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Universal aging properties at a disordered critical point

Gregory Schehr and Raja Paul
Theoretische Physik, Universität des Saarlandes, 66041 Saarbrücken, Germany
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We investigate, analytically near the dimension $d_{uc}=4$ and numerically in $d=3$, the nonequilibrium relaxational dynamics of the randomly diluted Ising model at criticality. Using the exact renormalization-group method to one loop, we compute the two times t, t_w correlation function and fluctuation dissipation ratio (FDR) for any Fourier mode of the order parameter, of finite wave vector q. In the large time separation limit, the FDR is found to reach a nontrivial value X^∞ independently of (small) q and coincide with the FDR associated to the *total* magnetization obtained previously. Explicit calculations in real space show that the FDR associated to the *local* magnetization converges, in the asymptotic limit, to this same value X^∞. Through a Monte Carlo simulation, we compute the autocorrelation function in three dimensions, for different values of the dilution fraction p at $T_c(p)$. Taking properly into account the corrections to scaling, we find, according to the renormalization-group predictions, that the autocorrelation exponent λ_c is independent of p. The analysis is complemented by a study of the nonequilibrium critical dynamics following a quench from a completely ordered state.

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The study of relaxational dynamics following a quench at a pure critical point has attracted much attention in the past few years [1–4]. Although simpler to study than glasses, critical dynamics display interesting nonequilibrium features such as aging, commonly observed in more complex disordered or glassy phases [5]. In this context, the computation of two times t, t_w response and correlation functions with associated universal exponents has been the subject of numerous analytical as well as numerical studies [4].

In addition, it has been proposed [6] that a nontrivial fluctuation dissipation ratio (FDR) X, originally introduced in the mean-field approach to glassy systems, which generalizes the fluctuation dissipation theorem (FDT) to nonequilibrium situations, is a new *universal* quantity associated with these critical points. As such, it has been computed using the powerful tools of renormalization group (RG), e.g., for pure $O(N)$ model at criticality in the vicinity of the upper critical dimension $d_{uc}=4$ and for various dynamics [4,7,8].

An important question related to the physical interpretation of X in terms of an effective temperature [9] $T_{\text{eff}}=T/X$ is its dependence on the observables [10,11]. In this respect, a heuristic argument [7] suggests that, for a wide class of critical systems, the *local* FDR associated with correlation and corresponding response of the local magnetization should be identical, in the large-time separation limit, to the FDR for the *total* magnetization, i.e., for the Fourier mode $q=0$. This argument relies strongly on the hypothesis that the time decay of the response function of the Fourier mode q is characterized by a single time scale $\tau_q \sim q^{-z}$, with z the dynamical exponent.

Characterizing the effects of quenched disorder on critical dynamics is a complicated task, and indeed the question of how quenched randomness modifies these properties has been much less studied. In particular, in this context of critical disordered systems, the question of universality, i.e., the dependence of the critical exponents on the strength of the disorder, is a controversial issue [12]. In this paper, we address these questions on the randomly diluted Ising model,

$$H = \sum_{\langle ij \rangle} \rho_i \rho_j s_i s_j, \tag{1}$$

where s_i are Ising spins on a d-dimensional hypercubic lattice and $\rho_i=1$ with probability p and 0 with probability $1-p$. For the experimentally relevant case of dimension $d=3$ [13], for which the specific-heat exponent of the pure model is positive, the disorder is expected, according to Harris criterion [14], to modify the universality class of the transition. For $1-p \ll 1$, the large-scale properties of Eq. (1) at criticality are then described by the following O(1) model with a random mass term, the so-called random Ising model (RIM):

$$H^{\psi}[\varphi] = \int d^d x \left[\frac{1}{2}(\nabla \varphi)^2 + \frac{1}{2}[r_0 + \psi(x)]\varphi^2 + \frac{g_0}{4!}\varphi^4 \right], \tag{2}$$

