Thermodynamic equilibrium and its stability for microcanonical systems described by the Sharma-Taneja-Mittal entropy

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It is generally assumed that the thermodynamic stability of equilibrium states is reflected by the concavity of entropy. We inquire, in the microcanonical picture, about the validity of this statement for systems described by the two-parametric entropy $S_{\kappa,r}$ of Sharma, Taneja, and Mittal. We analyze the "composability" rule for two statistically independent systems A and B, described by the entropy $S_{\kappa,r}$ with the same set of the deformation parameters. It is shown that, in spite of the concavity of the entropy, the "composability" rule modifies the thermodynamic stability conditions of the equilibrium state. Depending on the values assumed by the deformation parameters, when the relation $S_{\kappa,r}(\mathrm{A}\cup\mathrm{B})>S_{\kappa,r}(\mathrm{A})+S_{\kappa,r}(\mathrm{B})$ holds (superadditive systems), the concavity condition does imply thermodynamics stability. Otherwise, when the relation $S_{\kappa,r}(\mathrm{A}\cup\mathrm{B})<S_{\kappa,r}(\mathrm{A})+S_{\kappa,r}(\mathrm{B})$ holds (subadditive systems), the concavity condition does not imply thermodynamical stability of the equilibrium state.

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I. INTRODUCTION

The maximum entropy (MaxEnt) principle of thermodynamics, pioneered by Janes [1], implies that, at equilibrium, both $dS=0$ and $d^2S<0$. The first of these conditions states that entropy is an extremum, whereas the second condition states that this extremum is a maximum.

It is well known that from the second condition follow the concavity conditions for the Boltzmann-Gibbs entropy that are equivalent to the thermodynamic stability conditions of its equilibrium distribution [2,3].

Some interesting physical implications arise from the thermodynamic stability conditions. For instance, the positivity of the heat capacity assures that, for two bodies in thermal contact and with different temperatures, heat flows from the hot body to the cold one.

In the present day it is widely accepted that the Boltzmann-Gibbs distribution represents only a special case among the great diversity of statistical distributions observed in nature. In many cases such distributions show asymptotic long tails with a power-law behavior. Examples include anomalous diffusion and Levy flight [4,5], turbulence [6], self-gravitating systems [7], high-T_c superconductivity [8], Bose-Einstein condensation [9], kinetics of charge particles [10], biological systems [11] and others.

To deal with such *anomalous* statistical systems, some generalizations of the well-known Boltzmann-Gibbs entropy have been advanced, with the purpose, on one hand, of incorporating newly observed phenomenologies and, on the other hand, of mimicking the beautiful mathematical structure of the standard thermostatistics theory [12–14]. A possible way to do this is to replace the standard logarithm in the Boltzmann-Gibbs entropy, $S(p)=-k_{\mathrm{B}}\Sigma_i p_i \ln(p_i)$, with its generalized version [15,16].

In this work we investigate the relationship between the concavity conditions and the thermodynamic stability conditions of the equilibrium distribution of a conservative system, with fixed energy and volume, described by a generalized entropy. A preliminary investigation of this question can be found in Refs. [17–20].

As a working tool we employ the two-parameter entropy of Sharma, Taneja and Mittal [21–24]. Although an entropy containing two free parameters could sound unlike on the physical ground, the Sharma-Taneja-Mittal entropy includes, as special cases, some one-parameter entropies already proposed in the literature, like the Tsallis entropy [25,26], the Abe entropy [27,28] and the Kaniadakis entropy. [29,30] Consequently, the Sharma-Taneja-Mittal entropy enables us to consider all these one-parameter entropies in a unified scheme.

In Refs. [31,32] the question has been addressed of the existence of a generalized trace-form entropy

$$S(p)=-\sum_{i=1}^{W} p_i\Lambda(p_i) \tag{1.1}$$

(throughout this paper we use units with Boltzmann constant $k_{\mathrm{B}}=1$) preserving unaltered the epistemological structure of the standard statistical mechanics. In Eq. (1.1) $\Lambda(x)$ is a deformed logarithm [16] replacing the standard one, $\{p_i\}_{i=1,\ldots,W}$ is a discrete probability distribution function, and W is the number of microscopically accessible states. By requiring that the entropy (1.1) preserve the mathematical properties physically motivated [30], the following differential-functional equation has been obtained:

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$$\frac{d}{dp_j}[p_j\Lambda(p_j)] = \lambda\Lambda\left(\frac{p_j}{\alpha}\right). \tag{1.2}$$

A physically suitable solution $\Lambda(x) \equiv \ln_{(\kappa,r)}(x)$ can be written as

$$\ln_{\{\kappa,r\}}(x) = x^r \frac{x^\kappa - x^{-\kappa}}{2\kappa}, \tag{1.3}$$

which satisfies Eq. (1.2) with

$$\alpha = \left(\frac{1+r-\kappa}{1+r+\kappa}\right)^{1/2\kappa} \tag{1.4}$$

and

$$\lambda = \frac{(1+r-\kappa)^{(r+\kappa)/2\kappa}}{(1+r+\kappa)^{(r-\kappa)/2\kappa}} \tag{1.5}$$

Taking into account Eq. (1.3), the generalized entropy (1.1) assumes the form

$$S_{\kappa,r}(p) = -\sum_{i=1}^{W} (p_i)^{r+1}\frac{(p_i)^\kappa - (p_i)^{-\kappa}}{2\kappa} = -\sum_{i=1}^{W} p_i \ln_{\{\kappa,r\}}(p_i), \tag{1.6}$$

which was introduced in Refs. [21–23] and successively reconsidered in Ref. [24].

Eqation (1.6) mimics the expression of the Boltzmann-Gibbs entropy by replacing the standard logarithm $\ln(x)$ with the two-parametric deformed logarithm $\ln_{\{\kappa,r\}}(x)$.

The distribution obtained by optimizing Eq. (1.6), under the standard linear energy constraint $\Sigma_i p_i E_i = \langle E\rangle$ and the normalization constraint $\Sigma_i p_i = 1$, assumes the form

$$p_i = \alpha \exp_{\{\kappa,r\}}\left(-\frac{\beta}{\lambda}(E_i - \mu)\right), \tag{1.7}$$

where $\exp_{\{\kappa,r\}}(x)$ is the inverse function of $\ln_{\{\kappa,r\}}(x)$—namely the deformed exponential.

Remarkably, Eq. (1.7) exhibits an asymptotic power-law behavior, with $p_i \sim E_i^{1/(r-\kappa)}$, for large E_i. This entropy possesses positivity, continuity, symmetry, expandibility, decisivity, maximality, and concavity and is Lesche stable whenever $(\kappa, r) \in \mathcal{R}$, where the two-dimensional region $\mathcal{R}$ is defined by $-|\kappa| \leq r \leq |\kappa|$ for $|\kappa| < 1/2$ and $|\kappa| - 1 < r < 1 - |\kappa|$ for $1/2 \leq |\kappa| < 1$.

We remark that the deformed logarithm (1.3) reduces to the standard logarithm in the $(\kappa, r) \to (0,0)$ limit [$\ln_{\{0,0\}}(x) \equiv \ln x$] and, in the same limit, Eq. (1.6) reduces to the Boltzmann-Gibbs entropy. [Refer to Appendix A for the main mathematical properties of the deformed logarithm (1.3).]

The plan of the paper is the following. In the next section, we consider the Sharma-Taneja-Mittal entropy and its distribution in the microcanonical framework. In Sec. III we derive the "composability" rule for two statistically independent systems A and B with the same set of deformation parameters. In Sec. IV we inquire about the functional relationship between the Sharma-Taneja-Mittal entropy and the definitions of temperature and pressure obtained as equivalence relations at the equilibrium configuration. In Sec. V we examine the thermodynamic response produced by perturbing the system away from the equilibrium. The perturbations are generated by repartitioning the energy (heat transfer) and the volume (work transfer) between the two systems A and B. According to the MaxEnt principle such processes lead to a lower entropy, provided that the whole system $A \cup B$ is initially in a stable equilibrium. By analyzing the signs of the entropy changes for these processes we obtain the corresponding thermodynamic stability conditions. Finally, in Sec. VI we relate these results to some known one-parameter cases. Concluding remarks are reported in Sec. VII. In Appendix A we give some mathematical properties of the deformed logarithm, and Appendix B deals with a sketch of some proofs.

II. MICROCANONICAL SHARMA-TANEJA-MITTAL ENTROPY

According to the MaxEnt principle, the equilibrium distribution is the one that maximizes the entropy under the constraints imposed on the probability distribution.

In the microcanonical picture, the system has fixed total energy E and volume V, and the distribution $p \equiv \{p_i\}_{i=1,\ldots,W}$ is obtained by optimizing the entropy (1.6) under the only constraint on the normalization:

$$\sum_{i=1}^{W} p_i = 1. \tag{2.1}$$

Thus, we have to deal with the variational problem

$$\frac{\delta}{\delta p_j}\left(S_{\kappa,r}(p) - \gamma\sum_{i=1}^{W} p_i\right) = 0, \tag{2.2}$$

where γ is the Lagrange multiplier associated with the constraint (2.1). By taking into account Eqs. (1.1) and (1.2) it follows that

$$\lambda \ln_{\{\kappa,r\}}\left(\frac{p_j}{\alpha}\right) + \gamma = 0, \tag{2.3}$$

and by means of the deformed exponential $\exp_{\{\kappa,r\}}(x)$, we obtain

$$p_j = \alpha \exp_{\{\kappa,r\}}\left(-\frac{\gamma}{\lambda}\right). \tag{2.4}$$

Since this distribution does not depend on the index j, according to Eq. (2.1), it takes the form

$$p_j = \frac{1}{W(E,V)}, \text{ with } j = 1, \ldots, W, \tag{2.5}$$

where we took into account that the number of accessible states $W(E,V)$ is a function of the energy E and the volume V of the system.

By substituting Eq. (2.5) into Eq. (1.6) we obtain its expression in the microcanonical picture:

$$S_{\kappa,r}(E,V) = -\ln_{\{\kappa,r\}}\left(\frac{1}{W(E,V)}\right) = \ln_{\{\kappa,-r\}}[W(E,V)]. \tag{2.6}$$

This is evocative of the well-known Boltzmann formula $S=\ln(W)$, which is indeed recovered in the $(\kappa,r)\to(0,0)$ limit.

We observe that the concavity of the function $\ln_{\{\kappa,r\}}(x)$ with respect to its argument x does not necessarily imply the concavity of the entropy (2.6) with respect to E and V.

