Spatial coherence resonance in excitable media

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We study the phenomenon of spatial coherence resonance in a two-dimensional model of excitable media with FitzHugh-Nagumo local dynamics. In particular, we show that there exists an optimal level of additive noise for which an inherent spatial scale of the excitable media is best pronounced. We argue that the observed phenomenon occurs due to the existence of a noise robust excursion time that is characteristic for the local dynamics whereby the diffusion constant, representing the rate of diffusive spread, determines the actual resonant spatial frequency. Additionally, biological implications of presented results in the field of neuroscience are outlined.

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I. INTRODUCTION

Contradictory to the intuitive belief, noise has crystallized to be a welcomed ingredient of both local [1] and space extended [2] nonlinear systems, often facilitating their functioning and overall effectiveness. In particular, for temporal systems it has been discovered that the constructive role of noise depends resonantly on the noise intensity, which lead to terming this phenomenon stochastic resonance [3]. More precisely, stochastic resonance stands for the resonant noisy enhancement of a system's response to weak external stimuli [4–12]. Remarkably, stochastic resonance phenomena can also be observed in the absence of any deterministic external inputs in systems with no explicit time scales [13,14]. This striking phenomenon has been termed coherence resonance [15].

In systems with spatial degrees of freedom, spatiotemporal stochastic resonance has been first reported in Ref. [16] for excitable systems, while spatial coherence resonance has been introduced in Ref. [17] for systems near pattern forming instabilities. Moreover, there exist studies reporting noise-induced spiral growth and enhancement of spatiotemporal order [18–22], noise sustained coherence of space-time clusters and self-organized criticality [23], noise enhanced and induced excitability [24,25], noise induced propagation of harmonic signals [26], as well as noise sustained and controlled synchronization [27] in space extended systems. Recently, Busch and Kaiser [28] have shown that additive noise is also able to resonantly extract the spatiotemporal order in excitable media. Little attention, however, has been devoted to the explicit analysis of characteristic spatial frequencies of nonlinear media. In addition to the work of Carrillo *et al.* [17], there exist no studies reporting a resonant enhancement of an inherent spatial frequency in space extended systems.

In the present study, we analyze spatial frequency spectra of excitable media in dependence on different levels of additive noise. By calculating the average spatial structure function, we present first evidences for spatial coherence resonance in excitable media. Note that although stochastic [29] and coherence resonance [30–33] phenomena have been extensively studied in arrays of dynamical systems, our work focuses explicitly on the spatial rather than temporal system scale. In particular, we show that there exists an optimal level of additive noise for which a particular spatial frequency of the system is best pronounced. We emphasize that no additional deterministic inputs are introduced to the system, and the latter is locally initiated from steady state singular conditions. Hence, the studied spatial structures are induced solely by additive noise and reflect an inherent spatial scale of the media.

The excitable media under study is locally modeled by the FitzHugh-Nagumo equations [34,35] that were derived from the Hodgkin-Huxley model describing the excitable dynamics of electrical signal transmission along neuron axons [36]. Since neurons are known to be noisy analog units, which only if coupled can carry out highly complex and advanced computations with cognition and reliability [37], it is evident that excitable neural tissue combines features of being both noisy and spatially extended. Therefore, it is of great interest to study effects of noise on the spatial scale of such systems. Hopefully, our work outlines some possibilities for future experimental work, especially in the field of neuroscience, where excitability and noise in space extended systems appear to be universally present.

The paper is structured as follows. Section II is devoted to the description of the mathematical model and its main "local" characteristics. In Sec. III evidences for the spatial coherence resonance are presented, while in the last section we summarize the results and outline biological implications of our findings.

II. MATHEMATICAL MODEL

We study a mathematical model of excitable media given by

$$\frac{du}{dt} = f(u,v) + D\nabla^2 u + \xi, \tag{1}$$

$$\frac{dv}{dt} = g(u,v), \tag{2}$$

which is locally described by the FitzHugh-Nagumo equations [34,35]

$$f(u,v) = \frac{1}{\varepsilon}u(1-u)\left(u - \frac{v+b}{a}\right), \quad (3)$$