where $\varphi \equiv \varphi(x)$ and $\psi(x)$ is a Gaussian random variable $\overline{\psi(x)\psi(x')} = \Delta \delta^d(x-x')$, and r_0, the bare mass, is adjusted so that the renormalized one is zero. The static critical properties of this model have been intensively studied [15] both analytically, mainly using RG, within various schemes, and numerically [16]. The (perturbative) RG calculations below the upper critical dimension in $d=4-\epsilon$, which we will focus on here, confirm the qualitative Harris criterion and predict that the critical properties of these models (2) for different values of p close to 1 are described by a new disordered fixed point, which is independent of p. Therefore, an important statement of this RG analysis is that the critical exponents, which can be computed in an expansion of $\sqrt{\epsilon}$, e.g., $\eta=\mathcal{O}(\epsilon)$, are universal, i.e., independent of p. This was recently confirmed by Monte Carlo simulation in $d=3$ [16], over a wide range of concentration p above the percolation threshold $p_c=0.31$. Although quantitative discrepancies were found with perturbative RG calculations [17], universality was demonstrated by taking carefully into account the (strong) corrections to scaling [16]. In the equilibrium dynamics, at variance with the pure case, the perturbative expansion of the dynamical exponent z differs from its high-temperature value of 2 already at one loop [18] $z-2=\sqrt{(6\epsilon/53)}+\mathcal{O}(\epsilon)$ independently of p, corrections to two loops have been computed in Ref. [19], and up to three loops in Ref. [20]. After a long debate, a recent numerical simulation [21] where corrections to (dynamical) scaling were taken into account has confirmed the universality of z in $d=3$, leading to $z=2.62(7)$ independently of the spin concentration p above p_c.

By contrast, much less is known about the nonequilibrium dynamics of this disordered system at criticality. The critical initial slip exponent θ' vanishes to one loop [22], and correction to two loops have been computed [23]. This exponent has been recently computed up to two loops for the case of extended defects in [24]. The two times response in Fourier space, $\mathcal{R}^q_{tt_w}$ [22], and the correlation $\mathcal{C}^{q=0}_{tt_w}$ [25], including the associated scaling functions, are known up to one loop. But although the dynamical RG predicts a universal value of the autocorrelation exponent $\lambda_c=d-z\theta'$, for p close to 1, this statement remains an open question for a wider range of values of p. Furthermore, a nontrivial FDR [25], only for the total magnetization, was recently obtained to one loop, and it was argued, using the same aforementioned heuristic argument [7], to coincide with the *local* FDR. However, it was already noticed in Ref. [22] that, due to the disorder, $\mathcal{R}^q_{tt_w}$ decays as a *power law* for $q^z t \gg 1$. Therefore, the argument of Ref. [7] is challenged for this disordered critical point, and, already at one-loop order, the computation of the FDR needs a closer inspection, including an extension of the analysis of Ref. [25] beyond the "diffusive" $q=0$ mode.

In this paper, using RG to one loop, we obtain, for any finite Fourier mode q, the correlation function $\mathcal{C}^q_{tt_w}$ and the FDR $X^q_{tt_w}$, which are both characterized by scaling functions of the variables $q^z(t-t_w)$ and t/t_w. In the asymptotic large time separation regime $t \gg t_w$, the FDR reaches a nontrivial value X^∞, *independently* of (small) q. In addition, we explicitly compute the *local* FDR, which is a function of t/t_w and reaches the same nontrivial limit X^∞ when $t \gg t_w$, which thus establishes on firmer grounds the heuristic argument of Ref. [7] for the present disordered case. Besides, we perform a Monte Carlo simulation of the nonequilibrium relaxation of Eq. (1) following a quench from high temperature with initial magnetization $m_0=0$ at $T_c(p)$ and compute the autocorrelation function. In the asymptotic regime, it takes a scaling form compatible with the RG calculations. By taking into account corrections to scaling, we show that the exponent λ_c is independent of p. Finally, we compute numerically the autocorrelation function for the critical dynamics following a quench for a completely ordered initial condition with $m_0=1$. We observe that the system is also aging and show that the decaying exponent λ_c is strongly affected by this initial condition.

We study the relaxational dynamics of the randomly diluted Ising model in dimension $d=4-\epsilon$ described by a Langevin equation,

$$\eta\frac{\partial}{\partial t}\varphi(x,t) = -\frac{\delta H^{\psi}[\varphi]}{\delta\varphi(x,t)} + \zeta(x,t), \tag{3}$$

where $\langle\zeta(x,t)\rangle=0$ and $\langle\zeta(x,t)\zeta(x',t')\rangle=2\eta T\delta(x-x')\delta(t-t')$ is the thermal noise and η the friction coefficient. At initial time $t_i=0$, the system is in a random initial configuration with *zero* magnetization $m_0=0$ distributed according to a Gaussian with short-range correlations,

