Quantum scaling approach to nonequilibrium models

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Stochastic nonequilibrium exclusion models are treated using a real space scaling approach. The method exploits the mapping between nonequilibrium and quantum systems, and it is developed to accommodate conservation laws and duality symmetries, yielding exact fixed points for a variety of exclusion models. In addition, it is shown how the asymmetric simple exclusion process in one dimension can be written in terms of a classical Hamiltonian in two dimensions using a Suzuki-Trotter decomposition.

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I. INTRODUCTION

Stochastic models of lattice gas dynamics provide insight into the nonequilibrium behavior of a variety of physical processes, such as surface reactions and growth, catalysis and transport phenomena. These models are systems of many interacting particles—the dynamics of the particles are prescribed in the model definition—and the evolution is typically governed by a Master equation. They exhibit steady state phase transitions and very rich dynamics, but there exists no general framework in which to analyze nonequilibrium models. Exact treatments are scarce (see, e.g. [1–3]) and so approximate techniques are required. To this end we present a scaling treatment designed to capture universal and nonuniversal critical properties of nonequilibrium systems.

The scaling method was developed in detail in [4]. It exploits the well known equivalence between the master equation and the Schrödinger equation in imaginary time [1]—the stochastic lattice gas model is written as a quantum spin model. The scaling is achieved using a real space blocking procedure [5] in order to thin out the number of degrees of freedom. It has been applied to the contact process [6,7], where very accurate results for the critical point and certain critical exponents were obtained. Here, we show how to adapt the method to models which possess a conservation law (e.g., conserved particle number) in order to obtain exact fixed points. Further, from a stability analysis of the fixed points, we infer the role of bias in the model dynamics.

II. QUANTUM SCALING FOR EXCLUSION MODELS

In the following, we consider exclusion models—models where sites on a lattice are either occupied by a single particle or vacant. In the quantum formulation, these models are spin-1/2 quantum chains. The mapping is achieved by interpreting configurations of particles and vacancies in the nonequilibrium model as a configuration of quantum spins, where particles are replaced by an up-spin and vacancies are replaced by a down-spin, say. Since the dynamics in the nonequilibrium model become processes involving spin flips, they can be expressed in terms of a quantum Hamiltonian. Hence, for a lattice containing L sites, the configuration is written $|\{\sigma_l\}\rangle=\Pi_{l=1}^{L}\otimes|\sigma_l\rangle$, where $\{\sigma_l\}=\sigma_1,\ldots,\sigma_L$. We use the notation $\sigma_l=+_l$ to represent an up-spin at site l (i.e., a particle in the nonequilibrium system), and $\sigma_l=-_l$ to represent a down-spin (i.e., a vacancy in the nonequilibrium system). The steady state of the nonequilibrium system is equivalent to the ground state of the corresponding quantum problem.

A. Quantum renormalization group scheme

We now outline the renormalization group scheme for quantum systems [4]. The first step is to divide the lattice into adjacent blocks, each containing b sites, as indicated in Fig. 1. The lattice contains $L'=L/b$ blocks, labeled $\nu=1,\ldots,L'$, which will form the sites in the renormalized system. The Hamiltonian for the blocked system is written in terms of an intrablock part H_ν, containing all the interactions within block ν, and an interblock part $H_{\nu,\nu+1}$, containing all the interactions between block ν and block $\nu+1$ (we assume that there are nearest neighbor interactions only). Hence

$$H=\sum_{\nu=1}^{L'}[H_\nu+H_{\nu,\nu+1}]. \tag{1}$$

The renormalization is achieved as follows. We treat the intrablock Hamiltonian H_ν exactly, and regard the interblock part $H_{\nu,\nu+1}$ as a perturbation. Thus we find the eigenvectors of H_ν and use only those of lowest energy to form a truncated basis of states—in a spin-1/2 system we aim to keep the two lowest lying eigenstates. The configuration of the renormalized lattice is written as a direct product over blocks:

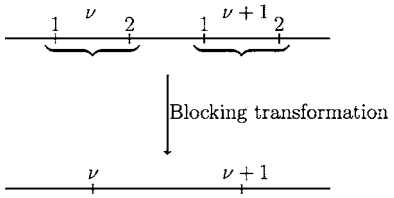

FIG. 1. Blocking of the lattice for $b=2$.

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$$|\{\sigma_\nu\}\rangle = \prod_{\nu=1}^{L'} \otimes |\sigma_\nu\rangle, \tag{2}$$

where $\{\sigma_\nu\}=\sigma_1,\ldots,\sigma_{L'}$ and the block spin states $|\sigma_\nu\rangle$ are the renormalized basis of states obtained from the lowest lying eigenstates of H_ν. Then the renormalized Hamiltonian H' is obtained by writing $H'=\Sigma_\nu[H'_\nu+H'_{\nu,\nu+1}]$. The matrix elements of H'_ν are given by

$$\langle\sigma_\nu|H'_\nu|\sigma'_\nu\rangle = \langle\sigma_\nu|H_\nu|\sigma'_\nu\rangle, \tag{3}$$

where, on the right-hand side, we use the fact that we have assigned left and right eigenvectors of H_ν to the block spin states (since these eigenvectors are orthogonal, i.e., $\langle\sigma|\sigma'\rangle = \delta_{\sigma,\sigma'}$ although H_ν is not necessarily Hermitian so $\langle\sigma|$ is not necessarily given by $|\sigma\rangle^T$, H'_ν is diagonal). Similarly, the matrix elements of $H'_{\nu,\nu+1}$ are given by

$$\langle\sigma_\nu,\sigma_{\nu+1}|H'_{\nu,\nu+1}|\sigma'_\nu,\sigma'_{\nu+1}\rangle = \langle\sigma_\nu,\sigma_{\nu+1}|H_{\nu,\nu+1}|\sigma'_\nu,\sigma'_{\nu+1}\rangle, \tag{4}$$

and thus $H'_{\nu,\nu+1}$ contributes the interaction terms in the renormalized Hamiltonian H'.

