy per unit time.

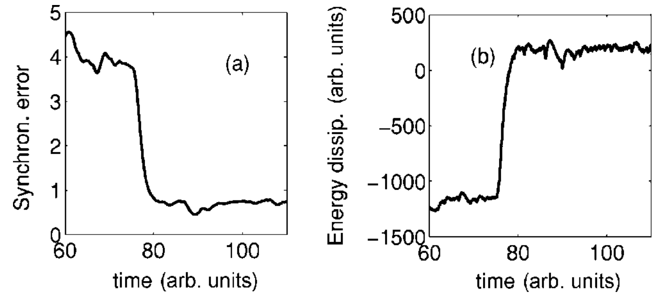

FIG. 7. Same caption as in Fig. 6. Only parameters ρ and β are adapted.

error quickly decreases to zero, reflecting the fact that the guided system has become structurally so close to the guiding system that they can reach a regime of practically identical synchronization. In Fig. 6(b) we can see how the dissipated energy changes from a value of about $\dot{H}=-1200$, corresponding to $k=30$, to the value $\dot{H}=0$ that corresponds to the identical synchronization regime.

In a second experiment we assume that only parameters ρ and β are accessible. The values of the adapted parameters are shown in Figs. 1(d)–1(f). This time, in spite of the adaptation of parameters ρ and β, the parameter σ remains different and identical synchronization cannot be reached. As can be appreciated in Fig. 7(a), the synchronization error stabilizes itself at a nonzero average regime. This new average error is smaller than it was before the adaptation took place. The energy balance of the synchronized regime also changes drastically with the adaptation process. As the systems remain different the final regime is not a regime of zero average energy exchange. As can be seen in Fig. 7(b), the adaptation process has led the guided Lorenz system to a synchronized regime with a positive average flow of energy. The net average flow of energy is also much smaller in the new regime than it was before the systems were adapted.

III. DISCUSSION AND CONCLUSION

When a chaotic oscillator moves freely on its natural attractor it moves endlessly through states of different energy in phase space. That is to say, its energy is continuously changing. Nevertheless, as it is trapped in the same region of the phase space there is no net change of energy on average. There is a balanced exchange of energy with its environment. What the environment is depends on additional hypotheses about the physical nature of the oscillator. For instance, Pasini and Pelino [20] transform the Lorenz equations in order to show that they can model a spinning rigid body involving an external angular momentum and friction plus an external forcing; they obtain, if the transformation is undone, the same energy function that we attribute here to the Lorenz system. For that Lorenz model the environment is a mechanical friction. The exchange of energy with the environment takes place in the dissipative elements of the chaotic system; for instance, in the resistor if it is a chaotic electric network. It should be noted that the resistor of a chaotic electric network must be sometimes passive and some other

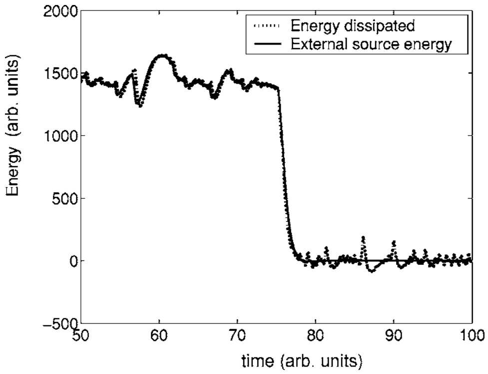

FIG. 8. Average over five units of time of the energy provided by the external source to synchronize two initially nonidentical Lorenz systems. Adaptation begins at $t=75$. Dots, the same average of the energy dissipated by the guided oscillator. All parameter values are the same as in Fig. 6.

times active, that is the necessary nature of a self-maintained oscillation. In conclusion, when a chaotic system moves freely on its natural attractor its oscillatory regime consists of a balanced exchange of energy between the system and its environment that occurs spontaneously through the divergent components of the system's structure without concurrence of any additional device. If, on the other hand, the system is forced to synchronize to a different guiding system its oscillatory regime occurs on an unnatural region of the state space where there is a nonzero net average exchange of energy with its environment. This net flow of energy per unit time requires the concurrence of a coupling device that includes an external source of energy. This flow of energy is necessary to maintain the synchronized regime and constitutes a cost for the synchronization process. In other words, synchrony cannot happen without a net energy consumption. This consumption of energy can be reduced if the guided system itself adapts its structure to become closer to the one of the guiding system. Ideally, if the systems become identical their joint dynamics is attracted toward a regime of zero error in the variables. This asymptotical limit regime of identical synchronization does not require a net flow from or toward an external source of energy.

To shed light on this discussion we have reconstructed the experiment of complete adaptation of two nonidentical Lorenz systems. This time the emphasis is placed on the energy per unit time required to compensate the dissipation of energy of the guided chaotic oscillator, but in the form that is provided by the external source through the coupling mechanism $[\nabla H^f(x_k)]^T K(y-x_k)$. Figure 8 shows by the solid line an average over five units of time of this energy per unit time. The energy dissipated per unit time by the guided system, $-[\nabla H^f(x_k)]^T f_d(x_k)$, is also shown in dots. It can be seen that before the adaptation occurs there is an oscillatory regime of dissipation of energy that has to be matched by the external source in order to preserve that degree of synchrony. If either the required net provision of energy or the required rhythm of the provision cannot be maintained by the external source, synchrony will be discontinued. Under this perspective, synchronization of nonidentical real systems does not seem to be a trivial phenomenon to achieve. This situation notably changes after adaptation. When the guided system becomes equal to the guiding system it still keeps an oscillatory pattern of energy dissipation but, this time, it is environmentally balanced on average. Its chaotic regime precisely consists in taking energy from its environment and giving it back to it, in a perfect balance. Its net average need of energy from the external source is zero. In Fig. 8 it can be seen how the external supply of energy responds to the new situation. The flow of energy from the source is zero. Not only is it zero on average, as it should be, but also becomes identically zero in value. To maintain synchrony in these circumstances is considerably easier. The coupling mechanism plays no active role. Nevertheless, it cannot be removed. The stable regime of zero error in the variables does not exist without a change in the dynamics introduced by the coupling and as soon as the coupling device is removed the synchronized regime will degrade.

Note that the fact of whether the function we have used is a real energy function or not is not absolutely relevant for our argument. Any nontrivial function of the coordinates of the phase space is useful to show that the chaotic movement cannot take place without a balancing movement in the variables of the coupling mechanism. This fact makes our argument robust in the sense that, whatever the energy of the real system is, it will behave in the way we have shown. The fact of having chosen the particular function we have used adds to our argument the important detail of its compatibility with a real energy.

Many real physical or biological processes involve synchronization between different members of the same family of systems that have similar, although not identical, values of some distinctive parameters. Under these circumstances to keep the process working involves a net flow of energy that can be costly to maintain. We have shown that the very fact of forcing synchronization between nonidentical systems creates appropriate conditions for an efficient actuation of adaptive laws able to make the systems structurally approach each other, with the final result of a decrease in the synchronization error and a decrease in the required flow of energy to maintain the process. Biological structures are particularly flexible in adapting their parameters and this mechanism of minimization could make some of the required collective behaviors energetically less costly and facilitate networking in arrays of coupled chaotic oscillators. As the Lorenz equations might not model every required aspect of a real biological oscillator [21], the analysis of specific problems, for instance the study of the synchronization process between single neurons, will require the particularization of the ideas expressed in this paper to a more specific model of the phenomenon under study.

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