The concavity conditions for the given problem follow from the analysis of the sign of the eigenvalues of the Hessian matrix associated with Eq. (2.6). In particular, by requiring that the following quadratic form be negative definite [33],

$$\phi(\boldsymbol{y};E,V) = \frac{\partial^2 S_{\kappa,r}}{\partial E^2}y_{\rm E}^2 + 2\frac{\partial^2 S_{\kappa,r}}{\partial E\,\partial V}y_{\rm E}y_{\rm V} + \frac{\partial^2 S_{\kappa,r}}{\partial V^2}y_{\rm V}^2, \tag{2.7}$$

for any arbitrary vector $\boldsymbol{y}\equiv(y_{\rm E},y_{\rm V})$, we obtain the relations

$$\frac{\partial^2 S_{\kappa,r}}{\partial E^2} < 0 \tag{2.8}$$

and

$$\frac{\partial^2 S_{\kappa,r}}{\partial E^2}\frac{\partial^2 S_{\kappa,r}}{\partial V^2} - \left(\frac{\partial^2 S_{\kappa,r}}{\partial E\,\partial V}\right)^2 > 0, \tag{2.9}$$

stating the concavity conditions for the entropy (2.6).

III. COMPOSED SYSTEMS

Let us consider two systems A and B described by the entropy (2.6), with the same set of deformation parameters.

We denote with $W_{\rm A}\equiv W(E_{\rm A},V_{\rm A})$ and $W_{\rm B}\equiv W(E_{\rm B},V_{\rm B})$ the number of accessible states of the two systems A and B, respectively, and hypothesize a statistical independence of A and B, in the sense that the number of accessible sates, $W_{\rm A\cup B}\equiv W(E_{\rm A\cup B},V_{\rm A\cup B})$, of the composed system $\rm A\cup B$ is given by $W_{\rm A\cup B}=W_{\rm A}W_{\rm B}$.

In Ref. [34] the most general form of pseudoadditivity of composable entropies, as prescribed by the existence of equilibrium, has been obtained. The main result reads

$$H(S(\rm A\cup B)) = H(S(A)) + H(S(B)) + \lambda\, H(S(A))H(\Sigma(B)), \tag{3.1}$$

where $H(x)$ is a certain differentiable function and λ denotes the set of deformation parameters, while $S(\rm A),S(B)$, and $S(\rm A\cup B)$ are the entropies of systems A, B, and $\rm A\cup B$, respectively. It is easy to show that Eq. (3.1) is fulfilled by the entropy (2.6) if we define

$$H[S_{\kappa,r}(W)] = -S_{\kappa,r}(W)\left\{\exp_{\{\kappa,r\}}[-S_{\kappa,r}(W)]\right\}^{-r-\kappa} = W^{r+\kappa}\ln_{\{\kappa,r\}}\left(\frac{1}{W}\right). \tag{3.2}$$

In fact, by using Eq. (A10) given in Appendix A, we have

$$\begin{aligned}(W_{\rm A}W_{\rm B})^{r+\kappa}\ln_{\{\kappa,r\}}\left(\frac{1}{W_{\rm A}W_{\rm B}}\right) &= (W_{\rm A})^{r+\kappa}\ln_{\{\kappa,r\}}\left(\frac{1}{W_{\rm A}}\right) + (W_{\rm B})^{r+\kappa}\ln_{\{\kappa,r\}}\left(\frac{1}{W_{\rm B}}\right)\\ &\quad - 2\kappa\left[(W_{\rm A})^{r+\kappa}\ln_{\{\kappa,r\}}\left(\frac{1}{W_{\rm A}}\right)\right]\left[(W_{\rm B})^{r+\kappa}\ln_{\{\kappa,r\}}\left(\frac{1}{W_{\rm B}}\right)\right],\end{aligned} \tag{3.3}$$

which has the same structure of Eq. (3.1) with $H[S_{\kappa,r}]$ given in Eq. (3.2) and $\lambda=-2\kappa$.

After multiplying Eq. (3.3) by $(W_{\rm A}W_{\rm B})^{-r-\kappa}$ and recalling the invariance of the entropy (2.6) under the interchange of $\kappa\leftrightarrow-\kappa$, Eq. (3.3) becomes

$$S_{\kappa,r}(\rm A\cup B) = S_{\kappa,r}(A)\mathcal{I}_{\kappa,r}(B) + \mathcal{I}_{\kappa,r}(A)S_{\kappa,r}(B), \tag{3.4}$$

where the function $\mathcal{I}_{\kappa,r}(p)$, for a given distribution function $p=\{p_i\}_{i=1,\ldots,W}$, is defined by

$$\mathcal{I}_{\kappa,r}(p) = \sum_{i=1}^{W}(p_i)^{r+1}\frac{(p_i)^{\kappa}+(p_i)^{-\kappa}}{2}, \tag{3.5}$$

with $\mathcal{I}_{0,0}(p)=1$, and reduces, for the uniform distribution (2.5), to

$$\mathcal{I}_{\kappa,r}\left(\frac{1}{W(E,V)}\right) = \frac{[W(E,V)]^{-r-\kappa}+[W(E,V)]^{-r+\kappa}}{2}. \tag{3.6}$$

We remark that Eq. (3.6) actually is a function of the entropy (2.6) through the relation

$$\mathcal{I}_{\kappa,r}(x) = \kappa\, S_{\kappa,r}(x) + \left\{\exp_{\{\kappa,r\}}[-S_{\kappa,r}(x)]\right\}^{r+\kappa}. \tag{3.7}$$

It is worthy to observe that Eq. (3.4) still holds, for a canonical distribution, also if the entropy (1.6), in this case, does not satisfy the criteria dictated by Eq. (3.1).

Equation (3.4) expresses the "composability" properties for a system described by the entropy (2.6), and, in the $(\kappa,r)\to(0,0)$ limit, we recover the well-known additivity rule of the Boltzmann entropy, $S(\rm A\cup B)=S(A)+S(B)$.

In the following we analyze this equation in more detail.

According to the results given in Appendix A, when $(\kappa,r)\in\mathcal{R}|_{r\leqslant0}$ it follows $\mathcal{I}_{\kappa,r}(1/W)>1$ for $W>1$. Consequently, from Eq. (3.4) we obtain

$$S_{\kappa,r}(\rm A\cup B) > S_{\kappa,r}(A) + S_{\kappa,r}(B), \tag{3.8}$$

and the entropy (2.6) exhibits a superadditive behavior.

The analysis of Eq. (3.4) becomes more complicated in the complementary region $\mathcal{R}|_{r>0}$. In fact, for $r>0$ there exists a threshold point $W_{\rm t}(\kappa,r)>1$, which is defined by

$$\mathcal{I}_{\kappa,r}\left(\frac{1}{W_{\rm t}}\right) = 1, \tag{3.9}$$

so that $\mathcal{I}_{\kappa,r}(1/W)\leqslant1$ when $1<W\leqslant W_{\rm t}$ whereas $\mathcal{I}_{\kappa,r}(1/W)>1$ when $W>W_{\rm t}$. Consequently, for $(\kappa,r)\in\mathcal{R}|_{r>0}$ we have a subadditive behavior

$$S_{\kappa,r}(\mathrm{A} \cup \mathrm{B}) < S_{\kappa,r}(\mathrm{A}) + S_{\kappa,r}(\mathrm{B}), \tag{3.10}$$

when both $1 < W_{\mathrm{A}} < W_{\mathrm{t}}$ and $1 < W_{\mathrm{B}} < W_{\mathrm{t}}$, whereas the superadditive behavior (3.8) is recovered whenever $W_{\mathrm{A}} > W_{\mathrm{t}}$ and $W_{\mathrm{B}} > W_{\mathrm{t}}$. In the intermediate situation $1 < W_{\mathrm{A}} < W_{\mathrm{t}}$ and $W_{\mathrm{B}} > W_{\mathrm{t}}$ or $W_{\mathrm{A}} > W_{\mathrm{t}}$ and $1 < W_{\mathrm{B}} < W_{\mathrm{t}}$, the character of the composition law is not well determined, depending on the values of W_{A} and W_{B}.

Thus, for $(\kappa, r) \in \mathcal{R}|_{r>0}$ the value of the entropy of a composed system $\mathrm{A} \cup \mathrm{B}$, with respect to the sum of the entropies of the two separate systems A and B, depends on the *size* of the two systems.[1] *Small* systems exhibit a subadditive behavior, which becomes superadditive when both the systems grow over the threshold point W_{t}.

As consequence, superadditivity behavior emerges with *larger* systems.

We observe that the threshold point W_{t} becomes larger and larger, for $r \to \kappa$, according to

$$\lim_{r\to\kappa} W_{\mathrm{t}}(\kappa, r) \to \infty. \tag{3.11}$$

As a consequence, $\mathcal{I}_{\kappa,\kappa} \leq 1$ and the entropy $S_{\kappa,\kappa}(\mathrm{A} \cup \mathrm{B})$ has always a subadditive behavior.

IV. THERMAL AND MECHANICAL EQUILIBRIUM

Possible definitions of temperature and pressure, in the construction of a generalized framework of thermodynamics, have been proposed in Refs. [35–37] through the study of the equilibrium configuration.

Such a method can be successfully applied to the generalized entropy under inspection.

We assume that both energy and volume are additive quantities—i.e., $E_{\mathrm{A}\cup\mathrm{B}} = E_{\mathrm{A}} + E_{\mathrm{B}}$ and $V_{\mathrm{A}\cup\mathrm{B}} = V_{\mathrm{A}} + V_{\mathrm{B}}$. A different approach, by utilizing nonadditive energy and volume, within the framework of nonextensive statistical mechanics, has been explored in Ref. [38].

Let us consider an isolated system $\mathrm{A} \cup \mathrm{B}$ composed of two statistically independent systems A and B in contact through an ideal wall. The wall permits the transfer of energy (heat) and/or volume (work) between the two systems but is adiabatic with respect to any other interaction.