$$g(u,v) = u - v. \quad (4)$$

The membrane potential $u(x,y,t)$ and time-dependent conductance of potassium channels $v(x,y,t)$ are considered as dimensionless two-dimensional scalar fields on a $n \times n$ square lattice with mesh size $\Delta x = \Delta y$, whereby the local dynamics of u is much faster ($\varepsilon \ll 1$) than that of v, whose diffusive spread is, for simplicity reasons, neglected. Moreover, ξ is additive Gaussian noise with zero mean, white in space and time, and variance σ^2 [2]. The Laplacian $D\nabla^2 u$, D being the diffusion coefficient, is integrated into the numerical scheme via a five-point finite-difference formula as described by Barkley [38] with no-flux boundary conditions. For parameter values $a=0.75$, $b=0.01$, and $\varepsilon=0.05$ the local FitzHugh-Nagumo system is governed by a Z-shaped u and a linear v nullcline as depicted in Fig. 1, where the point $u=v=0.0$ marks the only stable excitable steady state. Small perturbations of the excitable steady state evoke nontrivial spikelike behavior, which can induce various wave forms in the spatial domain of the space extended system [39]. Note also that since additive noise is introduced to the dynamics, it is possible to induce numerical instability in the model if $u > 1$ or $v < 0$ (see nullclines in Fig. 1). To ensure the model remains well behaved we use slightly modified FitzHugh-Nagumo equations as proposed in Ref. [40], which are given by

$$\tilde{f}(u,v) = \begin{cases} f(u,v), & u \leqslant 1, \\ -|f(u,v)|, & u > 1, \end{cases}$$
$$\tilde{g}(u,v) = \begin{cases} g(u,v), & v \geqslant 0, \\ |g(u,v)|, & v < 0, \end{cases} \quad (5)$$

whereby $f(u,v)$ and $g(u,v)$ are the same as in Eqs. (3) and (4). It is evident that Eq. (5) does not alter excitable properties of the system (trajectories depicted in Fig. 1 remain unchanged), but ensures that solutions of u and v remain close to the unit interval for arbitrary noise intensities [40]. In what follows, we will show that there exists an optimal level of additive noise for which a particular spatial frequency of the studied system is resonantly pronounced, thus providing evidences for spatial coherence resonance in excitable media.

III. SPATIAL COHERENCE RESONANCE

To quantify effects of various noise intensities on the spatial scale of the studied system we calculate the structure function according to the equation

$$P(k_x,k_y) = \langle H^2(k_x,k_y)\rangle / S, \quad (6)$$

where $H(k_x,k_y)$ is the spatial Fourier transform of the u field at a particular t [41], S is the area of the system, and $\langle \cdots \rangle$ is the ensemble average over noise realizations. Note that $P(k_x,k_y)$ can also be interpreted as the spatial power spectrum of the system. Figure 2 shows four spatial power spectra for various additive noise levels. It can be well observed that for small noise levels ($\sigma=0.11$, $\sigma=0.18$) the presented spectra show no particularly expressed spatial frequency, whereby a close examination of the spectrum at $\sigma=0.18$ already reveals the onset of structure formation in the system. For somewhat larger noise intensities ($\sigma=0.24$) the spectrum develops a well-expressed circularly symmetric ring, indicating the existence of a preferred spatial frequency induced by additive noise. As the noise level is further increased ($\sigma=0.48$) random fluctuations start to dominate the spatial scale and thus, similarly as by small noise levels, no preferred spatial frequency can be inferred.

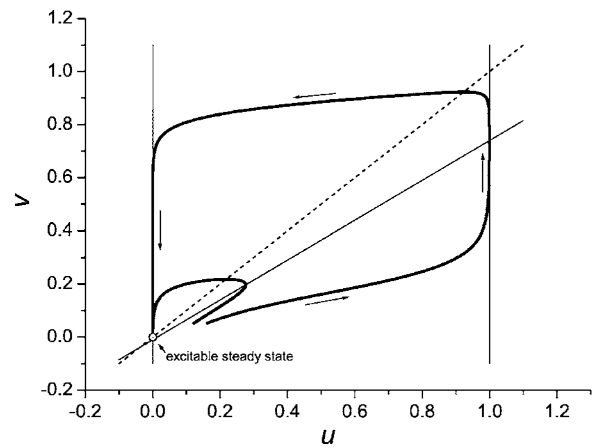

FIG. 1. Local dynamics of the FitzHugh-Nagumo system. Thin solid and dashed lines denote the u and v nullclines, respectively, while the circle marks the excitable steady state. Thick solid lines show possible excursions of the trajectory resulting from various perturbations acting upon $u=v=0.0$. Arrows indicate the direction of the flow.