Thus we have a prescription whereby we retain only the lowest lying eigenstates of a block Hamiltonian in order to thin out the number of degrees of freedom and still retain the features important for criticality—the renormalization is carried out near the ground state of the quantum system, equivalent to the steady state of the nonequilibrium system. By assigning this truncated basis of states to a block spin variable in an appropriate way, one hopes to obtain a rescaled Hamiltonian H' of the same form as the original Hamiltonian H but with rescaled parameters.

B. Scaling for the asymmetric simple exclusion process

In this section, we apply the above transformation to the asymmetric simple exclusion process (ASEP) in one dimension with periodic boundary conditions. In the ASEP, particles hop to the right (left) with rate p (q) provided that the target site is empty. These dynamics can be expressed in terms of a quantum Hamiltonian given by

$$H = \sum_{l=1}^{L} [p(P_l^+P_{l+1}^- - \sigma_l^-\sigma_{l+1}^+) + q(P_l^-P_{l+1}^+ - \sigma_l^+\sigma_{l+1}^-)], \tag{5}$$

where $P_l^\pm = \frac{1}{2}(1\pm\sigma_l^z)$ are projection operators and σ_l^+ (σ_l^-) creates (annihilates) a particle at site l. Thus the terms $\sigma_l^-\sigma_{l+1}^+$ and $\sigma_l^+\sigma_{l+1}^-$ generate particle hopping to the right and left, respectively—they derive from the gain terms of the original master equation (i.e., the terms due to particle hopping which increase the probability of finding the system in a particular configuration). The terms involving projection operators arise due to the loss terms in the master equation (i.e., the terms which contribute to the probability that the system is in a particular configuration provided that no particle performs a hop). Therefore, probability and particle number (which is related to the z component of the spin) are both conserved. This model also possesses a particle-hole symmetry under interchange of $p\leftrightarrow q$. The aim is to maintain this duality at all stages of scaling.

We begin by dividing the lattice into blocks of size $b=2$. Then we split H into a sum of intrablock Hamiltonians H_ν given by

$$H_\nu = p(P_{\nu,1}^+P_{\nu,2}^- - \sigma_{\nu,1}^-\sigma_{\nu,2}^+) + q(P_{\nu,1}^-P_{\nu,2}^+ - \sigma_{\nu,1}^+\sigma_{\nu,2}^-), \tag{6}$$

where the suffix ν,i indicates that the operator acts at site i in block ν, and a sum of interblock Hamiltonians $H_{\nu,\nu+1}$ given by

$$H_{\nu,\nu+1} = p(P_{\nu,2}^+P_{\nu+1,1}^- - \sigma_{\nu,2}^-\sigma_{\nu+1,1}^+) + q(P_{\nu,2}^-P_{\nu+1,1}^+ - \sigma_{\nu,2}^+\sigma_{\nu+1,1}^-), \tag{7}$$

such that both H_ν and $H_{\nu,\nu+1}$ possess the duality under interchange of $p\leftrightarrow q$ when $+\leftrightarrow-$.

The next step is to find the lowest lying eigenstates of H_ν and use them to form the renormalized basis of spin states. The ground state of H_ν is threefold degenerate—since the dynamics conserve particle number H_ν decomposes into $b+1$ disconnected sectors. This particle conservation can be respected in the renormalized Hamiltonian if it is present in the renormalized basis of states, therefore the ground eigenstates of H_ν are organized according to their eigenvalue of the block spin operator $(1/b)\Sigma_{i=1}^b\sigma_{\nu,i}^z$. This leads us to define block spin states

$$|+1_\nu\rangle = |+_{\nu,1},+_{\nu,2}\rangle, \tag{8}$$

$$|0_\nu\rangle = \frac{1}{p+q}[q|+_{\nu,1},-_{\nu,2}\rangle + p|-_{\nu,1},+_{\nu,2}\rangle], \tag{9}$$

$$|-1_\nu\rangle = |-_{\nu,1},-_{\nu,2}\rangle, \tag{10}$$

for each block ν. The corresponding left eigenstates of H_ν are assigned to the left block spin states

$$\langle+1_\nu| = \langle+_{\nu,1},+_{\nu,2}|, \tag{11}$$

$$\langle0_\nu| = \langle+_{\nu,1},-_{\nu,2}| + \langle-_{\nu,1},+_{\nu,2}|, \tag{12}$$

$$\langle-1_\nu| = \langle-_{\nu,1},-_{\nu,2}|. \tag{13}$$

The left ground eigenstates of a quantum Hamiltonian describing a nonequilibrium process are always given by such sums over vectors (where the coefficient of each vector is equal to one) due to conservation of probability. By defining the block states in this way, we aim to maintain particle conservation (which we cannot maintain by arranging these states in two linear superpositions forming a spin-1/2 basis) and the particle-hole duality of the model. Thus the renormalized Hamiltonian will describe a spin 1 quantum chain.