We suppose that the system, initially at the thermal and mechanical equilibrium, undergoes a small fluctuation of energy and volume between A and B. According to the MaxEnt principle the variation of the entropy evaluated at first order in δE and δV must vanish:

$$\delta S_{\kappa,r}(\mathrm{A} \cup \mathrm{B}) = 0, \tag{4.1}$$

where

$$\delta(E_{\mathrm{A}} + E_{\mathrm{B}}) = 0, \tag{4.2}$$

$$\delta(V_{\mathrm{A}} + V_{\mathrm{B}}) = 0. \tag{4.3}$$

From Eq. (3.4) we obtain (see Appendix B)

$$\frac{1}{\mathcal{I}_{\kappa,r} - r\, S_{\kappa,r}} \left.\frac{\partial S_{\kappa,r}}{\partial E}\right|_{\mathrm{A}} = \frac{1}{\mathcal{I}_{\kappa,r} - r S_{\kappa,r}} \left.\frac{\partial S_{\kappa,r}}{\partial E}\right|_{\mathrm{B}}, \tag{4.4}$$

and

$$\frac{1}{\mathcal{I}_{\kappa,r} - r S_{\kappa,r}} \left.\frac{\partial S_{\kappa,r}}{\partial V}\right|_{\mathrm{A}} = \frac{1}{\mathcal{I}_{\kappa,r} - r S_{\kappa,r}} \left.\frac{\partial S_{\kappa,r}}{\partial V}\right|_{\mathrm{B}}. \tag{4.5}$$

Actually Eqs. (4.4) and (4.5) state the analytical formulation of the zeroth law of the thermodynamics for the system under inspection and define, as equivalence relations, modulo of a multiplicative constant, the temperature

$$\frac{1}{T} = \frac{1}{\mathcal{I}_{\kappa,r} - r S_{\kappa,r}} \frac{\partial S_{\kappa,r}}{\partial E} \tag{4.6}$$

and the pressure

$$P = \frac{T}{\mathcal{I}_{\kappa,r} - r S_{\kappa,r}} \frac{\partial S_{\kappa,r}}{\partial V}, \tag{4.7}$$

which, accounting Eq. (3.7), are given only through the entropy $S_{\kappa,r}$.

The standard relations of classical thermodynamics

$$\frac{1}{T} = \frac{\partial S}{\partial E} \tag{4.8}$$

and

$$P = T\frac{\partial S}{\partial V}, \tag{4.9}$$

are recovered in the $(\kappa, r) \to (0, 0)$ limit.

It is worth observing, by using Eq. (2.6), Eqs. (4.6) and (4.7) can be written as

$$\frac{1}{T} = \frac{\partial}{\partial E}\ln(W), \tag{4.10}$$

and

$$P = T\frac{\partial}{\partial V}\ln(W), \tag{4.11}$$

which are positive-definite quantities if $W(E, V)$ is a monotonic increasing function with respect to both E and V.

In Refs. [39,40] it has been noted that in the microcanonical framework of Tsallis thermostatistics, the definitions of T and P, obtained through a study of the equilibrium configuration, lead to expressions which coincide with those obtained by using the standard Boltzmann formalism of statistical mechanics. This result still hold in the presence of the entropy (2.6), as can be seen from Eqs. (4.10) and (4.11), which define the temperature and pressure as a function of W and coincide with the standard definitions adopted in Boltzmann theory.

V. THERMODYNAMIC STABILITY

In this section we examine the thermodynamic response produced by perturbing the system which is assumed initially

[1] Here *size* is used to indicate the value W of the accessible states of the system.

in equilibrium. By analyzing the signs of thermodynamic changes, we obtain the thermodynamic stability conditions.

Let us consider a small perturbation of the system through a transfer of an amount of energy and/or volume between A and B: $S_{\kappa,r}(\mathrm{A}\cup\mathrm{B})\rightarrow S_{\kappa,r}((\mathrm{A}+\delta\mathrm{A})\cup(\mathrm{B}+\delta\mathrm{B}))$.

According to the MaxEnt principle, such a perturbation leads the system to in a new state with a lower entropy:

$$S_{\kappa,r}(\mathrm{A}\cup\mathrm{B}) > S_{\kappa,r}((\mathrm{A}+\delta\mathrm{A})\cup(\mathrm{B}+\delta\mathrm{B})). \tag{5.1}$$

In Eq. (5.1) we denote $S_{\kappa,r}(\mathrm{A}\cup\mathrm{B})\equiv S_{\kappa,r}((E_\mathrm{A}+E_\mathrm{B},V_\mathrm{A}+V_\mathrm{B}))$ and $S_{\kappa,r}((\mathrm{A}+\delta\mathrm{A})\cup(\mathrm{B}+\delta\mathrm{B}))\equiv S_{\kappa,r}((E_\mathrm{A}+\delta E_\mathrm{A},V_\mathrm{A}+\delta V_\mathrm{A})\cup(E_\mathrm{B}+\delta E_\mathrm{B},V_\mathrm{B}+\delta V_\mathrm{B}))$ where $\delta E_\mathrm{A}=-\delta E_\mathrm{B}\equiv\delta E$ and $\delta V_\mathrm{A}=-\delta V_\mathrm{B}\equiv\delta V$.

Recalling Eq. (3.4), Eq. (5.1) can be written as

$$S_{\kappa,r}(\mathrm{A})\mathcal{I}_{\kappa,r}(\mathrm{B})+\mathcal{I}_{\kappa,r}(\mathrm{A})S_{\kappa,r}(\mathrm{B}) > S_{\kappa,r}(\mathrm{A}+\delta\mathrm{A})\mathcal{I}_{\kappa,r}(\mathrm{B}+\delta\mathrm{B}) + \mathcal{I}_{\kappa,r}(\mathrm{A}+\delta\mathrm{A})S_{\kappa,r}(\mathrm{B}+\delta\mathrm{B}), \tag{5.2}$$

and after expanding the right-hand side of Eq. (5.2) up to second order in δE and δV, we obtain (see Appendix B)

$$\frac{1}{2}(\mathcal{I}_{\kappa,r}-rS_{\kappa,r})\bigg|_{\mathrm{A}\cup\mathrm{B}}\left[\frac{\mathcal{S}_{\mathrm{EE}}(\delta E)^2+2\mathcal{S}_{\mathrm{EV}}\delta E\,\delta V+\mathcal{S}_{\mathrm{VV}}(\delta V)^2}{\mathcal{I}_{\kappa,r}-rS_{\kappa,r}}\bigg|_{\mathrm{A}}+\frac{\mathcal{S}_{\mathrm{EE}}(\delta E)^2+2\mathcal{S}_{\mathrm{EV}}\delta E\,\delta V+\mathcal{S}_{\mathrm{VV}}(\delta V)^2}{\mathcal{I}_{\kappa,r}-rS_{\kappa,r}}\bigg|_{\mathrm{B}}\right]<0, \tag{5.3}$$

where we have posed

$$\mathcal{S}_{\mathrm{XY}}=\frac{\partial^2 S_{\kappa,r}}{\partial X\,\partial Y}-\frac{(\kappa^2+r^2)S_{\kappa,r}-2r\mathcal{I}_{\kappa,r}}{(\mathcal{I}_{\kappa,r}-rS_{\kappa,r})^2}\frac{\partial S_{\kappa,r}}{\partial X}\frac{\partial S_{\kappa,r}}{\partial Y}. \tag{5.4}$$

Equation (5.3) is fulfilled if the inequalities

$$\mathcal{S}_{\mathrm{EE}}<0, \tag{5.5}$$

$$\mathcal{S}_{\mathrm{EE}}\mathcal{S}_{\mathrm{VV}}-\mathcal{S}_{\mathrm{EV}}^2>0 \tag{5.6}$$

are separately satisfied by both systems A and B.

We remark that Eqs. (5.5) and (5.6) have the same structure as Eqs. (2.8) and (2.9).

Explicitly, Eqs. (5.5) and (5.6) read

$$\frac{\partial^2 S_{\kappa,r}}{\partial E^2}<\mathcal{A}_{\kappa,r}\left(\frac{\partial S_{\kappa,r}}{\partial E}\right)^2 \tag{5.7}$$

and

$$\frac{\partial^2 S_{\kappa,r}}{\partial^2 E}\frac{\partial^2 S_{\kappa,r}}{\partial^2 V}-\left(\frac{\partial^2 S_{\kappa,r}}{\partial E\,\partial V}\right)^2>\mathcal{A}_{\kappa,r}\mathcal{B}_{\kappa,r}, \tag{5.8}$$

where

$$\mathcal{A}_{\kappa,r}=\frac{(\kappa^2+r^2)S_{\kappa,r}-2r\mathcal{I}_{\kappa,r}}{(\mathcal{I}_{\kappa,r}-rS_{\kappa,r})^2} \tag{5.9}$$

and

$$\mathcal{B}_{\kappa,r}=\left(\frac{\partial^2 S_{\kappa,r}}{\partial E^2}\right)^{-1}\left\{\left(\frac{\partial^2 S_{\kappa,r}}{\partial E^2}\frac{\partial S_{\kappa,r}}{\partial V}-\frac{\partial^2 S_{\kappa,r}}{\partial E\,\partial V}\frac{\partial S_{\kappa,r}}{\partial E}\right)^2+\left(\frac{\partial S_{\kappa,r}}{\partial E}\right)^2\left[\frac{\partial^2 S_{\kappa,r}}{\partial E^2}\frac{\partial^2 S_{\kappa,r}}{\partial V^2}-\left(\frac{\partial^2 S_{\kappa,r}}{\partial E\,\partial V}\right)^2\right]\right\}. \tag{5.10}$$

In particular, the quantity $\mathcal{B}_{\kappa,r}$ is negative definite for a concave entropy, as a consequence of Eqs. (2.8) and (2.9).

From Eqs. (5.5) and (5.6) it is trivial to obtain the further inequality $\mathcal{S}_{\mathrm{vv}}<0$, which can be written as

$$\frac{\partial^2 S_{\kappa,r}}{\partial V^2}<\mathcal{A}_{\kappa,r}\left(\frac{\partial S_{\kappa,r}}{\partial V}\right)^2. \tag{5.11}$$

Equation (5.7), (5.8), and (5.11) are the announced thermodynamic stability conditions for the family of entropies given in Eq. (1.6) and reduce to Eqs. (2.8) and (2.9) in the $(\kappa,r)\rightarrow(0,0)$ limit. Depending on the sign of the function $\mathcal{A}_{\kappa,r}$ this inequality is satisfied if the concavity condition is also satisfied.