$$[\varphi(x,t=0)\varphi(x',t=0)]_i = \tau_0^{-1}\delta^d(x-x'). \tag{4}$$

Notice that it has been shown that τ_0^{-1} is irrelevant here (in the RG sense) in the large time regime studied here [26]. We will focus on the correlation $\mathcal{C}^q_{tt_w}$ in Fourier space and the autocorrelation C_{tt_w},

$$\mathcal{C}^q_{tt_w} = \overline{\langle\varphi(q,t)\varphi(-q,t_w)\rangle}, \quad C_{tt_w} = \overline{\langle\varphi(x,t)\varphi(x,t_w)\rangle} \tag{5}$$

and the response $\mathcal{R}^q_{tt_w}$ to a small external field $f(-q,t_w)$ as well as on the local response function R_{tt_w} respectively defined, for $t>t_w$, as

$$\mathcal{R}^q_{tt_w} = \overline{\frac{\delta\langle\varphi(q,t)\rangle}{\delta f(-q,t_w)}}, \quad R_{tt_w} = \overline{\frac{\delta\langle\varphi(x,t)\rangle}{\delta f(x,t_w)}}, \tag{6}$$

where $\overline{\cdot\cdot}$ and $\langle\cdot\cdot\rangle$ denote, respectively, averages with respect to disorder and thermal fluctuations. We focus also on the FDR $X^q_{tt_w}$ associated to the observable φ [5],

$$\frac{1}{X^q_{tt_w}} = \frac{\partial_{t_w}\mathcal{C}^q_{tt_w}}{T\mathcal{R}^q_{tt_w}} \tag{7}$$

defined such that $X^q_{tt_w}=1$ at equilibrium. Notice also that for this choice of initial conditions (4), connected and nonconnected correlations do coincide for large system size.

A convenient way to study the Langevin dynamics defined by Eq. (3) is to use the Martin-Siggia-Rose generating functional. Using the Ito prescription, it can be readily averaged over the disorder. The correlations (5) and response (6) are then obtained from a dynamical (disorder-averaged) generating functional or, equivalently, as functional derivatives of the corresponding dynamical *effective* action Γ. This functional can be perturbatively computed [27] using the exact RG equation associated with the multilocal operators expansion introduced in [28,29]. It allows us to handle arbitrary cutoff functions $c(q^2/2\Lambda_0^2)$ and check universality, independence with respect to $c(x)$, and the ultraviolet scale Λ_0. It describes the evolution of Γ when an additional infrared cutoff Λ_l is lowered from Λ_0 to its final value $\Lambda_l \to 0$, where a fixed point of order $\mathcal{O}(\sqrt{\epsilon})$ is reached. In this limit, one obtains $\mathcal{R}^q_{tt_w}$ and $\mathcal{C}^q_{tt_w}$ (for $t>t_w$) from

$$\partial_t\mathcal{R}^q_{tt_w} + [q^2+\mu(t)]\mathcal{R}^q_{tt_w} + \int_{t_i}^{t} dt_1 \Sigma_{tt_1}\mathcal{R}^q_{t_1t_w} = 0, \tag{8}$$

$$\mathcal{C}^q_{tt_w} = 2T\int_{t_i}^{t_w} dt_1 \mathcal{R}^q_{tt_1}\mathcal{R}^q_{t_wt_1} + \int_{t_i}^{t} dt_1 \int_{t_i}^{t_w} dt_2 \mathcal{R}^q_{tt_1} D_{t_1t_2}\mathcal{R}^q_{t_wt_2} \tag{9}$$

with $\mu(t)=-\int_{t_i}^t dt_1 \Sigma_{tt_1}$ and where the self-energy $\Sigma_{t_1t_2}$ and the noise-disorder kernel $D_{t_1t_2}$ are directly obtained from Γ at the fixed point. One finds

$$\Sigma_{tt'} = -\frac{1}{2}\sqrt{\frac{6\epsilon}{53}}\int_a [\gamma_a(t-t')]^2, \quad (10)$$

$$D_{tt'} = \frac{T_c}{2}\sqrt{\frac{6\epsilon}{53}}\int_a [\gamma_a(t-t') - \gamma_a(t+t')], \quad (11)$$

where $\gamma_a(x)=[x+a/(2\Lambda_0^2)]^{-1}$. For concrete calculations, we have used the decomposition of the cutoff function $c(x) = \int da\hat{c}(a)e^{-ax} \equiv \int_a e^{-ax}$.