To study results presented in Fig. 2 in more detail, we exploit the circular symmetry of the presented spatial power spectra as proposed in Ref. [17]. In particular, we calculate the circular average of the structure function according to the equation

$$p(k) = \int_{\Omega_k} P(\mathbf{k})\, d\Omega_k, \quad (7)$$

where $\mathbf{k}=(k_x,k_y)$, and Ω_k is a circular shell of radius $k=|\mathbf{k}|$. Figure 3(a) shows results for three different σ. It can be observed that there indeed exists a particular spatial frequency, marked with the thin solid line at $k=k_{max}$, that is resonantly enhanced for some intermediate level of additive noise. To quantify the ability of each particular noise level to extract the characteristic spatial periodicity in the system more precisely, we calculate the quantity $\delta p = p(k_{max})/\tilde{p}$, where $\tilde{p}=[p(k_{max}-\Delta k_a)+p(k_{max}+\Delta k_b)]/2$ is an approximation for the level of background fluctuations in the system, whereby $\Delta k_a=0.071$ and $\Delta k_b=0.18$ mark the estimated width of the peak around k_{max} at the optimal σ. Thus, δp measures the normalized height of the peak at k_{max} for each particular σ. This is the spatial counterpart of the measure frequently used for quantifying constructive effects of noise in the temporal domain of dynamical systems [42], whereas a similar

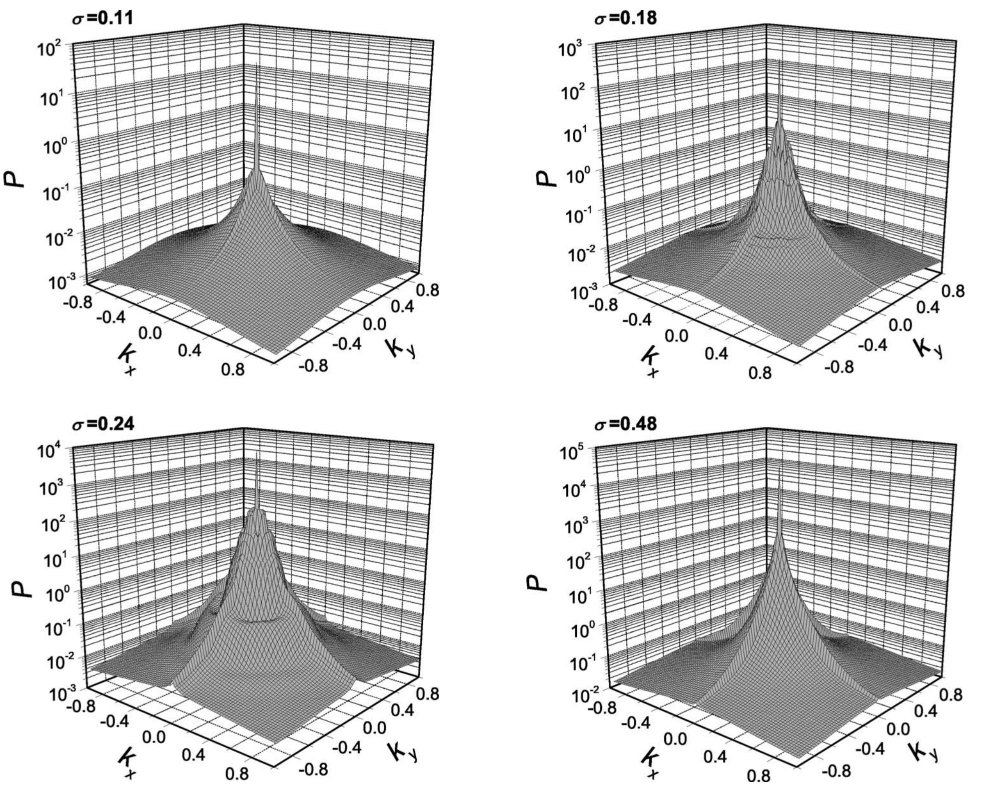

FIG. 2. Two-dimensional power spectra of the spatial profile of u for various σ. Parameter values used for the calculation where $D=0.75$, $n=128$, and $\Delta x=0.3125$, whereby the system was initiated from steady state singular conditions $u(t=0)=v(t=0)=0.0$ at all lattice sites.

measure for quantifying effects of noise on the spatial scale of space extended systems was also used in Ref. [17]. Figure 3(b) shows how δp varies with σ for three different diffusion constants D. It is evident that there always exists an optimal level of additive noise for which the peak of the circularly averaged structure function is best resolved, thus indicating the existence of spatial coherence resonance in the studied excitable media.