We observe that the equation $\mathcal{A}_{\kappa,r}(\tilde{x}_\mathrm{t})=0$ has a unique solution given by

$$\tilde{x}_\mathrm{t}=\left(\frac{\kappa-r}{\kappa+r}\right)^{1/\kappa}. \tag{5.12}$$

By inspection it follows that $\mathcal{A}_{\kappa,r}(x)>0$ when $0\leqslant x\leqslant\tilde{x}_\mathrm{t}$ while $\mathcal{A}_{\kappa,r}(x)<0$ when $\tilde{x}_\mathrm{t}\leqslant x\leqslant+\infty$. In the parametric region $(\kappa,r)\in\mathcal{R}|_{r\leqslant 0}$, accounting for Eq. (5.12), it follows that $1/\widetilde{W}_\mathrm{t}=\tilde{x}_\mathrm{t}>1$. Thus, being $W\geqslant 1$, it follows $1/W<1<1/\widetilde{W}_\mathrm{t}$ so that $\mathcal{A}_{\kappa,r}>0$ and consequently Eqs. (5.7), (5.8), and (5.11) are fulfilled if the concavity conditions are accomplished.

Differently, in the parametric region $(\kappa,r)\in\mathcal{R}_{r>0}$, we have $1/\widetilde{W}_t<1$. In this case $\mathcal{A}_{\kappa,r}\geqslant 0$ when $W\geqslant\widetilde{W}_t$. As a consequence we obtain that the concavity conditions imply the thermodynamic stability conditions if and only if both $W_A\geqslant\widetilde{W}_t$ and $W_B\geqslant\widetilde{W}_t$ are satisfied. At this point we observe that, when $r>0$,

$$\mathcal{I}_{\kappa,r}\left(\frac{1}{\widetilde{W}_t}\right)<1, \tag{5.13}$$

so that

$$\widetilde{W}_t<W_t, \tag{5.14}$$

where W_t is the threshold point defined in Eq. (3.9) and the system becomes superadditive when the number of accessible states, W_A and W_B, is beyond W_t. Thus, we can conclude that, whenever the system exhibits a superadditive behavior, the concavity conditions are sufficient to guarantee thermodynamic stability of the equilibrium configuration.

VI. EXAMPLES

In this section we specify our results to some one-parameter entropies, already known in the literature and belonging to the family of Sharma-Taneja-Mittal entropies.

In Fig. 1 we depict the log-linear plots for the three one-parameter entropies discussed in this section for different values of the deformation parameter. The solid line shows the Boltzmann entropy.

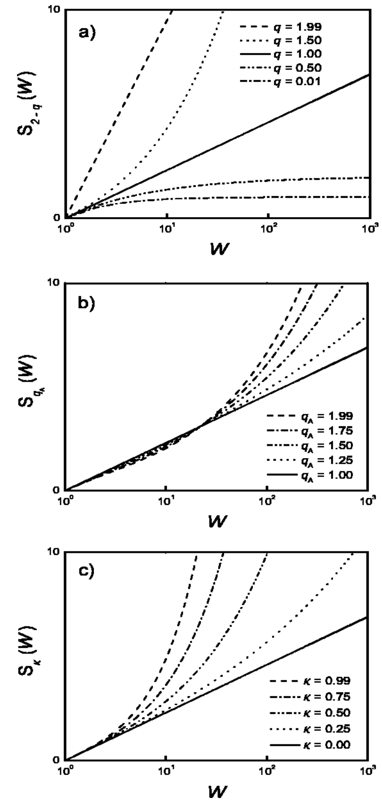

FIG. 1. Log-linear plot of some one-parameter entropies (in arbitrary units) belonging to the family (2.6) for several values of the deformation parameter: (a) Tsallis entropy [25], (b) Abe entropy [27], and (c) Kaniadakis entropy [30]. The solid curves show the Boltzmann entropy.

A. Tsallis entropy

As a first example, we consider the Tsallis entropy [25]

$$S_{2-q}=-\sum_{i=1}^{W}\frac{p_i^{2-q}-p_i}{q-1}=-\sum_{i=1}^{W}p_i\ln_q(p_i), \tag{6.1}$$

with $0<q<2$, which follows from Eq. (6.1) by posing $r=\pm|\kappa|$ and introducing the parameter $q=1\mp 2|\kappa|$. We remark that Eq. (6.1) differs from the usual definition adopted in the Tsallis framework which is recovered by replacing $q\rightarrow 2-q$.

In Eq. (6.1) the q logarithm $\ln_q(x)$ is defined by

$$\ln_q(x)=\frac{x^{1-q}-1}{1-q}, \tag{6.2}$$

whereas its inverse function—namely, the q exponential—is given by

$$\exp_q(x)=\left[1+(1-q)x\right]^{1/(1-q)}. \tag{6.3}$$

Both Eqs. (6.2) and (6.3) fulfill the relations

$$\exp_q(x)\exp_q(y)=\exp_q(x\oplus_q y), \tag{6.4}$$

$$\ln_q(xy)=\ln_q(x)\oplus_q\ln_q(y), \tag{6.5}$$

where the q deformed sum, introduced in [41,42], is defined as

$$x\oplus_q y=x+y+(1-q)xy. \tag{6.6}$$

Equations (6.4) and (6.5) reduce, in the $q\rightarrow 1$ limit, to the well-known standard relations $\exp(x)\exp(y)=\exp(x+y)$ and $\ln(xy)=\ln(x)+\ln(y)$, respectively, according to $x\oplus_1 y=x+y$.

After its introduction in 1988, Tsallis entropy has been largely applied, as a paradigm, in the study of complex systems having a probability distribution function with a power-law behavior in the tail. Typically, these systems are characterized by long-range interactions or long-time memory effects that establish a space-time interconnection by the parts, which causes a strong interdependence and the existence of a rich structure over several scales. All these induce correlations between the parts of the system, which gives the origin of a dynamical equilibrium rather than a static equilibrium: the system remains in a metastable configuration that could persist for a long period of time as compared with the characteristic time scale of the underlying microscopic dynamical process.

As is known, the Tsallis entropy exhibits many interesting properties which make it a suitable substitute for the Boltzmann Gibbs entropy in the study of these anomalous systems. Among them we recall that it is concave for $q>0$, Lesche stable [43], a basic property which must be satisfied in order to represent a well-defined physical observable [44] and fulfill the Pesin identity [45] stating a relation between the sensitivity to the initial conditions and the (finite) entropy production per unit time. Many other physical properties about the Tsallis entropy can be found in [46].

In the microcanonical picture, with a uniform distribution $p_i=1/W$, Eq. (6.1) reduces to

$$S_{2-q}=-\ln_q\left(\frac{1}{W}\right), \tag{6.7}$$

and we introduce the function $\mathcal{I}_{2-q}$ which, according to Eq. (3.7), can be expressed as a linear function of the entropy:

$$\mathcal{I}_{2-q}=\frac{1}{2}(q-1)S_{2-q}+1. \tag{6.8}$$

From Eq. (6.8) it readily follows that $\mathcal{I}_{2-q}>1$ for $q>1$ and $\mathcal{I}_{2-q}<1$ for $q<1$, and from Eq. (3.4) we obtain the well-known "composability" rule

$$S_{2-q}(\mathrm{A}\cup\mathrm{B})=S_{2-q}(\mathrm{A})+S_{2-q}(\mathrm{B})+(q-1)S_{2-q}(\mathrm{A})S_{2-q}(\mathrm{B}), \tag{6.9}$$

which shows that S_{2-q} is subadditive for $q\in(0,1)$ and superadditive for $q\in(1,2)$. We remark that Eq. (6.9) can be readily obtained also from the properties (6.5) of the q logarithm.

Temperature and pressure are defined through [36,37]

$$\frac{1}{T}=\frac{1}{1+(q-1)S_{2-q}}\frac{\partial S_{2-q}}{\partial E}, \tag{6.10}$$

$$P=\frac{T}{1+(q-1)S_{2-q}}\frac{\partial S_{2-q}}{\partial V}, \tag{6.11}$$

respectively, whereas the thermodynamic stability conditions are obtained through Eqs. (5.7)–(5.11) and read [19,20]

$$\frac{\partial^2 S_{2-q}}{\partial E^2}<\mathcal{A}_{2-q}\left(\frac{\partial S_{2-q}}{\partial E}\right)^2, \tag{6.12}$$

$$\frac{\partial^2 S_{2-q}}{\partial E^2}\frac{\partial^2 S_{2-q}}{\partial V^2}-\left(\frac{\partial^2 S_{2-q}}{\partial E\,\partial V}\right)^2>\mathcal{A}_{2-q}\mathcal{B}_{2-q}, \tag{6.13}$$

where

$$\mathcal{A}_{2-q}=\frac{q-1}{1+(q-1)S_{2-q}}. \tag{6.14}$$

Equation (6.14) is a positive quantity for $q\in(1,2)$. Consequently, it follows that both Eqs. (6.12) and (6.13) are fulfilled if the concavity conditions are satisfied. Differently, for $q\in(0,1)$, Eq. (6.14) is a negative quantity. For this range of values of q the thermodynamical stability of the equilibrium configuration does not follow merely from the concavity conditions of S_{2-q}. Such a conclusion is in accordance with the results discussed in Ref. [47].

In spite of the success obtained by the Tsallis entropy in the study of anomalous systems, others entropic forms, with probability distribution function exhibiting an asymptotic power-law behavior, have been proposed by different authors. Some of them belong to the family of Sharma-Taneja-Mittal entropies and we explore them in the next examples.

B. Abe entropy

In Ref. [27] an entropy has been presented containing the quantum group deformation structure, through the requirement of the invariance under the interchange $q\leftrightarrow q^{-1}$. This can be accomplished by posing $r=\sqrt{1+\kappa^2}-1>0$ and $q_\mathrm{A}=\sqrt{1+\kappa^2}+|\kappa|$, so that Eq. (1.6) becomes

$$S_{q_\mathrm{A}}=-\sum_{i=1}^{W}\frac{p_i^{q_\mathrm{A}^{-1}}-p_i^{q_\mathrm{A}}}{q_\mathrm{A}^{-1}-q_\mathrm{A}}=-\sum_{i=1}^{W}p_i\ln_{q_\mathrm{A}}(p_i), \tag{6.15}$$

with $1/2<q_\mathrm{A}\leqslant 2$ and

$$\ln_{q_\mathrm{A}}(x)=\frac{x^{(q_\mathrm{A}^{-1})-1}-x^{q_\mathrm{A}-1}}{q_\mathrm{A}^{-1}-q_\mathrm{A}}. \tag{6.16}$$

We remark that the inverse function of Eq. (6.16)—namely, $\exp_{q_\mathrm{A}}(x)$—exists because Eq. (6.16) is a monotonic function, but its expression cannot be given in terms of elementary functions.