The computation of the correlation function $\mathcal{C}^q_{tt_w}$ requires the knowledge of the response, which we first focus on. By solving perturbatively to order $\mathcal{O}(\sqrt{\epsilon})$ the differential equation (8), similarly to what is done in Ref. [27], one recovers, in the limit $q/\Lambda_0 \ll 1$ keeping the scaling variables $v=q^z(t-t_w)$ and $u=t/t_w$ fixed, the solution obtained in Ref. [22], consistent with the scaling form

$$\mathcal{R}^q_{tt_w} = q^{-2+z+\eta}\left(\frac{t}{t_w}\right)^\theta F_R(q^z(t-t_w),t/t_w), \quad (12)$$

where $\theta=\frac{1}{2}\sqrt{(6\epsilon/53)}+\mathcal{O}(\epsilon)$ and the universal [30] scaling function $F_R(v,u)$ admits also an expansion in powers of $\sqrt{\epsilon}$ with [22] $F_R(v,u)\equiv F_R^{\mathrm{eq}}(v)=e^{-v}+\frac{1}{2}\sqrt{6\epsilon/53}[(v-1)\mathrm{Ei}(v)e^{-v}+e^{-v}-1]$, where $\mathrm{Ei}(v)$ is the exponential integral function. At variance with the pure model at one loop [7], the large-v behavior of $F_R^{\mathrm{eq}}(v)$ is a power law, $F_R(v)\propto v^{-2}$, which already indicates that the heuristic argument of Ref. [7] cannot be applied here. Besides, when computing the local response R_{tt_w}, one is left with an integral over momentum which is logarithmically divergent, indicating that this integral has to be handled with care to obtain the correct result, as the scaling form in Eq. (12) is valid only for $q/\Lambda_0 \ll 1$. We thus solve perturbatively Eq. (8) for any fixed q and obtain an expression for $\mathcal{R}^q_{tt_w}$ consistent with the scaling form,

$$\mathcal{R}^q_{tt_w} = \tilde{g}_1(q)\left(\frac{t}{t_w}\right)^\theta F_R(\tilde{g}_2(q)q^2(t-t_w),t/t_w), \quad (13)$$

where $\tilde{g}_1(q)$, $\tilde{g}_2(q)$ are nonuniversal functions, i.e., which depend explicitly on the cutoff function $c(x)$ and Λ_0, with the universal [30] small-q behavior,

$$\tilde{g}_1(q) \sim q^{z-2+\eta}, \quad \tilde{g}_2(q) \sim q^{z-2}, \quad (14)$$

which thus allows us to recover the previous expression in the asymptotic limit $q/\Lambda_0 \ll 1$ (12). By computing the Fourier transform of $\mathcal{R}^q_{tt_w}$ as given in Eq. (13), we explicitly check that the local response R_{tt_w} is consistent with the scaling form,

$$R_{tt_w} = \frac{K_d}{2}\frac{A^0_{\mathcal{R}} + A^1_{\mathcal{R}}\ln(t-t_w)}{(t-t_w)^{1+(d-2+\eta)/z}}\left(\frac{t}{t_w}\right)^\theta \quad (15)$$

with $K_d=S_d/(2\pi)^d$ and where the nonuniversality is left in the amplitudes $A^0_{\mathcal{R}}$ and $A^1_{\mathcal{R}}$,

$$A^0_R = 1 - \frac{3}{2}\sqrt{\frac{6\epsilon}{53}} + \rho_R, \quad A^1_R = \frac{1}{2}\sqrt{\frac{6\epsilon}{53}}, \quad (16)$$

$$\rho_R = \sqrt{\frac{6\epsilon}{53}}\int_a \ln\left(\frac{2\Lambda_0^2}{a}\right).$$

At the order of our calculations $\mathcal{O}(\sqrt{\epsilon})$, although $z\neq 2$, this scaling form (15) is compatible with local scale invariance arguments [31]. Notice also that, at this order, the scaling form obtained for R_{tt_w} could be written as

$$R_{tt_w} = \frac{K_d}{2}A_R\frac{1}{(t-t_w)^{(1+a)}}\left(\frac{t}{t_w}\right)^\theta \quad (17)$$

with $a\neq(d-2+\eta)/z$. Although this scaling form (17) cannot be ruled out at this stage, which would in principle require a two-loop calculation, it seems rather unlikely given the scaling form obtained in Fourier space (12), where instead a logarithmic correction as in Eq. (15) is suggested by the large argument behavior of the function $F_R^{\mathrm{eq}}(v)$.