The existence of a preferred spatial periodicity in the studied excitable media for a certain level of additive noise can be well corroborated by studying snapshots of typical u-field configurations for various σ and D, as presented in Fig. 4. It is evident that small noise levels are unable to excite the system strong enough to evoke any particular spatial dynamics in the media. On the other hand, optimal noise levels at each particular D clearly enhance a particular spatial scale, thus providing visible evidences that corroborate results presented in Figs. 2 and 3. For large noise levels the pattern formation becomes violent so that the spatial profile again lacks any visible structure. Taken together, our results provide firm evidence for spatial coherence resonance in the studied excitable media.

In order to provide insights into mechanisms that guarantee the observed noise-induced spatial dynamics, we outline some important aspects of results presented in Figs. 3(b) and 4. First, it is evident that larger D require larger σ for the resonant enhancement of a particular spatial frequency. Also, the overall peak of δp that can be achieved with additive noise decreases with increasing D. Moreover, the middle row of Fig. 4 clearly shows that the characteristic scale of the system, i.e., the inverse of $k_{\max}$, increases with increasing D.

To explain these results, we first briefly summarize findings regarding the temporal coherence resonance in excitable

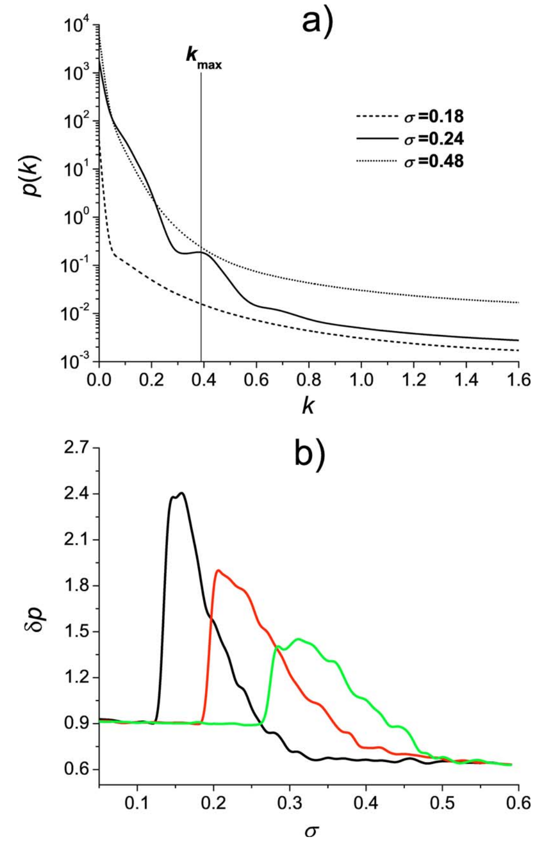

FIG. 3. (Color online.) Spatial coherence resonance in the studied excitable media. (a) Circular average of the structure function for three different σ at $D=0.75$. (b) δp in dependence σ for various diffusion constants (black $D=0.375$, red $D=0.75$, green $D=1.5$). Other parameter values are the same as in Fig. 2.