The entropy (6.15) has been applied in [28] to a generalized statistical mechanics study of q-deformed oscillators. The basic idea is to incorporate the nonadditive feature of the energies of the systems having quantum group structures with generalized statistics. It has been shown that for large values of $\partial S_{q_\mathrm{A}}/\partial E$ the deformation of the entropy gives rise to significative deviations of the Planck distribution with respect to the standard (undeformed) behavior.

In Ref. [27] it has been shown that the Abe entropy can be expressed as a combination of Tsallis entropy with different deformation parameters. Consequently, many physical properties of the former follow from the physical properties of the latter. In particular, it can be shown that it is Lesche stable [48] and fulfills the Pesin equality [49].

In the microcanonical picture, the entropy (6.15) becomes

$$S_{q_\mathrm{A}}=-\ln_{q_\mathrm{A}}\left(\frac{1}{W}\right), \tag{6.17}$$

and we introduce the function $\mathcal{I}_{q_\mathrm{A}}$, through Eq. (3.6), which assumes the expression

$$\mathcal{I}_{q_\mathrm{A}}=\frac{1}{2}\left(W^{1-(q_\mathrm{A}^{-1})}+W^{1-q_\mathrm{A}}\right). \tag{6.18}$$

We recall that, according to Eq. (3.7), Eq. (6.18) is a function of the entropy S_{q_A}.

By taking into account the results of Appendix A, we have $1/2\leqslant\mathcal{I}_{q_\mathrm{A}}\leqslant+\infty$, depending on the value of W. After introducing the threshold point through $\mathcal{I}_{q_\mathrm{A}}(W_\mathrm{t})=1$, it follows that for $W_\mathrm{A}>W_\mathrm{t}$ and $W_\mathrm{B}>W_\mathrm{t}$:

$$S_{q_\mathrm{A}}(\mathrm{A}\cup\mathrm{B})>S_{q_\mathrm{A}}(\mathrm{A})+S_{q_\mathrm{A}}(\mathrm{B}). \tag{6.19}$$

In the same way, for $W_\mathrm{A}<W_\mathrm{t}$ and $W_\mathrm{B}<W_\mathrm{t}$, we obtain

$$S_{q_\mathrm{A}}(\mathrm{A}\cup\mathrm{B})<S_{q_\mathrm{A}}(\mathrm{A})+S_{q_\mathrm{A}}(\mathrm{B}). \tag{6.20}$$

Temperature and pressure are given by

$$\frac{1}{T}=\frac{1}{\mathcal{I}_{q_\mathrm{A}}-\tilde{q}_\mathrm{A}S_{q_\mathrm{A}}}\frac{\partial S_{q_\mathrm{A}}}{\partial E}, \tag{6.21}$$

$$P = \frac{T}{\mathcal{I}_{q_A} - \tilde{q}_A S_{q_A}} \frac{\partial S_{q_A}}{\partial V}, \tag{6.22}$$

respectively, where $\tilde{q}_A = (q_A^{1/2} - q_A^{-1/2})^2/2$. They reduce to the standard definition of temperature and pressure in the $q_A \to 1$ limit.

The thermodynamic stability conditions now read

$$\frac{\partial^2 S_{q_A}}{\partial E^2} < \mathcal{A}_{q_A} \left(\frac{\partial S_{q_A}}{\partial E} \right)^2, \tag{6.23}$$

$$\frac{\partial^2 S_{q_A}}{\partial E^2} \frac{\partial^2 S_{q_A}}{\partial V^2} - \left(\frac{\partial^2 S_{q_A}}{\partial E \, \partial V} \right)^2 > \mathcal{A}_{q_A} \mathcal{B}_{q_A}, \tag{6.24}$$

with

$$\mathcal{A}_{q_A} = 2\tilde{q}_A \frac{(\tilde{q}_A + 1) S_{q_A} - \mathcal{I}_{q_A}}{\left(\mathcal{I}_{q_A} - \tilde{q}_A S_{q_A} \right)^2}. \tag{6.25}$$

The sign of Eq. (6.25) changes at the point

$$\tilde{W}_t = q_A^{2/(q_A - q_A^{-1})}, \tag{6.26}$$

so that $\mathcal{A}_{q_A} < 0$ for $W < \tilde{W}_t$ and $\mathcal{A}_{q_A} > 0$ for $W > \tilde{W}_t$. On the other hand, observing that $\tilde{W}_t < W_t$, it follows that for superadditive systems with $W_A > W_t$ and $W_B > W_t$, the concavity conditions imply the thermodynamic stability conditions.

It is worthy to observe that by posing $r = 1 - \sqrt{1+\kappa^2} < 0$ and $q_A = \sqrt{1+\kappa^2} - |\kappa|$ we obtain another family of entropies embodying the symmetry $q \leftrightarrow 1/q$ given by

$$S^*_{q_A}(W) = -\sum_{i=1}^{W} \frac{p_i^{2-q_A^{-1}} - p_i^{2-q_A}}{q_A - q_A^{-1}} = -\sum_{i=1}^{W} p_i \ln^*_{q_A}(p_i), \tag{6.27}$$

with $1/2 < q_A \leqslant 2$, where now

$$\ln^*_{q_A}(x) = \frac{x^{1-q_A^{-1}} - x^{1-q_A}}{q_A - q_A^{-1}}. \tag{6.28}$$

$\ln_{q_A}(x)$ and $\ln^*_{q_A}(x)$ are related as [16]

$$\ln_{q_A}(x) = -\ln^*_{q_A}\left(\frac{1}{x}\right), \tag{6.29}$$

and the entropies (6.15) and (6.27) are dual to each other.

In the microcanonical picture Eq. (6.27) becomes

$$S^*_{q_A}(W) = -\ln^*_{q_A}\left(\frac{1}{W}\right), \tag{6.30}$$

and because now the function $\mathcal{I}^*_{q_A} > 1$, the entropy (6.30) describes superadditive systems: $S^*_{q_A}(A \cup B) > S^*_{q_A}(A) + S^*_{q_A}(B)$. The function $\mathcal{A}^*_{q_A} > 0$ and the concavity conditions for the entropy (6.30) are enough to guarantee the thermodynamic stability conditions of the system for any values of the deformation parameter.

C. Kaniadakis entropy

As a last example, we discuss the entropic form introduced previously in Ref. [29] which follows from Eq. (1.6) after we posed $r=0$:

$$S_\kappa = -\sum_{i=1}^{W} \frac{p_i^\kappa - p_i^{-\kappa}}{2\kappa} = -\sum_{i=1}^{W} p_i \ln_{\{\kappa\}}(p_i), \tag{6.31}$$

where $|\kappa| < 1$ and the κ logarithm $\ln_{\{\kappa\}}(x)$ is defined by

$$\ln_{\{\kappa\}}(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}. \tag{6.32}$$

Its inverse function the κ exponential, is given by

$$\exp_{\{\kappa\}}(x) = (\sqrt{1+\kappa^2 x^2} + \kappa x)^{1/\kappa} \tag{6.33}$$

and satisfies the relation

$$\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(-x) = 1, \tag{6.34}$$

which means that it increases for $x \to \infty$ and decreases for $x \to -\infty$ with the same steepness. Remarkably, the κ logarithm and κ exponential fulfill the two mathematical proprieties

$$\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(y) = \exp_{\{\kappa\}}(x \overset{\kappa}{\oplus} y), \tag{6.35}$$

$$\ln_{\{\kappa\}}(xy) = \ln_{\{\kappa\}}(x) \overset{\kappa}{\oplus} \ln_{\{\kappa\}}(y), \tag{6.36}$$

where the κ-deformed sum is defined as [50]

$$x \overset{\kappa}{\oplus} y = x\sqrt{1+\kappa^2 y^2} + y\sqrt{1+\kappa^2 x^2}. \tag{6.37}$$

Equations (6.35) and (6.36) reduce, in the $\kappa \to 0$ limit, to the well-known relations $\exp(x)\exp(y) = \exp(x+y)$ and $\ln(xy) = \ln(x) + \ln(y)$, respectively, according to $x \overset{0}{\oplus} y = x + y$.

In Ref. [30] it has been shown that the κ sum emerges naturally within Einstein's theory of special relativity. In fact, following the same argument presented in [30] it is possible to link the relativistic sum of the velocities,

$$v_1 \oplus^c v_2 = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}, \tag{6.38}$$

with the κ sum (6.37) in the sense of

$$p(v_1) \overset{\kappa}{\oplus} p(v_2) = p(v_1 \oplus^c v_2), \tag{6.39}$$

where $\kappa = 1/mc$ and $p(v) = mv/\sqrt{1 - v^2/c^2}$ is the relativistic momentum of a particle with rest mass m. In this way, the origin of the κ deformation is related to a finite value of the light speed c. In particular, in the classical limit $c \to \infty$ the parameter κ approaches zero and the κ entropy reduces to the Boltzmann-Gibbs one.

In agreement with this interpretation, we consider statistical systems (physical or not) that can achieve an equilibrium configuration through an exchange of information between the parts of the system, propagating with a limiting velocity, like the light speed in relativity theory. For these systems it is reasonable to suppose that a mechanism similar to the one described above, in the framework of special relativity, can arise, so that the κ deformation can occur. In this way, the κ entropy can be successfully applied in the study of the statistical proprieties of these systems.

An important physical example where the κ distribution has been successfully applied is in the reproduction of the energy distribution of the fluxes of cosmic rays [30] (see also [51]). Moreover, the κ entropy has been applied to the study of fracture propagation in brittle material, showing good agreement with results obtained experimentally and with ones obtained through numerical simulations [52].

Finally, we recall that, like the previous one-parameter deformed entropic forms, also the κ entropy fulfills many physically relevant properties. In particular, in Ref. [53], it has been shown experimentally by its stability whereas in Ref. [49] the finite entropy production in time units in connection with the Pesin identity for a system described by the entropy (6.31) has been discussed.