We are now able to calculate the matrix elements of H'_ν and $H'_{\nu,\nu+1}$. First, we note that because our basis states are degenerate eigenstates of H_ν, the contribution due to H'_ν is a constant and, moreover, because the ground state eigenvalue is zero (which is always the case for the ground state eigenvalue of quantum systems representing nonequilibrium models) this constant is zero. The task then is to evaluate the matrix elements of $H'_{\nu,\nu+1}$, as prescribed by Eq. (4). For example, the term in $H_{\nu,\nu+1}$ given by $\sigma_{\nu,2}^-\sigma_{\nu+1,1}^+$ contributes the matrix elements

$$\langle \sigma_\nu, \sigma_{\nu+1}|H'_{\nu,\nu+1}|\sigma'_\nu, \sigma'_{\nu+1}\rangle = \langle \sigma_\nu|\sigma^-_{\nu,2}|\sigma'_\nu\rangle\langle \sigma_{\nu+1}|\sigma^+_{\nu+1,1}|\sigma'_{\nu+1}\rangle. \tag{14}$$

Thus the operator $\sigma^-_{\nu,2}$ is replaced by a renormalized operator $\sigma^{-\prime}_{\nu,2}$ given by

$$\sigma^{-\prime}_{\nu,2} = \begin{pmatrix} \langle +1_\nu|\sigma^-_{\nu,2}|+1_\nu\rangle & \langle +1_\nu|\sigma^-_{\nu,2}|0_\nu\rangle & \langle +1_\nu|\sigma^-_{\nu,2}|-1_\nu\rangle \\ \langle 0_\nu|\sigma^-_{\nu,2}|+1_\nu\rangle & \langle 0_\nu|\sigma^-_{\nu,2}|0_\nu\rangle & \langle 0_\nu|\sigma^-_{\nu,2}|-1_\nu\rangle \\ \langle -1_\nu|\sigma^-_{\nu,2}|+1_\nu\rangle & \langle -1_\nu|\sigma^-_{\nu,2}|0_\nu\rangle & \langle -1_\nu|\sigma^-_{\nu,2}|-1_\nu\rangle \end{pmatrix}_\nu,$$

where the ν suffix on the matrix represents an operator acting on the truncated basis of states in block ν. Evaluating such matrix elements for all the operators appearing in $H_{\nu,\nu+1}$ yields

$$P^{+\prime}_{\nu,2} = P^{+1}_\nu + \frac{p}{p+q}P^0_\nu, \quad P^{+\prime}_{\nu+1,1} = P^{+1}_{\nu+1} + \frac{q}{p+q}P^0_{\nu+1},$$

$$P^{-\prime}_{\nu,2} = \frac{q}{p+q}P^0_\nu + P^{-1}_\nu, \quad P^{-\prime}_{\nu+1,1} = \frac{p}{p+q}P^0_{\nu+1} + P^{-1}_{\nu+1},$$

$$\sigma^{+\prime}_{\nu,2} = \frac{q}{p+q}\sigma^{0,+1}_\nu + \sigma^{-1,0}_\nu, \quad \sigma^{+\prime}_{\nu+1,1} = \frac{p}{p+q}\sigma^{0,+1}_{\nu+1} + \sigma^{-1,0}_{\nu+1},$$

$$\sigma^{-\prime}_{\nu,2} = \sigma^{+1,0}_\nu + \frac{p}{p+q}\sigma^{0,-1}_\nu, \quad \sigma^{-\prime}_{\nu+1,1} = \sigma^{+1,0}_{\nu+1} + \frac{q}{p+q}\sigma^{0,-1}_{\nu+1},$$

where P^i_ν projects into the block spin state i and $\sigma^{i,j}_\nu$ raises or lowers the spin from block spin state i to block spin state j. These expressions are substituted into $H_{\nu,\nu+1}$ leading to a renormalized Hamiltonian given by $H' = \Sigma_\nu H_{\nu,\nu+1}$.

Thus the renormalized Hamiltonian describes a three-state stochastic process (probability is still conserved) where the z component of the spin is still conserved. In order to obtain scaling equations for the rates p and q this Hamiltonian has to be projected onto a basis of spin-1/2 states, spin up or down corresponding to a rescaled particle or vacancy. To do this, we note that raising (lowering) operators in the spin-1/2 system are written only in terms of operators which raise (lower) the spin in the spin-1 system. Further, each of the dynamical processes, generated by the raising and lowering operators in the spin-1 system, can be identified with a corresponding term involving projection operators due to no transition. Hence we rewrite

$$P^{+\prime}_{\nu,2} = \left(1 + \frac{p}{p+q}\right)P^+_\nu, \quad P^{+\prime}_{\nu+1,1} = \left(1 + \frac{q}{p+q}\right)P^+_{\nu+1},$$

$$P^{-\prime}_{\nu,2} = \left(1 + \frac{q}{p+q}\right)P^-_\nu, \quad P^{-\prime}_{\nu+1,1} = \left(1 + \frac{p}{p+q}\right)P^-_{\nu+1},$$

$$\sigma^{+\prime}_{\nu,2} = \left(1 + \frac{q}{p+q}\right)\sigma^+_\nu, \quad \sigma^{+\prime}_{\nu+1,1} = \left(1 + \frac{p}{p+q}\right)\sigma^+_{\nu+1},$$

$$\sigma^{-\prime}_{\nu,2} = \left(1 + \frac{p}{p+q}\right)\sigma^-_\nu, \quad \sigma^{-\prime}_{\nu+1,1} = \left(1 + \frac{q}{p+q}\right)\sigma^-_{\nu+1}.$$