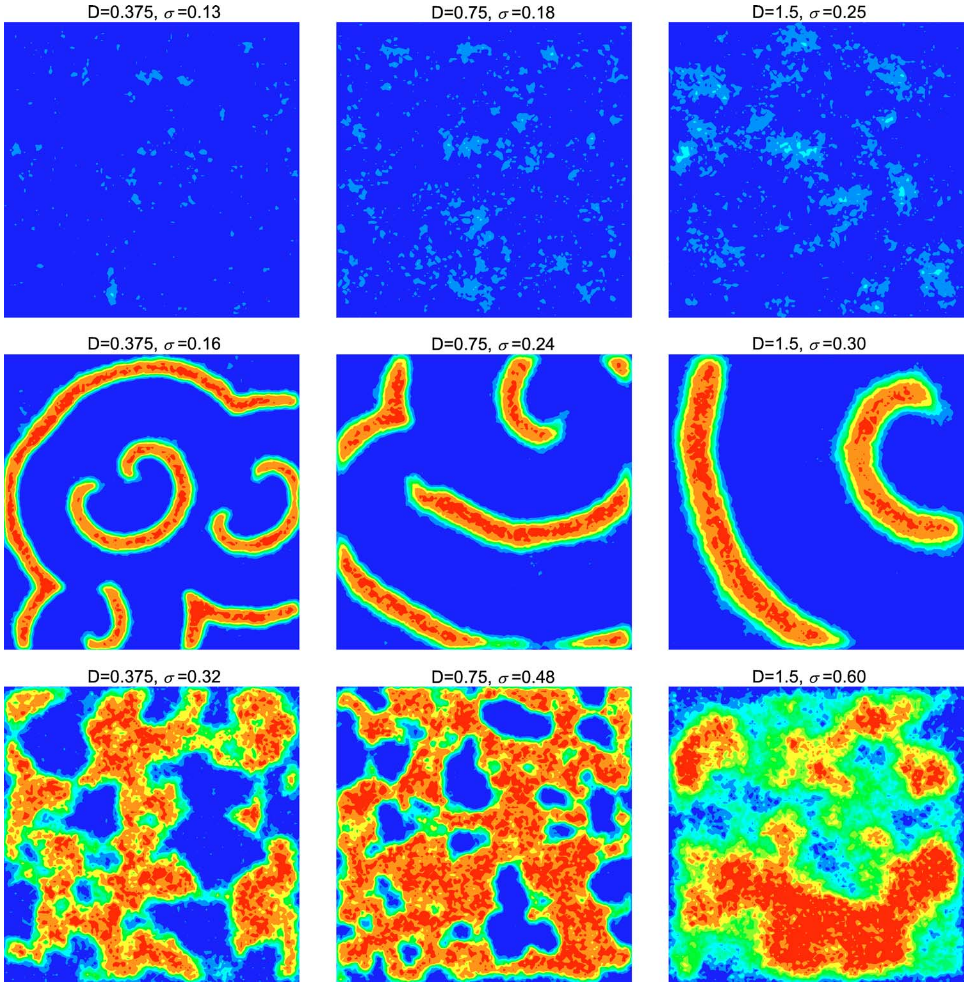

FIG. 4. (Color online.) Characteristic snapshots of the spatial profile of u for various D at small (top), near optimal (middle), and large (bottom) noise levels. Note that all figures are depicted on 128×128 square grid with a linear color profile, red marking 1.0, and blue 0.0 values of u.

systems [15]. It is known that excitable systems have a characteristic firing time t_e, termed excursion time, which is well preserved under variable noisy perturbations. Contrary, the average time between consecutive firings t_a, termed activation time, depends heavily on the level of additive noise, i.e., decreases with increasing σ. The time coherence of the system is best pronounced when the noise level is large enough so that $t_a \ll t_e$, but still small enough so that fluctuations of t_e remain moderate and thus the outline excursion phase smooth [15].

These different noise dependencies of t_e and t_a, together with the rate of diffusive spread that is proportional to $\sqrt{D}$ [43], hold also the key to understanding the spatial coherence resonance in excitable media. We argue that during t_e each particular lattice site acts like a circular (after local initialization all directions for spreading are equally probable) front initiator. After initialization the front starts to spread through the media with a rate proportional to $\sqrt{D}$. When embarking on neighboring sites the front can, depending on the level of additive noise, cause new excitation or eventually die out. In particular, if σ is large enough, i.e., t_a short enough, neighboring sites have a large probability to also become excited, which eventually nucleates a wave that propagates through the media. Analogous to the time domain, for this to happen the noise level also has to be sufficiently small so that the outline of the excursion phase remains smooth, which constitutes a nearly deterministic nucleus formation in the spatial domain and guarantees that locally initiated excitations can merge into a spatially coherent structure. Since larger D constitute faster diffusive spread, it is understandable that the characteristic spatial scale of coherent structures induced by increasing D increases (see middle row of Fig. 4 from left to right). However, since for larger D local excitations tend to die out more quickly, and larger coherent structures also require a higher rate of local excitations to propagate through the media, it is evident that shorter t_a (larger σ) are required to produce sustained waves. This explains the increasing σ that is required for the optimal response at larger D, as shown in Fig. 3(b). Furthermore, since larger noise intensities blur local excursion phases (t_e) as well, the maximal spatial coherence that can be achieved by additive noise decreases with increasing D.