In the microcanonical picture, Eq. (6.31) becomes

$$S_\kappa = \ln_{\{\kappa\}}(W), \tag{6.40}$$

and we introduce the function $\mathcal{I}_\kappa$ which, according to Eq. (3.7), can be written as

$$\mathcal{I}_\kappa = \sqrt{1+\kappa^2 S_\kappa^2}, \tag{6.41}$$

so that $\mathcal{I}_\kappa \geqslant 1$. As a consequence Eq. (3.3) becomes

$$S_\kappa(\mathrm{A}\cup\mathrm{B}) = S_\kappa(\mathrm{A})\sqrt{1+\kappa^2[S_\kappa(\mathrm{B})]^2} + S_\kappa(\mathrm{B})\sqrt{1+\kappa^2[S_\kappa(\mathrm{A})]^2}, \tag{6.42}$$

which can be also written as

$$S_\kappa(\mathrm{A}\cup\mathrm{B}) = S_\kappa(\mathrm{A}) \overset{\kappa}{\oplus} S_\kappa(\mathrm{B}), \tag{6.43}$$

according to Eq. (6.37). Equation (6.42) or (6.43) implies the relation

$$S_\kappa(\mathrm{A}\cup\mathrm{B}) > S_\kappa(\mathrm{A}) + S_\kappa(\mathrm{B}), \tag{6.44}$$

stating that the κ entropy in the microcanonical picture is always superadditive.

Temperature and pressure are given by

$$\frac{1}{T} = \frac{1}{\sqrt{1+\kappa^2 S_\kappa^2}}\frac{\partial\, S_\kappa}{\partial E}, \tag{6.45}$$

$$P = \frac{T}{\sqrt{1+\kappa^2 S_\kappa^2}}\frac{\partial\, S_\kappa}{\partial V}, \tag{6.46}$$

respectively, and reduce to the standard definitions in the $\kappa \to 0$ limit.

Finally, the thermodynamic stability conditions become [19,20]

$$\frac{\partial^2 S_\kappa}{\partial E^2} < \mathcal{A}_\kappa \left(\frac{\partial S_\kappa}{\partial E}\right)^2, \tag{6.47}$$

$$\frac{\partial^2 S_\kappa}{\partial E^2}\frac{\partial^2 S_\kappa}{\partial V^2} - \left(\frac{\partial^2 S_\kappa}{\partial E\,\partial V}\right)^2 > \mathcal{A}_\kappa \mathcal{B}_\kappa, \tag{6.48}$$

where

$$\mathcal{A}_\kappa = \frac{\kappa^2 S_\kappa}{1+\kappa^2 S_\kappa^2}. \tag{6.49}$$

The function (6.49) is always positive and, as a consequence, the concavity conditions for the entropy (6.40) are enough to guarantee the thermodynamic stability conditions of the system for any values of the deformation parameter.

VII. CONCLUDING REMARKS

In the present work we have investigated the thermodynamic stability conditions for a microcanonical system described by the Sharma-Taneja-Mittal entropy and their relation with the concavity conditions for this entropy.

The main results can be summarized in the following two points.

First, we have analyzed the "composability" rule for statistically independent systems described by the entropy (2.6). It has been shown that the parameter space $\mathcal{R}$ can be split into two disjoint regions. In the region $\mathcal{R}|_{r\leqslant 0}$ the entropy $S_{\kappa,r}$ shows a superadditivity behavior: $S_{\kappa,r}(\mathrm{A}\cup\mathrm{B}) > S_{\kappa,r}(\mathrm{A}) + S_{\kappa,r}(\mathrm{B})$. Otherwise, in the region $\mathcal{R}|_{r>0}$ the behavior of the entropy is not well defined, depending on the *size* of the two systems A and B. In particular it has been shown that, given the threshold point $W_{\rm t}(\kappa,r) > 1$, when the *size* of the two parts A and B are smaller than $W_{\rm t}$, in the sense of $W_{\rm A} < W_{\rm t}$ and $W_{\rm B} < W_{\rm t}$, the system exhibits a subadditive behavior $S_{\kappa,r}(\mathrm{A}\cup\mathrm{B}) < S_{\kappa,r}(\mathrm{A}) + S_{\kappa,r}(\mathrm{B})$, becoming superadditive when both $W_{\rm A} > W_{\rm t}$ and $W_{\rm B} > W_{\rm t}$.

Second, we have inquired on the thermodynamic stability conditions of the equilibrium configuration. In the Boltzmann theory the concavity conditions imply the thermodynamic stability conditions. Such a situation changes when the system is described by the entropy $S_{\kappa,r}$. We have shown that, starting from an equilibrium configuration of the system $\mathrm{A}\cup\mathrm{B}$ and supposing an exchange of a small quantity of heat and/or work between the two parts A and B, assumed statistically independent, if the entropy of the system $S_{\kappa,r}(\mathrm{A}\cup\mathrm{B})$ is larger than the sum of the entropy of the two systems $S_{\kappa,r}(\mathrm{A})$ and $S_{\kappa,r}(\mathrm{B})$, the concavity conditions still imply the thermodynamic stability conditions. In the opposite situation, in spite of the concavity of $S_{\kappa,r}$, stability requires *large* systems, in the sense of $W_{\rm A} > W_{\rm t}$ and $W_{\rm B} > W_{\rm t}$.

APPENDIX A

In this appendix we summarize some mathematical properties of the deformed logarithm [31,32]:

$$\ln_{\{\kappa,r\}}(x) = \frac{x^{r+\kappa} - x^{r-\kappa}}{2\ \kappa}. \tag{A1}$$

Let $\mathcal{R}$ be the region in parametric space, defined by

$$\mathbb{R}^2 \supset \mathcal{R} = \begin{cases} -|\kappa| \leqslant r \leqslant |\kappa| & \text{if } 0 \leqslant |\kappa| < \frac{1}{2}, \\ |\kappa| - 1 < r < 1 - |\kappa| & \text{if } \frac{1}{2} \leqslant |\kappa| < 1. \end{cases} \tag{A2}$$

For any $(\kappa, r) \in \mathcal{R}$, the $\ln_{\{\kappa,r\}}(x) = \ln_{\{-\kappa,r\}}(x)$ is a continuous, monotonic, increasing, and concave function for $x \in \mathbb{R}^+$, with $\ln_{\{\kappa,r\}}(\mathbb{R}^+) \subseteq \mathbb{R}$, fulfilling the relation $\int_0^1 \ln_{\{\kappa,r\}}(x^{\pm 1})dx = \mp 1/[(1 \pm r)^2 - \kappa^2]$. The standard logarithm is recovered in the $(\kappa, r) \to (0,0)$ limit $\ln_{\{0,0\}}(x) = \ln(x)$.

Equation (A1) satisfies the relation

$$\ln_{\{\kappa,r\}}(x) = -\ln_{\{\kappa,-r\}}\left(\frac{1}{x}\right) \tag{A3}$$

and, for $r=0$, reproduces the well-known properties of the standard logarithm $\ln_{\{\kappa,0\}}(x) = -\ln_{\{\kappa,0\}}(1/x)$. From Eq. (A3) it follows that the behavior of $\ln_{\{\kappa,r\}}(x)$ for $r>0$ and $0 \leqslant x \leqslant 1$ is related to the one for $r<0$ and $x>1$.

We observe that the deformed logarithm is a solution of the differential-functional equation

$$\frac{d}{dx}\left[x \ln_{\{\kappa,r\}}(x)\right] = \lambda \ln_{\{\kappa,r\}}\left(\frac{x}{\alpha}\right), \tag{A4}$$

with the boundary conditions $\ln_{\{\kappa,r\}}(1) = 0$ and $d\ln_{\{\kappa,r\}}(x)/dx|_{x=1} = 1$ and the constants α and λ given in Eqs. (1.4) and (1.5).

The inverse function of Eq. (A1)—namely, the deformed exponential $\exp_{\{\kappa,r\}}(x)$—exists for any $(\kappa, r) \in \mathcal{R}$, since $\ln_{\{\kappa,r\}}(x)$ is a strictly monotonic function. Its analytical properties are well defined and follow from the corresponding ones of the deformed logarithm. Nevertheless, the explicit expression of $\exp_{\{\kappa,r\}}(x)$ can be obtained only for particular relationships between r and κ.

By inspection there exists a point $x_0(\kappa, r)$ such that the inequality $\ln_{\{\kappa,r\}}(x) \geqslant \ln x$ holds for $x > x_0(\kappa, r)$. We see that $x_0(\kappa, r)$ is a monotonic decreasing function with respect to r with $x_0(\kappa, r) > 1$ for $r<0$ and $x_0(\kappa, r) < 1$ for $r>0$. In particular, $x_0(\kappa, -\kappa) = +\infty, x_0(\kappa, 0) = 1$, and $x_0(\kappa, \kappa) = 0$.

The entropy $S_{\kappa,r}(p)$, introduced in Eq. (1.6), is related to the function $\ln_{\{\kappa,r\}}(x)$ through the relation

$$S_{\kappa,r}(p) = -\sum_{i=1}^{W} p_i \ln_{\{\kappa,r\}}(p_i). \tag{A5}$$

The deformed logarithm (A1) can be obtained from the two-parametric generalization of the Jackson derivative, previously proposed in Ref. [54]

$$\frac{d_{\kappa,r} f(x)}{d_{\kappa,r} x} = \frac{f((r+\kappa)x) - f((r-\kappa)x)}{2\kappa x}, \tag{A6}$$

by posing

$$\ln_{\{\kappa,r\}}(x) = \left. \frac{d_{\kappa,r} x^y}{d_{\kappa,r} y} \right|_{y=1}. \tag{A7}$$

Some properties of the deformed logarithm can be naturally understood as those of the generalized Jackson derivative (A6). For instance, from the generalized Leibnitz rule

$$\frac{d_{\kappa,r} f(x) g(x)}{d_{\kappa,r} x} = \frac{d_{\kappa,r} f(x)}{d_{\kappa,r} x} g((r+\kappa)x) + f((r-\kappa)x)\frac{d_{\kappa,r} g(x)}{d_{\kappa,r} x}, \tag{A8}$$

we obtain the useful relation

$$\ln_{\{\kappa,r\}}(xy) = x^{r+\kappa} \ln_{\{\kappa,r\}}(y) + y^{r-\kappa} \ln_{\{\kappa,r\}}(x), \tag{A9}$$

and by using the identity $y^{r-\kappa} = y^{r+\kappa} - 2\kappa \ln_{\{\kappa,r\}}(y)$, Eq. (A9) becomes

$$\ln_{\{\kappa,r\}}(xy) = x^{r+\kappa} \ln_{\{\kappa,r\}}(y) + y^{r+\kappa} \ln_{\{\kappa,r\}}(x) - 2\kappa \ln_{\{\kappa,r\}}(x) \ln_{\{\kappa,r\}}(y). \tag{A10}$$