We now turn to the computation of the correlation function in Fourier space $\mathcal{C}^q_{tt_w}$, which was only computed for the $q=0$ mode [25]. Solving Eq. (9), one obtains an explicit expression, which, in the aforementioned scaling limit, is compatible with the scaling form,

$$\mathcal{C}^q_{tt_w} = T_c q^{-2+\eta}\left(\frac{t}{t_w}\right)^\theta F_C(q^z(t-t_w),t/t_w) \quad (18)$$

with the full expression

$$F_C(v,u) = F_C^0(v,u) + \sqrt{\epsilon}F_C^1(v,u) + \mathcal{O}(\epsilon), \quad (19)$$

$$F_C^0(v,u) = e^{-v} - e^{-v[(u+1)/(u-1)]},$$

$$F_C^1(v,u) = F_C^{1\mathrm{eq}}(v) - F_C^{1\mathrm{eq}}\left(v\frac{u+1}{u-1}\right) + \sqrt{\frac{6}{53}}e^{-v[(u+1)/(u-1)]} \times\left[\mathrm{Ei}\left(\frac{2v}{u-1}\right) - \ln\left(\frac{2v}{u-1}\right) - \gamma_E\right], \quad (20)$$

$$F_C^{1\mathrm{eq}}(v) = \frac{1}{2}\sqrt{\frac{6}{53}}[e^{-v} + ve^{-v}\mathrm{Ei}(v)], \quad (21)$$

where γ_E is the Euler constant. In the limit $q\to 0$, our full expression (19) gives back the result of Ref. [25]. In the large time separation limit $u\gg 1$, keeping v fixed, one obtains the result

$$F_C(v,u) = \frac{1}{u}F_{C,\infty}(v) + \mathcal{O}(u^{-2}), \quad (22)$$

$$F_{C\infty}(v) = A_{C\infty}vF_R^{\mathrm{eq}}(v), \quad A_{C\infty} = 2 + 2\sqrt{\frac{6\epsilon}{53}}. \quad (23)$$

This $\mathcal{O}(u^{-1})$ decay in Eq. (22) is expected from RG arguments, and has been explicitly checked for different pure critical systems [4]. However, this relation (23) is *a priori* nontrivial and cannot be obtained from general arguments. This identity was also found for the pure $O(N)$ model at criticality to one-loop order [7] as well as in the glass phase of the sine-Gordon model with random phase shifts [32], and it would be interesting to investigate whether such a behavior (23) can be obtained from more general arguments.

The full expression for $\mathcal{C}^q_{tt_w}$ (19) also allows us to compute the structure factor $\mathcal{C}^q_{tt}$. It is obtained from Eq. (19) in the limit $v\to 0$, $u\to 1$ keeping $v/(u-1)=q^z t$ fixed, and we check that one recovers the previous result obtained in Ref. [22]. Thus, one explicitly checks, at order $\mathcal{O}(\sqrt{\epsilon})$, that the dynamical exponent z associated with dynamical *equilibrium* fluctuations is the same as the one associated with *nonequilibrium* relaxation.

As noticed previously for the response function, the large-v behavior of $F_C(v,u)$ is a power law $F_C(v,u)\propto v^{-1}$. Therefore, given the scaling form (18), the computation of the autocorrelation C_{tt_w} has to be done carefully. Just as for the response, we thus compute the correlation function $\mathcal{C}^q_{tt_w}$ from Eq. (9) for any fixed q and then perform the Fourier transformation. One obtains the scaling form

$$C_{tt_w} = K_d \frac{A_C^0 + A_C^1 \ln(t-t_w)}{(t-t_w)^{(d-2+\eta)/z}} \left(\frac{t}{t_w}\right)^{\theta} \mathcal{F}(t/t_w) \tag{24}$$

with

$$A_C^0 = 1 - \frac{1}{2}\sqrt{\frac{6\epsilon}{53}} + \rho_R, \quad A_C^1 = \frac{1}{2}\sqrt{\frac{6\epsilon}{53}},$$

$$\mathcal{F}(u) = \frac{1}{1+u} + \mathcal{O}(\epsilon). \tag{25}$$

The same remarks, concerning the response, made before Eq. (17) also hold here for the autocorrelation.