At the level of the stochastic processes, this rewriting represents the following approximation: In the three-state system, the dynamics are those of a biased exclusion process where the exclusion interaction prohibits more than two particles to occupy the same site. These particles still represent the original particles of the unscaled model. In a coarse-grained sense, a hop to the right of an original particle in the three state system is interpreted as a hop to the right of a rescaled particle in the two-state (spin-1/2) exclusion process. We expect that this approximation is adequate to capture the scaling of the bias, p/q. Using this approximation, the renormalized Hamiltonian assumes the same form as the original:

$$H' = \sum_\nu [p'(P^+_\nu P^-_{\nu+1} - \sigma^-_\nu\sigma^+_{\nu+1}) + q'(P^-_\nu P^+_{\nu+1} - \sigma^+_\nu\sigma^-_{\nu+1})], \tag{15}$$

where the rescaled rates p' and q' are given by

$$p' = p\left(1 + \frac{p}{p+q}\right)^2, \quad q' = q\left(1 + \frac{q}{p+q}\right)^2. \tag{16}$$

In order to exploit this renormalization, we consider the ratio $\gamma = p/q$. Stable fixed points for γ are found at $\gamma^* = 0$ and ∞, and these are separated by an unstable fixed point at $\gamma^* = 1$. Hence symmetric diffusion is unstable with respect to bias. At the symmetric fixed point the Hamiltonian (5) describes a spin-1/2 Heisenberg chain whose dynamics are governed by a dynamic exponent $z=2$. This behavior therefore is unstable and in the presence of any bias the dynamics are described by a new exponent. A Bethe ansatz calculation for $q=0$ shows that this exponent is $z=3/2$ [8]. That we find the exact value for the unstable fixed point is a consequence of our preservation of duality at all stages in the blocking.

C. Scaling for the pair evaporation and deposition model

Another model possessing a duality is a process whereby pairs of particles evaporate from adjacent lattice sites with a rate ϵ or are deposed onto adjacent vacancies with a rate δ. The quantum Hamiltonian representing these processes takes the form

$$H = \sum_l [\delta(P^-_l P^-_{l+1} - \sigma^+_l\sigma^+_{l+1}) + \epsilon(P^+_l P^+_{l+1} - \sigma^-_l\sigma^-_{l+1})]. \tag{17}$$

The duality in this model is again a particle-hole symmetry under the interchange of $\delta \leftrightarrow \epsilon$. There is also a conservation law similar to the particle conservation in the ASEP: if we label the sublattice of odd (even) sites by A (B), then the density of particles on sublattice A (B) is ρ_A (ρ_B). The conserved quantity is the difference in the sublattice densities, $\rho_A - \rho_B$.

The duality of this Hamiltonian is preserved if we divide it into a sum of intrablock parts, each given by

$$H_\nu = \delta(P^-_{\nu,1}P^-_{\nu,2} - \sigma^+_{\nu,1}\sigma^+_{\nu,2}) + \epsilon(P^+_{\nu,1}P^+_{\nu,2} - \sigma^-_{\nu,1}\sigma^-_{\nu,2}), \tag{18}$$

and a sum of interblock parts, each given by

$$H_{\nu,\nu+1} = \delta(P^-_{\nu,2}P^-_{\nu+1,1} - \sigma^+_{\nu,2}\sigma^+_{\nu+1,1}) + \epsilon(P^+_{\nu,2}P^+_{\nu+1,1} - \sigma^-_{\nu,2}\sigma^-_{\nu+1,1}). \quad (19)$$

Again, the ground state of H_ν is threefold degenerate with eigenvectors given by

$$|1_\nu\rangle = |-_{\nu,1}, +_{\nu,2}\rangle, \quad (20)$$

$$|2_\nu\rangle = \frac{1}{\delta+\epsilon}[\delta|+_{\nu,1}, +_{\nu,2}\rangle + \epsilon|-_{\nu,1}, -_{\nu,2}\rangle], \quad (21)$$

$$|3_\nu\rangle = |+_{\nu,1}, -_{\nu,2}\rangle. \quad (22)$$

We can maintain the conservation of the sublattice densities with respect to both the intrablock *and* the interblock Hamiltonians if we assign spin-1 block states in the following way:

$$|+1_\nu\rangle = |1_\nu\rangle, \quad |+1_{\nu+1}\rangle = |3_{\nu+1}\rangle,$$

$$|0_\nu\rangle = |2_\nu\rangle, \quad |0_{\nu+1}\rangle = |2_{\nu+1}\rangle,$$

$$|-1_\nu\rangle = |3_\nu\rangle, \quad |-1_{\nu+1}\rangle = |1_{\nu+1}\rangle. \quad (23)$$

Using these states to calculate $H'_{\nu,\nu+1}$ (H'_ν is zero as before), and then forcing the resulting spin-1 Hamiltonian back into a spin-1/2 basis in the same way as was done in the previous section, one obtains a renormalized Hamiltonian of the same form as (17) with rescaled rates δ' and ϵ' given by

$$\delta' = \delta\left(1 + \frac{\epsilon}{\delta+\epsilon}\right)^2, \quad \epsilon' = \epsilon\left(1 + \frac{\delta}{\delta+\epsilon}\right)^2. \quad (24)$$

Now, in terms of the ratio $\gamma \equiv \delta/\epsilon$, we again find three fixed points at $\gamma^*=0$, 1 and ∞ but now the symmetric fixed point $\gamma^*=1$ is stable. Therefore we find no dynamic transition in this model—the dynamics are independent of bias. Again, the Hamiltonian (17) of the symmetric problem is given by that of the spin-1/2 Heisenberg chain. Thus the dynamic exponent is $z=2$ in the biased and unbiased cases. This result is consistent with a perturbation analysis which predicts that the relaxation is diffusive independent of bias, and it is supported by numerical simulation [9].