Moreover, recalling the $\kappa \leftrightarrow -\kappa$ symmetry, Eq. (A10) can be rewritten as

$$\ln_{\{\kappa,r\}}(xy) = u_{\{\kappa,r\}}(x) \ln_{\{\kappa,r\}}(y) + u_{\{\kappa,r\}}(x) \ln_{\{\kappa,r\}}(y), \tag{A11}$$

where we have introduced the function

$$u_{\{\kappa,r\}}(x) = \frac{x^{r+\kappa} + x^{r-\kappa}}{2}. \tag{A12}$$

For any $(\kappa, r) \in \mathcal{R}$ the function $u_{\{\kappa,r\}}(x) = u_{\{-\kappa,r\}}(x)$ is continuous for $x \in \mathbb{R}^+$, with $u_{\{\kappa,r\}}(\mathbb{R}^+) \subseteq \mathbb{R}^+$, and $u_{\{\kappa,r\}}(0) = u_{\{\kappa,r\}}(+\infty) = +\infty$ for $r \neq |\kappa|$ satisfies the relation $u_{\{\kappa,r\}}(x) = u_{\{\kappa,-r\}}(1/x)$ and reduces to unity in the $(\kappa, r) \to (0,0)$ limit: $u_{\{0,0\}}(x) = 1$. It reaches the minimum values

$$u_{\{\kappa,r\}}(x_m) = \kappa \frac{(\kappa - r)^{(r-\kappa)/2\,\kappa}}{(\kappa + r)^{(r+\kappa)/2\,\kappa}} \tag{A13}$$

at

$$x_m = \left(\frac{\kappa - r}{\kappa + r}\right)^{1/2\,\kappa}. \tag{A14}$$

In particular, for any $(\kappa, r) \in \mathcal{R}|_{r \leqslant 0}, x_m \geqslant 1$, and taking into account that $u_{\{\kappa,r\}}(1) = 1$, it follows $1 \leqslant u_{\{\kappa,r\}}(x) \leqslant \infty$ when $x \in (0,1)$. For any $(\kappa, r) \in \mathcal{R}|_{r>0}$, from Eq. (A14) we obtain $0 \leqslant x_m \leqslant 1$ and from Eq. (A13) it follows that $1/2 \leqslant u_{\{\kappa,r\}}(x_m) \leqslant 1$.

By inspection, it follows that there exists a threshold point $x_t(\kappa, r)$, defined by $u_{\{\kappa,r\}}(x_t) = 1$. $x_t(\kappa, r)$ is monotonic decreasing function with respect to r, with $x_t(\kappa, -\kappa) = +\infty, x_t(\kappa, 0) = 1$, and $x_t(\kappa, \kappa) = 0$, such that $1/2 \leqslant u_{\{\kappa,r\}}(x_m) \leqslant u_{\{\kappa,r\}}(x) \leqslant 1$ for $x \geqslant x_t(\kappa, r)$ and $1 \leqslant u_{\{\kappa,r\}}(x) \leqslant +\infty$ for $0 \leqslant x \leqslant x_t(\kappa, r)$.

Finally, we remark that the function (A12) fulfills the relation

$$u_{\{\kappa,r\}}(xy) = u_{\{\kappa,r\}}(x) u_{\{\kappa,r\}}(y) + \kappa^2 \ln_{\{\kappa,r\}}(x) \ln_{\{\kappa,r\}}(y). \tag{A15}$$

We observe that, like the deformed logarithm, the function $u_{\{\kappa,r\}}(x)$ is a solution of the differential-functional equation (1.2) with the boundary conditions $u_{\{\kappa,r\}}(1) = 1$ and $d\,u_{\{\kappa,r\}}(x)/d\,x|_{x=1} = r$ and the constants α and λ given in Eqs. (1.4) and (1.5). Moreover, the two functions $\ln_{\{\kappa,r\}}(x)$ and $u_{\{\kappa,r\}}(x)$ are related by the relation

$$u_{\{\kappa,r\}}(x) = x^{r+\kappa} - \kappa \ln_{\{\kappa,r\}}(x). \tag{A16}$$

The function $\mathcal{I}_{\kappa,r}(p)$, introduced in Eq. (3.6), is related to the function $u_{\{\kappa,r\}}(x)$ through the relation

$$\mathcal{I}_{\kappa,r}(p) = \sum_{i=1}^{W} p_i u_{\{\kappa,r\}}(p_i). \tag{A17}$$

From the definitions (A5) and (A17) and Eqs. (A11) and (A15), we obtain the useful relations

$$S_{\kappa,r}(\mathrm{A} \cup \mathrm{B}) = S_{\kappa,r}(\mathrm{A})\mathcal{I}_{\kappa,r}(\mathrm{B}) + \mathcal{I}_{\kappa,r}(\mathrm{A})S_{\kappa,r}(\mathrm{B}), \tag{A18}$$

$$\mathcal{I}_{\kappa,r}(\mathrm{A} \cup \mathrm{B}) = \mathcal{I}_{\kappa,r}(\mathrm{A})\mathcal{I}_{\kappa,r}(\mathrm{B}) + \kappa^2 S_{\kappa,r}(\mathrm{A})S_{\kappa,r}(\mathrm{B}), \tag{A19}$$

stating the additivity of $S_{\kappa,r}$ and $\mathcal{I}_{\kappa,r}$ for statistically independent systems $p^{\mathrm{A}\cup\mathrm{B}} = \{p_i^{\mathrm{A}} p_j^{\mathrm{B}}\}$ with $i=1,\ldots,W_{\mathrm{A}}$ and $j=1,\ldots,W_{\mathrm{B}}$.

APPENDIX B

In this appendix we derive the equilibrium conditions given in Eqs. (4.4) and (4.5) and the thermodynamic stability conditions given in Eqs. (5.7) and (5.8).

Let us suppose that the whole system $\mathrm{A}\cup\mathrm{B}$ initially at equilibrium undergoes a small transfer of heat and/or work between the two parts A and B, with the constraints

$$\delta(E_{\mathrm{A}} + E_{\mathrm{B}}) = 0, \tag{B1}$$

$$\delta(V_{\mathrm{A}} + V_{\mathrm{B}}) = 0. \tag{B2}$$

Recalling that the entropy evaluated at an equilibrium configuration is a maximum, we have

$$S_{\kappa,r}(\mathrm{A} \cup \mathrm{B}) > S_{\kappa,r}((\mathrm{A} + \delta\mathrm{A}) \cup (\mathrm{B} + \delta\mathrm{B})), \tag{B3}$$

and taking into account of Eq. (A18) it follows that

$$S_{\kappa,r}(\mathrm{A})\mathcal{I}_{\kappa,r}(\mathrm{B}) + \mathcal{I}_{\kappa,r}(\mathrm{A})S_{\kappa,r}(\mathrm{B}) > S_{\kappa,r}(\mathrm{A}+\delta\mathrm{A})\mathcal{I}_{\kappa,r}(\mathrm{B}-\delta\mathrm{B}) + \mathcal{I}_{\kappa,r}(\mathrm{A}+\delta\mathrm{A})S_{\kappa,r}(\mathrm{B}-\delta\mathrm{B}). \tag{B4}$$

We expand the right-hand side of Eq. (B4) up to the second order in δE and δV, where $\delta E \equiv \delta E_{\mathrm{A}} = -\delta E_{\mathrm{B}}$ and $\delta V \equiv \delta V_{\mathrm{A}} = -\delta V_{\mathrm{B}}$. According to the MaxEnt principle, the first-order terms must vanish:

$$\left(\frac{\partial\, S_{\kappa,r}(\mathrm{A})}{\partial\, E_{\mathrm{A}}}\mathcal{I}_{\kappa,r}(\mathrm{B}) + \frac{\partial\, \mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial\, E_{\mathrm{A}}}S_{\kappa,r}(\mathrm{B}) - S_{\kappa,r}(\mathrm{A})\frac{\partial\, \mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial\, E_{\mathrm{B}}} - \mathcal{I}_{\kappa,r}(\mathrm{A})\frac{\partial\, S_{\kappa,r}(\mathrm{B})}{\partial\, E_{\mathrm{B}}}\right)\delta E + \left(\frac{\partial\, S_{\kappa,r}(\mathrm{A})}{\partial\, V_{\mathrm{A}}}\mathcal{I}_{\kappa,r}(\mathrm{B}) + \frac{\partial\, \mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial\, V_{\mathrm{A}}}S_{\kappa,r}(\mathrm{B}) - S_{\kappa,r}(\mathrm{A})\frac{\partial\, \mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial\, V_{\mathrm{B}}} - \mathcal{I}_{\kappa,r}(\mathrm{A})\frac{\partial\, S_{\kappa,r}(\mathrm{B})}{\partial\, V_{\mathrm{B}}}\right)\delta V = 0. \tag{B5}$$

By using the relations

$$\frac{\partial\, \mathcal{I}_{\kappa,r}}{\partial X} = \frac{\kappa^2 S_{\kappa,r} - r\, \mathcal{I}_{\kappa,r}}{\mathcal{I}_{\kappa,r} - r\, S_{\kappa,r}}\frac{\partial S_{\kappa,r}}{\partial X}, \tag{B6}$$

with $X \equiv E$ or $X \equiv V$, Eq. (B5) becomes

$$(\mathcal{I}_{\kappa,r} - r\, S_{\kappa,r})\Big|_{\mathrm{A}\cup\mathrm{B}}\left[\frac{1}{\mathcal{I}_{\kappa,r} - r\, S_{\kappa,r}}\left.\frac{\partial S_{\kappa,r}}{\partial E}\right|_{\mathrm{A}}\delta E + \frac{1}{\mathcal{I}_{\kappa,r} - r\, S_{\kappa,r}}\left.\frac{\partial S_{\kappa,r}}{\partial V}\right|_{\mathrm{A}}\delta V - \frac{1}{\mathcal{I}_{\kappa,r} - r\, S_{\kappa,r}}\left.\frac{\partial S_{\kappa,r}}{\partial E}\right|_{\mathrm{B}}\delta E - \frac{1}{\mathcal{I}_{\kappa,r} - r\, S_{\kappa,r}}\left.\frac{\partial S_{\kappa,r}}{\partial V}\right|_{\mathrm{B}}\delta V\right] = 0, \tag{B7}$$

where we use relations (A18) and (A19).