Finally, it is of interest to explain the existence of a particular spatial periodicity. We argue that the characteristic noise robust excursion time t_e, combined with the diffusive spread rate proportional to $\sqrt{D}$, marks a characteristic spatial scale of the system that is indicated by the resonantly enhanced spatial wave number $k_{\max}$. Since the characteristic spatial scale is determined by the inverse of the resonantly enhanced spatial wave number, our reasoning thus predicts the dependence $k_{\max}=1/\sqrt{\tau D}$, whereby $\tau \propto t_e \approx$ constant. Figure 5 shows numerically obtained $k_{\max}$ for different D. It is

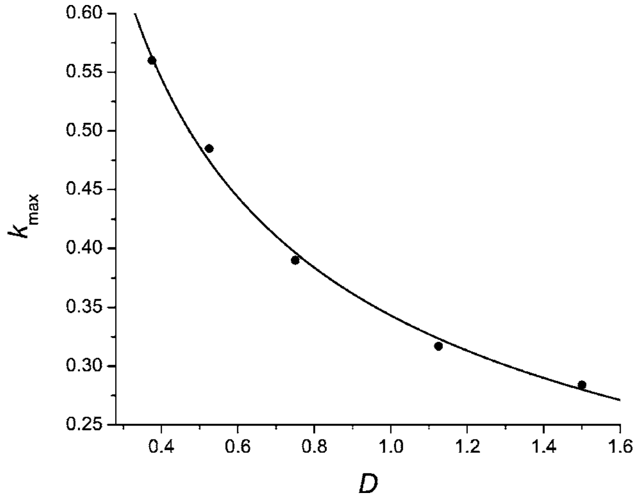

FIG. 5. Dependence of the optimal spatial wave number $k_{\max}$, corresponding to the maximum of $p(k)$ at the optimal σ, on different values of D. Dots indicate numerically obtained values, whereas the solid line indicates the predicted $k_{\max}=1/\sqrt{\tau D}$ dependence for $\tau=8.5$.

evident that obtained values are in excellent agreement with the inverse square root function, thereby validating our above explanation. Nevertheless, an open question remains how the constant τ is explicitly linked to t_e. Since a particular lattice site acts like a front initiator only when the variable u crosses a certain threshold value (not during the whole t_e), and also because other constants determining the exact rate of diffusive spread are not known, the task of explicitly linking τ and t_e is left for future studies. The main point is that the inverse square root function fits to the numerically obtained values with a *constant* τ, which reflects a noise robust t_e that is characteristic for excitable systems [15]. Together with a given D, this property of excitable systems constitutes an inherent spatial scale that can be resonantly enhanced by additive noise, thus explaining the existence of spatial coherence resonance in excitable media.

IV. SUMMARY

We show that additive Gaussian noise is able to extract a characteristic spatial scale of excitable media in a resonant manner. In particular, there exist an optimal level of additive noise for which the spatial periodicity of the system is best pronounced. Thereby, no additional deterministic inputs were introduced to the system and the latter was initiated from steady state singular conditions. Thus, the presented results offer convincing evidence for the existence of spatial coherence resonance in the studied excitable media. We argue that the observed phenomenon occurs due to existence of a noise robust excursion time that is characteristic for the local dynamics whereby the diffusion constant, representing the rate of diffusive spread, determines the actual resonant spatial frequency, which decreases with increasing D.

Since excitability is ubiquitous in all areas of science [44], the present study might also have important biological implications [45]. In the nervous system, for example, it has been discovered that excitable systems guarantee robust signal propagation through the tissue in a substantially noisy environment [46]. Moreover, studies evidencing the existence of stochastic resonance in the human brain have recently been mounting [8–10,12]. Thus, it would be very interesting to elucidate if spatial coherence resonance in the nervous system can be confirmed experimentally also. The above theoretical results indicate that such experimental findings might be feasible and indeed not attributed to serendipity.

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