D. Scaling for the pair evaporation and deposition process with diffusion in one dimension

In this section, we apply the renormalization group transformation to a model incorporating both the dynamics of the ASEP and of the pair evaporation and deposition process. These dynamics, on a chain with periodic boundary conditions, are represented by a quantum Hamiltonian given by

$$H = \sum_l [\delta(P^-_l P^-_{l+1} - \sigma^+_l\sigma^+_{l+1}) + \epsilon(P^+_l P^+_{l+1} - \sigma^-_l\sigma^-_{l+1}) + p(P^+_l P^-_{l+1} - \sigma^-_l\sigma^+_{l+1}) + q(P^-_l P^+_{l+1} - \sigma^+_l\sigma^-_{l+1})]. \quad (25)$$

Exact results have been obtained when $\delta=\epsilon$ and $p=q$ in which case the model is equivalent to a spin-1/2 *XXZ* ferromagnet, and also for $\delta+\epsilon=p+q$ which is the condition that the evolution operator can be written as a free-fermion Hamiltonian [2,9].

A blocking, with a dilation factor $b=2$, is implemented by dividing H into a sum of intrablock parts H_ν given by

$$H_\nu = \delta(P^-_{\nu,1}P^-_{\nu,2} - \sigma^+_{\nu,1}\sigma^+_{\nu,2}) + \epsilon(P^+_{\nu,1}P^+_{\nu,2} - \sigma^-_{\nu,1}\sigma^-_{\nu,2}) + p(P^+_{\nu,1}P^-_{\nu,2} - \sigma^-_{\nu,1}\sigma^+_{\nu,2}) + q(P^-_{\nu,1}P^+_{\nu,2} - \sigma^+_{\nu,1}\sigma^-_{\nu,2}), \quad (26)$$

and a sum of interblock parts given by

$$H_{\nu,\nu+1} = \delta(P^-_{\nu,2}P^-_{\nu+1,1} - \sigma^+_{\nu,2}\sigma^+_{\nu+1,1}) + \epsilon(P^+_{\nu,2}P^+_{\nu+1,1} - \sigma^-_{\nu,2}\sigma^-_{\nu+1,1}) + p(P^+_{\nu,2}P^-_{\nu+1,1} - \sigma^-_{\nu,2}\sigma^+_{\nu+1,1}) + q(P^-_{\nu,2}P^+_{\nu+1,1} - \sigma^+_{\nu,2}\sigma^-_{\nu+1,1}). \quad (27)$$

The blocking proceeds as in the previous two sections. The ground state of H_ν is twofold degenerate with eigenvectors denoted

$$|1_\nu\rangle = \frac{1}{\delta+\epsilon}[\delta|+_{\nu,1}+_{\nu,2}\rangle + \epsilon|-_{\nu,2}-_{\nu,2}\rangle], \quad (28)$$

$$|2_\nu\rangle = \frac{1}{p+q}[q|+_{\nu,1}-_{\nu,2}\rangle + p|-_{\nu,1}+_{\nu,2}\rangle]. \quad (29)$$

Effective spin states for the block ν are identified by taking the block up-spin $|+_\nu\rangle = |1_\nu\rangle$ and the block down-spin $|-_\nu\rangle = |2_\nu\rangle$ for all ν. Thus the block spin states observe two symmetries of the model: the particle-hole symmetry under interchange of rates $\delta \leftrightarrow \epsilon$ and $p \leftrightarrow q$, and a symmetry whereby the spins on the even sublattice (say) of sites are flipped and the pair evaporation and deposition processes are transformed into the hopping processes and vice versa. This assignment of block spin states yields a renormalized Hamiltonian H' of the same form as (25), but with rescaled rates δ', ϵ', p' and q' given by

$$\delta' = \frac{p^3+q^3+pq(\delta+\epsilon)}{(p+q)^2}, \quad (30)$$

$$\epsilon' = \frac{\delta\epsilon(\delta+\epsilon+p+q)}{(\delta+\epsilon)^2}, \quad (31)$$

$$p' = \frac{\delta\epsilon(p+q)+\delta p^2+\epsilon q^2}{(\delta+\epsilon)(p+q)}, \quad (32)$$

$$q' = \frac{\delta\epsilon(p+q)+\epsilon p^2+\delta q^2}{(\delta+\epsilon)(p+q)}. \quad (33)$$

A flow diagram is obtained where the symmetric fixed point (i.e., where all rates are equal) is fully stable. This suggests that the dynamic transition in the ASEP is removed when the pair evaporation and deposition processes are included—the dynamics in this model are described by the exponent $z=2$ for all choices of δ, ϵ, p and q (provided δ and ϵ are not both equal to zero). Also, all the fixed points satisfy the free-fermion condition $\delta+\epsilon=p+q$ for which exact results are available [2,9]. Moreover, this condition is stable: after iterating the transformation an infinite number of times the rescaled rates always satisfy the free-fermion condition, regardless of the original choice of rates.