Taking into account that

$$\mathcal{I}_{\kappa,r}\left(\frac{1}{W}\right) - r\, S_{\kappa,r}\left(\frac{1}{W}\right) = \frac{1}{W}\frac{d}{d(1/W)}\ln_{\{\kappa,r\}}\left(\frac{1}{W}\right) > 0, \tag{B8}$$

through Eq. (B7), the two equilibrium conditions follow:

$$\frac{1}{\mathcal{I}_{\kappa,r} - r\, S_{\kappa,r}}\left.\frac{\partial S_{\kappa,r}}{\partial E}\right|_{\mathrm{A}} = \frac{1}{\mathcal{I}_{\kappa,r} - rS_{\kappa,r}}\left.\frac{\partial S_{\kappa,r}}{\partial E}\right|_{\mathrm{B}}, \tag{B9}$$

$$\frac{1}{\mathcal{I}_{\kappa,r} - rS_{\kappa,r}}\left.\frac{\partial S_{\kappa,r}}{\partial V}\right|_{\mathrm{A}} = \frac{1}{\mathcal{I}_{\kappa,r} - rS_{\kappa,r}}\left.\frac{\partial S_{\kappa,r}}{\partial V}\right|_{\mathrm{B}}, \tag{B10}$$

which coincide with Eqs. (4.4) and (4.5).

In order to obtain the thermodynamic stability conditions let us consider the second-order terms in the expansion of Eq. (B4):

$$\frac{1}{2}\left[\mathcal{I}_{\kappa,r}(\mathrm{A})\frac{\partial^2 S_{\kappa,r}(\mathrm{B})}{\partial E_{\mathrm{B}}^2} - 2\frac{\partial \mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial E_{\mathrm{A}}}\frac{S_{\kappa,r}(\mathrm{B})}{\partial E_{\mathrm{B}}} + \frac{\partial^2 \mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial E_{\mathrm{A}}^2}S_{\kappa,r}(\mathrm{B}) + \mathcal{I}_{\kappa,r}(\mathrm{B})\frac{\partial^2 S_{\kappa,r}(\mathrm{A})}{\partial E_{\mathrm{A}}^2} - 2\frac{\partial \mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial E_{\mathrm{B}}}\frac{S_{\kappa,r}(\mathrm{A})}{\partial E_{\mathrm{A}}} + \frac{\partial^2 \mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial E_{\mathrm{B}}^2}S_{\kappa,r}(\mathrm{A})\right](\delta E)^2$$
$$+ \frac{1}{2}\left[\mathcal{I}_{\kappa,r}(\mathrm{A})\frac{\partial^2 S_{\kappa,r}(\mathrm{B})}{\partial V_{\mathrm{B}}^2} - 2\frac{\partial \mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial V_{\mathrm{A}}}\frac{S_{\kappa,r}(\mathrm{B})}{\partial V_{\mathrm{B}}} + \frac{\partial^2 \mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial V_{\mathrm{A}}^2}S_{\kappa,r}(\mathrm{B}) + \mathcal{I}_{\kappa,r}(\mathrm{B})\frac{\partial^2 S_{\kappa,r}(\mathrm{A})}{\partial V_{\mathrm{A}}^2} - 2\frac{\partial \mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial V_{\mathrm{B}}}\frac{S_{\kappa,r}(\mathrm{A})}{\partial V_{\mathrm{A}}} + \frac{\partial^2 \mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial V_{\mathrm{B}}^2}S_{\kappa,r}(\mathrm{A})\right](\delta V)^2 + \left[\mathcal{I}_{\kappa,r}(\mathrm{A})\frac{\partial^2 S_{\kappa,r}(\mathrm{B})}{\partial E_{\mathrm{B}}\,\partial V_{\mathrm{B}}} - \frac{\partial \mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial V_{\mathrm{A}}}\frac{\partial S_{\kappa,r}(\mathrm{B})}{\partial E_{\mathrm{B}}} - \frac{\partial \mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial E_{\mathrm{A}}}\frac{\partial S_{\kappa,r}(\mathrm{B})}{\partial V_{\mathrm{B}}}\right.$$

$$+\frac{\partial^2\mathcal{I}_{\kappa,r}(\mathrm{A})}{\partial E_\mathrm{A}\,\partial V_\mathrm{A}}S_{\kappa,r}(\mathrm{B})+\mathcal{I}_{\kappa,r}(\mathrm{B})\frac{\partial^2 S_{\kappa,r}(\mathrm{A})}{\partial E_\mathrm{A}\,\partial V_\mathrm{A}}-\frac{\partial\mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial V_\mathrm{B}}\frac{\partial S_{\kappa,r}(\mathrm{A})}{\partial E_\mathrm{A}}-\frac{\partial\mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial E_\mathrm{B}}\frac{\partial S_{\kappa,r}(\mathrm{A})}{\partial V_\mathrm{A}}+\frac{\partial^2\mathcal{I}_{\kappa,r}(\mathrm{B})}{\partial E_\mathrm{B}\,\partial V_\mathrm{B}}S_{\kappa,r}(\mathrm{A})\Bigg]\delta E\ \delta V<0. \tag{B11}$$

By using the relation

$$\frac{\partial^2\mathcal{I}_{\kappa,r}}{\partial X\ \ \partial Y}=\frac{\kappa^2 S_{\kappa,r}-r\mathcal{I}_{\kappa,r}}{\mathcal{I}_{\kappa,r}-rS_{\kappa,r}}\frac{\partial^2 S_{\kappa,r}}{\partial X\ \ \partial Y}+(\kappa^2-r^2)\frac{\mathcal{I}_{\kappa,r}^2-\kappa^2 S_{\kappa,r}^2}{(\mathcal{I}_{\kappa,r}-rS_{\kappa,r})^3}\frac{\partial S_{\kappa,r}}{\partial X}\frac{\partial S_{\kappa,r}}{\partial Y}, \tag{B12}$$

Eq. (B11) becomes

$$\frac{1}{2}(\mathcal{I}_{\kappa,r}-rS_{\kappa,r})\Bigg|_{\mathrm{A}\cup\mathrm{B}}\left[\ \frac{\mathcal{S}_{\mathrm{EE}}(\delta E)^2+2\mathcal{S}_{\mathrm{EV}}\delta E\,\delta V+\mathcal{S}_{\mathrm{VV}}(\delta V)^2}{\mathcal{I}_{\kappa,r}-rS_{\kappa,r}}\Bigg|_\mathrm{A}+\ \frac{\mathcal{S}_{\mathrm{EE}}(\delta E)^2+2\mathcal{S}_{\mathrm{EV}}\delta E\,\delta V+\mathcal{S}_{\mathrm{VV}}(\delta V)^2}{\mathcal{I}_{\kappa,r}-rS_{\kappa,r}}\Bigg|_\mathrm{B}\right]<0, \tag{B13}$$

where we have defined

$$\mathcal{S}_{\mathrm{XY}}=\frac{\partial^2 S_{\kappa,r}}{\partial X\ \ \partial Y}-\frac{(\kappa^2+r^2)S_{\kappa,r}-2\ r\ \mathcal{I}_{\kappa,r}}{(\mathcal{I}_{\kappa,r}-r\ S_{\kappa,r})^2}\frac{\partial S_{\kappa,r}}{\partial X}\frac{\partial S_{\kappa,r}}{\partial Y}, \tag{B14}$$

and taking into account Eq. (B8), from Eq. (B13) it follows that the inequality

$$\mathcal{S}_{\mathrm{EE}}(\delta E)^2+2\ \mathcal{S}_{\mathrm{EV}}\delta E\ \ \delta V+\mathcal{S}_{\mathrm{VV}}(\delta V)^2<0 \tag{B15}$$

must holds for both systems A and B.

In Eq. (B15) (by posing $\delta V=0$ and $\delta E=0$, respectively), it follows that

$$\mathcal{S}_{\mathrm{EE}}<0,\quad \mathcal{S}_{\mathrm{VV}}<0, \tag{B16}$$

and, multiplying Eq. (B15) by $\mathcal{S}_{\mathrm{EE}}$, we obtain

$$(\mathcal{S}_{\mathrm{EE}}\delta E+\mathcal{S}_{\mathrm{EV}}\delta V)^2+(\mathcal{S}_{\mathrm{EE}}\mathcal{S}_{\mathrm{VV}}-\mathcal{S}_{\mathrm{EV}}^2)(\delta V)^2>0, \tag{B17}$$

which implies

$$\mathcal{S}_{\mathrm{EE}}\mathcal{S}_{\mathrm{VV}}-\mathcal{S}_{\mathrm{EV}}^2>0. \tag{B18}$$

Equations (B16) and (B18) are the thermodynamic stability conditions.

In particular, Eqs. (B16) can be written as

$$\frac{\partial^2 S_{\kappa,r}}{\partial X^2}<\mathcal{A}_{\kappa,r}\left(\frac{\partial S_{\kappa,r}}{\partial X}\right)^2, \tag{B19}$$

with

$$\mathcal{A}_{\kappa,r}=\frac{(\kappa^2+r^2)S_{\kappa,r}-2\ r\ \mathcal{I}_{\kappa,r}}{(\mathcal{I}_{\kappa,r}-r\ S_{\kappa,r})^2}, \tag{B20}$$

while from Eq. (B18) we obtain

$$\frac{\partial^2 S_{\kappa,r}}{\partial E^2}\frac{\partial^2 S_{\kappa,r}}{\partial V^2}-\left(\frac{\partial^2 S_{\kappa,r}}{\partial E\ \ \partial V}\right)^2>\mathcal{A}_{\kappa,r}\mathcal{B}_{\kappa,r},$$

where

$$\mathcal{B}_{\kappa,r}=\left(\frac{\partial^2 S_{\kappa,r}}{\partial E^2}\right)^{-1}\left\{\left(\frac{\partial^2 S_{\kappa,r}}{\partial E^2}\frac{\partial S_{\kappa,r}}{\partial V}-\frac{\partial^2 S_{\kappa,r}}{\partial E\ \ \partial V}\frac{\partial S_{\kappa,r}}{\partial E}\right)^2+\left(\frac{\partial S_{\kappa,r}}{\partial E}\right)^2\left[\frac{\partial^2 S_{\kappa,r}}{\partial E^2}\frac{\partial^2 S_{\kappa,r}}{\partial V^2}-\left(\frac{\partial^2 S_{\kappa,r}}{\partial E\ \ \partial V}\right)^2\right]\right\}, \tag{B21}$$

and, according to Eq. (2.8) and (2.9), it follows that $\mathcal{B}_{\kappa,r}<0$.

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