We now turn to the FDR, first in Fourier space. Given the scaling forms for the response $\mathcal{R}^q_{tt_w}$ (12) and for the correlation $\mathcal{C}^q_{tt_w}$ (18) that we have explicitly checked here, the FDR $X^q_{tt_w}$ takes the simple scaling form in the regime $q/\Lambda_0 \ll 1$,

$$(X^q_{tt_w})^{-1} = F_X(q^z(t-t_w), t/t_w). \tag{26}$$

We have obtained the complete expression for the scaling function $F_X(v,u)$, which at variance with the pure O(N) model at criticality is a function of both $q^z t$ and $q^z t'$. In the large time separation limit $u \gg 1$, keeping v fixed, one obtains, as a consequence of Eq. (23),

$$\lim_{u\to\infty} (X^q_{tt_w})^{-1} = 2 + \sqrt{\frac{6\epsilon}{53}} + \mathcal{O}(\epsilon), \tag{27}$$

independently of v, i.e., of (small) wave vector q, which coincides of course with the asymptotic value for the $q=0$ mode obtained in Ref. [25]. We can check easily, using the result of Ref. [7], that this property, independent of v on the asymptotic limit, holds also for the pure model at one loop, and it was also found in the glass phase of the sine-Gordon model with random phase shifts [32].

As we saw previously, the large-v power-law behavior of the scaling function $F_R^{\mathrm{eq}}(v)$ prevents us from using the argument of Ref. [7] for the present case. Therefore, one computes directly the FDR for the local correlation and associated response $X^{x=0}_{tt_w}$. It is also characterized by a scaling function of t/t_w, which can be simply written as

$$(X^{x=0}_{tt_w})^{-1} = \mathcal{F}_X(t/t_w), \tag{28}$$

$$\mathcal{F}_X(u) = 2\frac{u^2+1}{(u+1)^2} + \sqrt{\frac{6\epsilon}{53}}\left(\frac{u-1}{u+1}\right)^2 + \mathcal{O}(\epsilon), \tag{29}$$

where $\mathcal{F}_X(u)$ is a monotonic increasing function of u. It interpolates between 1, in the quasiequilibrium regime for $u\to 1$, and its asymptotic value for $u\to\infty$ given by

$$\lim_{t/t_w\to\infty} (X^{x=0}_{tt_w})^{-1} = \lim_{t/t_w\to\infty} (X^{q=0}_{tt_w})^{-1} = 2 + \sqrt{\frac{6\epsilon}{53}} + \mathcal{O}(\epsilon), \tag{30}$$

which shows explicitly, at order $\mathcal{O}(\sqrt{\epsilon})$, that the asymptotic FDR for both the total and the local magnetization are indeed in the same.

Let us next present results from our Monte Carlo simulations of the relaxational dynamics of the randomly diluted Ising model (1) in dimension $d=3$, which were done on $L\times L\times L$ cubic lattices with periodic boundary conditions. We first focus on the following situation where the system is initially prepared in a random initial configuration with zero magnetization $m_0=0$. At each time step, the L^3 sites are then sequentially updated: for each site i, the move $s_i\to -s_i$ is accepted or rejected according to METROPOLIS rule. If one gradually decreases p, the fraction of magnetic sites will be reached below which the system no longer exhibits a transition to ferromagnetic order at any finite temperature. This happens at the percolation threshold, for which $T_c(p_c)=0$ [12,16]. For different values of $p>p_c$, we compute the spin-spin autocorrelation function defined as

$$C_{tt_w} = \frac{1}{L^3}\sum_i \overline{\langle s_i(t)s_i(t_w)\rangle}. \tag{31}$$

In the following, we will also be interested in the connected correlation function $\widetilde{C}(t,t_w)$ defined as

$$\widetilde{C}(t,t_w) = \frac{1}{L^3}\sum_i \overline{\langle s_i(t)s_i(t_w)\rangle - \langle s_i(t)\rangle\langle s_i(t_w)\rangle}. \tag{32}$$

In order to obtain better statistics, C_{tt_w} [or $\widetilde{C}(t,t_w)$] is averaged over a suitably chosen time window Δ_t around t, with $\Delta_t \ll t$. All our data are obtained for a lattice linear size $L=100$, as an average over 500 independent initial conditions and disorder configurations. We also produced data (not shown here) for the spatial correlation function, for the same system size, to ensure that our results are not influenced by finite-size effects.