As an illustration, and test of the results of the scaling, consider the case where $\epsilon=\epsilon'$ and $p=q$. This case is not free-fermion—it is equivalent to an *XXZ* ferromagnet, the spectrum of which has a gap and so the model exhibits exponential relaxation of the density for example. The scaling equations take this model to the equal-rate case, equivalent to an Ising ferromagnet, which also has a gap. This is consistent with the exponential relaxation of the unscaled model. This particular case has been discussed and simulated in detail in [9].

III. TROTTER DECOMPOSITION FOR THE ASYMMETRIC SIMPLE EXCLUSION PROCESS

One difficulty that arises when constructing renormalization group transformations for quantum systems is associated with the noncommutation of operators appearing in the quantum Hamiltonian. In this section, we remove this problem by exploiting the Suzuki-Trotter decomposition [10–13] to rewrite the quantum Hamiltonian representing the ASEP as a classical Hamiltonian for Ising spin variables in two dimensions. As a by-product, direct contact is made between stochastic nonequilibrium models in one dimension and vertex models in two dimensions [14].

The general scheme for the mapping is to split the Hamiltonian into a set of operators $\{H_i\}$ such that

$$H=\sum_{i=0}^{j} H_i, \tag{34}$$

where each term in H_i commutes with every other (or, at least, where any noncommutation can be easily dealt with), but $[H_l,H_m]\neq 0$ for $l\neq m$. The exponential of a sum of operators is then expanded as a product of exponentials by dealing with the noncommutation through the Trotter formula [10]

$$e^{\sum_{i=0}^{j} H_i}=\lim_{n\to\infty}[e^{H_0/n}\cdots e^{H_j/n}]^n. \tag{35}$$

For the ASEP, we begin with the quantum Hamiltonian (5), and divide it up in the following way:

$$H_0=\sum_{l\ \mathrm{odd}}[p\ P_l^+P_{l+1}^- + q\ P_l^-P_{l+1}^+], \tag{36}$$

$$H_1=-\sum_{l\ \mathrm{odd}}[p\ \sigma_l^-\sigma_{l+1}^+ + q\ \sigma_l^+\sigma_{l+1}^-], \tag{37}$$

$$H_2=\sum_{l\ \mathrm{even}}[p\ P_l^+P_{l+1}^- + q\ P_l^-P_{l+1}^+], \tag{38}$$

$$H_3=-\sum_{l\ \mathrm{even}}[p\ \sigma_l^-\sigma_{l+1}^+ + q\ \sigma_l^+\sigma_{l+1}^-]. \tag{39}$$

Ultimately, this choice must be made to reflect the update mechanism—here we consider a parallel sublattice update. According to Eq. (35) the partition function for the quantum system is written

$$Z=\lim_{n\to\infty} Z_{(n)}, \tag{40}$$

where we have defined $Z_{(n)}$ by

$$Z_{(n)}\equiv \mathrm{Tr}[e^{-\beta H_0/n}e^{-\beta H_1/n}e^{-\beta H_2/n}e^{-\beta H_3/n}]^n. \tag{41}$$

The next step is to insert complete sets of basis states into (41). We choose the basis $|s\rangle=|\sigma_1,\sigma_2,\ldots,\sigma_L\rangle$, where σ_i is the eigenvalue of the Pauli matrix σ_i^z; a basis which diagonalizes H_0 and H_2. Hence $Z_{(n)}$ is expressed as follows:

$$\begin{aligned} Z_{(n)} &= \sum_{\{s_i\}} \langle s_1|e^{-\beta H_0/n}e^{-\beta H_1/n}|s_2\rangle \\ &\quad\times\langle s_2|e^{-\beta H_2/n}e^{-\beta H_3/n}|s_3\rangle\cdots\langle s_{2n}|e^{-\beta H_2/n}e^{-\beta H_3/n}|s_1\rangle \\ &= \sum_{\{s_i\}}\left[\prod_{\tau\ \mathrm{odd}}\langle s_\tau|e^{-\beta H_0/n}e^{-\beta H_1/n}|s_{\tau+1}\rangle\right] \\ &\quad\times\left[\prod_{\tau\ \mathrm{even}}\langle s_\tau|e^{-\beta H_2/n}e^{-\beta H_3/n}|s_{\tau+1}\rangle\right]. \end{aligned} \tag{42}$$

This insertion of basis states also reflects the update mechanism of the original particle process. Now we interpret the τ label as a label for a new spatial axis, which we will refer to as the Trotter axis (see Fig. 3). This axis has its origin in the time axis of the quantum representation of the master equation. Usually, the Trotter axis is an imaginary time dimension, but since our quantum problem is governed by a Schrödinger equation in imaginary time, here the Trotter axis represents a real time evolution axis.

For simplicity, we shall consider the fully ASEP: $q=0$. Our task now is to evaluate the matrix elements appearing in (42). Since we chose our basis states to diagonalize H_0, the contribution from its exponential factor is trivially evaluated. The contribution from the off-diagonal part, H_1, proceeds as follows:

$$\begin{aligned} &\prod_{\tau\ \mathrm{odd}}\langle s_\tau|\exp\left[\sum_{l\ \mathrm{odd}}\left(\frac{\beta p}{n}\sigma_l^-\sigma_{l+1}^+\right)\right]|s_{\tau+1}\rangle \\ &\quad=\lim_{\Delta\to\infty}\exp\left(-\sum_{l,\tau\ \mathrm{odd}} h_{l,\tau}\right), \end{aligned} \tag{43}$$

where $h_{l,\tau}$ is given by

$$h_{l,\tau}=\Delta\Phi_{l,\tau}-\ln\left(\frac{\beta p}{n}\right)P_{l,\tau}^+P_{l,\tau+1}^-P_{l+1,\tau}^-P_{l+1,\tau+1}^+, \tag{44}$$

with $P_{l,\tau}^{\pm}=\frac{1}{2}(1\pm\sigma_{l,\tau})$ and $\sigma_{l,\tau}$ is an Ising spin variable associated with the site (l,τ), and where

$$\begin{aligned} \Phi_{l,\tau} &= P_{l,\tau}^+P_{l,\tau+1}^+P_{l+1,\tau}^+P_{l+1,\tau}^- + P_{l,\tau}^+P_{l,\tau+1}^+P_{l+1,\tau}^-P_{l+1,\tau}^+P_{l+1,\tau+1}^+ \\ &\quad + P_{l,\tau}^+P_{l,\tau+1}^-P_{l+1,\tau}^+P_{l+1,\tau+1}^+ + P_{l,\tau}^-P_{l,\tau+1}^+P_{l+1,\tau}^+P_{l+1,\tau+1}^+ \\ &\quad + P_{l,\tau}^+P_{l,\tau+1}^-P_{l+1,\tau}^-P_{l+1,\tau+1}^- + P_{l,\tau}^-P_{l,\tau+1}^+P_{l+1,\tau}^-P_{l+1,\tau+1}^- \\ &\quad + P_{l,\tau}^-P_{l,\tau+1}^-P_{l+1,\tau}^+P_{l+1,\tau+1}^- + P_{l,\tau}^-P_{l,\tau+1}^-P_{l+1,\tau}^-P_{l+1,\tau+1}^+ \\ &\quad + P_{l,\tau}^+P_{l,\tau+1}^-P_{l+1,\tau}^+P_{l+1,\tau+1}^- + P_{l,\tau}^-P_{l,\tau+1}^+P_{l+1,\tau}^-P_{l+1,\tau+1}^+ \\ &\quad + P_{l,\tau}^-P_{l,\tau+1}^+P_{l+1,\tau}^+P_{l+1,\tau+1}^-. \end{aligned} \tag{45}$$

The parameter Δ has been introduced in order to project away unwanted configurations, i.e., those which do not rep-

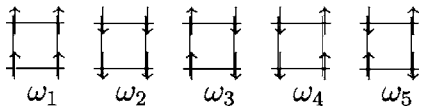

FIG. 2. The allowed plaquette configurations with their weights.

resent processes allowed under the original stochastic dynamics; for instance, the final term in Eq. (45) corresponds to a particle hopping to the left, which is forbidden for $q=0$.

To obtain the full effective Hamiltonian for the two-dimensional classical system, $Z_{(n)}=\lim_{\Delta\to\infty}\mathrm{Tr}\, e^{-H^{(eff)}}$, we must include the contributions from the the Hamiltonians H_0, H_2, and H_3. Then the effective Hamiltonian can be written

$$H^{(eff)} = \sum_{l,\tau \text{ odd}} h_{l,\tau}^{(eff)} + \sum_{l,\tau \text{ even}} h_{l,\tau}^{(eff)}, \tag{46}$$

with

$$h_{l,\tau}^{(eff)} = \Delta\Phi_{l,\tau} - \ln\left(\frac{\beta p}{n}\right) P_{l,\tau}^{+} P_{l,\tau+1}^{-} P_{l+1,\tau}^{-} P_{l+1,\tau+1}^{+} - \ln\left(1-\frac{\beta p}{n}\right) P_{l,\tau}^{+} P_{l,\tau+1}^{+} P_{l+1,\tau}^{-} P_{l+1,\tau+1}^{-}, \tag{47}$$

where the final term here corresponds to the "stay-put" probability that a particle, with a vacancy to its right, does not perform a hop.

The quantum Hamiltonian (5) has now been rewritten as a classical Hamiltonian (46) for an Ising system in two dimensions. The nonequilibrium steady state of the original model is characterized by the zero temperature behavior of the quantum model. This limit is recovered in the Ising system by taking the extent of the Trotter axis to be infinite. The temperature in the quantum system is not the same as the temperature in the classical Ising system. Instead, the inverse temperature of the quantum model translates into the extent of the classical system in the Trotter direction. Thus the critical behavior in the quantum ground state is expressed through the critical behavior of the finite temperature Ising system in equilibrium. Zero temperature transitions in the quantum model, which occur as a function of the couplings, are caused in the infinite Ising system by varying the temperature.

A. "Plaquette" structure of Ising Hamiltonian

The Ising Hamiltonian $H^{(eff)}$ representing the ASEP contains a parameter Δ which is infinite. This constraint can be incorporated in a compact fashion into a vertex model description. In the representation we have chosen, $H^{(eff)}$ contains only four-spin interactions. These are shown in Fig. 2 as the five possible arrangements of spins around a plaquette.