Figure 1 shows the autocorrelation function C_{tt_w} as a function of $t-t_w$ for different values of the waiting time $t_w=2^4, 2^5, 2^6, 2^7$, and 2^8 at $p=0.8$. One observes a clear dependence on t_w, which indicates a nonequilibrium dynamical regime. We have also checked that for this choice of initial conditions, $C(t,t_w)$ and $\widetilde{C}(t,t_w)$ do coincide. The scaling form obtained from the RG analysis (24) suggests, discarding the logarithmic correction, to plot $t_w^{(1+\eta)/z}C_{tt_w}$ as a function of t/t_w. Taking the values $\eta=0.0374$ from Ref. [16] and

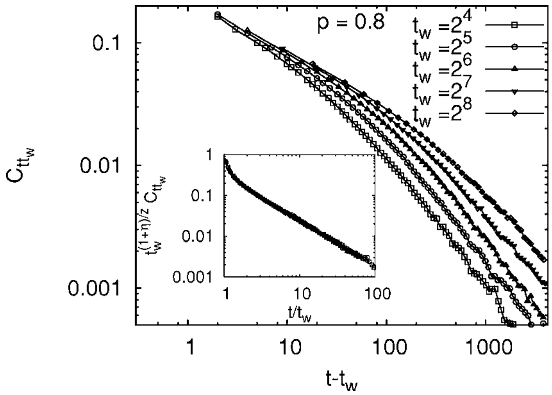

FIG. 1. Log-log plot of autocorrelation function $\mathcal{C}_{tt_w}$ vs $t-t_w$. Inset: Scaling plot of $\mathcal{C}_{tt_w}$ as a function of t/t_w. Here, the system is initially prepared in a random initial configuration with zero magnetization.

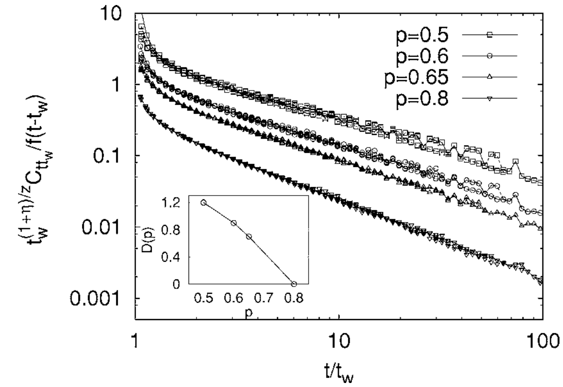

FIG. 2. $t_w^{(1+\eta)/z} \mathcal{C}_{tt_w}/f(t-t_w)$ as a function of t/t_w for different p $=0.5$, 0.6, 0.65, and 0.8. Waiting times t_w corresponding to $p=0.5$, 0.6, 0.65 are $2^6,\ldots,2^{10}$ whereas for $p=0.8$, $t_w=2^4,\ldots,2^8$. $f(x)$ is defined in the text. Inset : Nonuniversal amplitude $D(p)$ as a function of p. Here, the system is initially prepared in a random initial configuration with zero magnetization.

$z=2.62$ from Ref. [21], we see in the inset of Fig. 1 that, for $p=0.8$, one obtains a good collapse of the curves for different t_w. Notice that such scaling forms are also obtained in more complicated disordered systems like three-dimensional spin glasses [33].

However, for different values of p, the best collapse, under this form (24), would be obtained for a p-dependent exponent $(1+\eta)/z$. Thus one would conclude that this exponent is nonuniversal [12]. Nevertheless, it is known [21] that such p dependence occurs due to corrections to scaling. Therefore, to include them, we extend the scaling form (24) as

$$\mathcal{C}_{tt_w} = \frac{1}{(t-t_w)^{(1+\eta)/z}} \times \left(\tilde{F}_p(t/t_w) - \frac{D(p)}{(t-t_w)^b} \tilde{G}_p(t/t_w) \right) \tag{33}$$