The weights ω_1 to ω_5 of the allowed vertices are

$$\omega_1 = \omega_2 = \omega_4 = 1, \quad \omega_3 = 1 - \frac{\beta p}{n}, \quad \omega_5 = \frac{\beta p}{n}. \tag{48}$$

All other vertices have zero weight. A lattice configuration is then specified by placing the allowed plaquette configurations on the shaded squares of the lattice shown in Fig. 3.

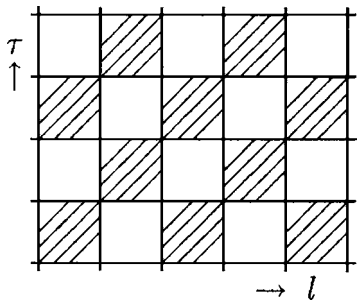

FIG. 3. Lattice on which the $2d$ classical Ising model is defined. The original spatial axis is labeled by l and τ labels the Trotter axis. The configurations around the shaded squares are determined by the allowed plaquette configurations shown in Fig. 2.

The open plaquettes in this diagram play the role of passive plaquettes, whose configurations are determined only by the surrounding configurations shown in Fig. 2. Every configuration is allowed for an open plaquette and each occurs with weight 1. The partition function for this system is

$$Z = \sum_{\substack{\text{allowed} \\ \text{configurations}}} \prod_{\text{plaquettes } i} \omega_i. \tag{49}$$

As mentioned previously, the division of the quantum Hamiltonian into sums over commuting operators must be chosen with a specific update mechanism in mind. From Fig. 3, we see that the division (36)–(39) combined with the insertion of basis states in Eq. (42) describes a parallel sublattice update mechanism—the shaded plaquettes in a row of the diagram represent the bonds chosen for an update in one time step. Alternatively, we may have chosen to describe sequential update. This is achieved by dividing the quantum Hamiltonian into L pairs of local bond Hamiltonians. One of each pair contains only diagonal terms, the other contains only nondiagonal terms, and there is one pair for every bond in the system. Then, inserting basis states between each pair of local Hamiltonians leads to a sequential update mechanism.

We also note that in the chosen basis, we can write a transfer matrix for the evolution of configurations along the Trotter axis. Its matrix elements are identical to those appearing in the matrix representation of the discrete time master equation (for a specified update mechanism) with a hopping rate given by $\beta p/n$.

B. Scaling

By rewriting the quantum Hamiltonian as a classical Hamiltonian, the problem of noncommutation of operators in the quantum Hamiltonian has been removed and the classical Hamiltonian provides Boltzmann weights for physical processes (in the sense that each plaquette configuration has a direct physical interpretation in terms of the original stochastic dynamics). The restricted number of plaquette configurations and the way they are placed on the lattice suggest that we can devise direct and straightforward scaling procedures. Configurations (and their weights) are built up by piecing plaquettes together in an allowed way. Then one can coarse-grain by matching configurations under a change of length

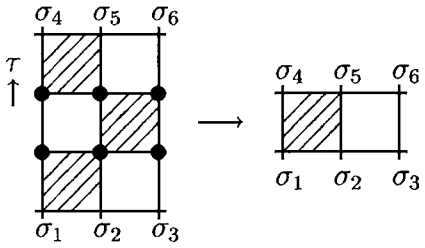

FIG. 4. Decimation of the Trotter lattice along the time axis. Sites labeled by a circle are traced over.

scale. This dilation of scale may be applied to either the time axis or the spatial axis individually, or to both at once. One can also explore how changing the update mechanism effects the scaling.

To illustrate these ideas, consider a putative decimation eliminating sites along the time axis for a parallel sublattice update mechanism. This is shown, for a particular matching of active and inactive plaquettes, in Fig. 4. The weight for the unrenormalized system is obtained by tracing over all allowed plaquettes consistent with a given configuration $\{\sigma_i\}$. A scaling equation may be obtained, for example, by matching this weight with the renormalized weight for a plaquette with spins given by $\sigma_1, \sigma_2, \sigma_4$ and σ_5.

We also remark that the classical Hamiltonian may be amenable to a version of the density matrix renormalization group scheme, as described in [15,16].

IV. CONCLUSION

Scaling techniques which are simple to implement have been described within the quantum formulation of nonequilibrium exclusion models. In particular, the exact fixed points are obtained for the ASEP in which the known dependence of the dynamics on asymmetry is recovered. For a model involving the evaporation and deposition of adjacent pairs exact fixed points are obtained; the resulting flow diagram indicates that the dynamics are independent of the ratio of evaporation and deposition rates. Further, the stability of the free-fermion condition in the model combining the dynamics of the ASEP with pair evaporation and deposition indicates that the exact solutions provide a complete account of the large scale behavior. Again, the dynamics are found to be independent of bias.

We note that the projections of spin-1 operators onto a basis of spin-1/2 operators in Sec. II B and II C are not necessary. Indeed, one could continue scaling the system to renormalized bases of higher and higher spin. An approach to yield the continuum limit of the ASEP in this way [17] is under investigation.

We have also shown how to map a quantum Hamiltonian representing a nonequilibrium exclusion process onto a classical Hamiltonian in one higher dimension. In this way, we have shown how to write the steady state probabilities for configurations of the nonequilibrium system in terms of classical Boltzmann weights. This should enable one to borrow real-space renormalization group techniques for classical Hamiltonian systems and apply them to nonequilibrium systems.

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