with $b=\omega_d/z$, where ω_d corresponds to the biggest irrelevant eigenvalue of the RG in the dynamics, which is *a priori* different from the leading corrections in the statics [21]. Unfortunately, we do not have any information on the function $\tilde{G}_p(x)$. We will thus propose the simplest hypothesis $\tilde{G}_p(x)$ $=\tilde{F}_p(x)$. In Fig. 2, we show a plot of $t_w^{(1+\eta)/z} \mathcal{C}_{tt_w}/f(t-t_w)$, with $f(x)=1-D(p)x^{-b}$: this results in a reasonably good data collapse of the curves for different t_w, for $p=0.5$, 0.6, 0.65, and 0.8. For each value of p, this data collapse is obtained via the fitting of three parameters: the exponents b, z and the amplitude $D(p)$. We found a quite stable value of the exponents $z=2.6\pm0.1$ and $b=0.23\pm0.02$, which are *both independent* of p. Our value of z, together with $\omega_d=0.61\pm0.06$, is consistent with the value obtained by Parisi *et al.* [21]. All the p dependence is thus contained in the nonuniversal amplitude $D(p)$, as shown in the inset of Fig. 2. According to our data, the corrections to scaling in the quasiequilibrium regime vanish for $p=0.8$, i.e., $D(p=0.8)\simeq 0$, in agreement with a previous numerical computation of the equilibrium autocorrelation function [12]. Notice that this value $p=0.8$ is also known [16], in the statics, to minimize the corrections to scaling.

As shown on the log-log plot in Fig. 2, and consistently with the RG prediction (24), $\tilde{F}_p(t/t_w)$ (33) decays as a power law for $t \gg t_w$. However, this plot in Fig. 2 would suggest that the decaying exponent depends, namely decreases, with p. We expect instead that this p dependence is again due to corrections to scaling [21]. Consistently with the corrections we introduced in the quasiequilibrium part of $\mathcal{C}_{tt_w}$ in Eq. (33), we propose the form

$$\tilde{F}_p(x) = A(p) x^{(1+\eta-\lambda_c)/z} [1 + B(p) x^{-b}], \tag{34}$$

where we (reasonably) assume that the dynamical corrections to scaling are characterized by the *same*, p-independent, exponent $b=0.23\pm0.02$ as obtained previously (33). Therefore, for each value of p one has three parameters to fit: the exponent λ_c/z and the amplitudes $A(p)$, $B(p)$. We obtain a quite stable fit for the different values of p, with the *p-independent* value of the decaying exponent λ_c/z,

$$\frac{\lambda_c}{z} = 1.05 \pm 0.03, \tag{35}$$

all the p dependence being contained in the nonuniversal amplitudes $A(p)$, $B(p)$ (see the inset in Fig. 3). As shown in Fig. 3, the curves for different values of p (and different t_w) in Fig. 2 collapse on a master curve when we plot $t_w^{(1+\eta)/z} \mathcal{C}_{tt_w}/[f(t-t_w)g(t/t_w)]$, with $g(x)=A(p)[1+B(p)x^{-b}]$, as a function of t/t_w. This fact supports universality of the long-time nonequilibrium relaxation in this model. Our value for the exponent λ_c/z, together with $z=2.6\pm0.1$, gives for the initial slip exponent $\theta'=0.1\pm0.035$, which is in rather good agreement with the two-loop RG result $\theta'_{2\text{loops}}=0.0868$ [23]. Alternatively, this exponent could be measured by studying the initial stage of the relaxational dynamics starting from a nonzero magnetization: this is left for future investigations [34].

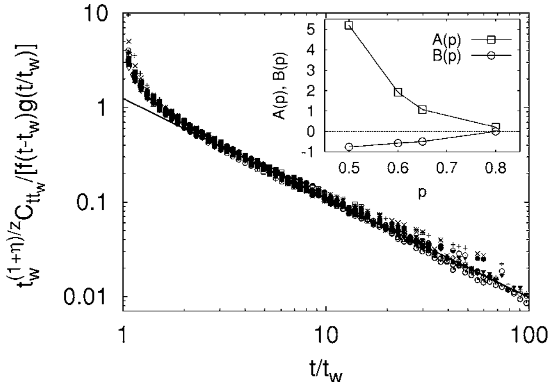

FIG. 3. Universality of C_{tt_w} for $p=0.6,0.65,0.8$. The function $g(x)$ is defined in the text. Inset: Nonuniversal amplitudes $A(p),B(p)$ as functions of concentration p. Here, the system is initially prepared in a random initial configuration with zero magnetization.

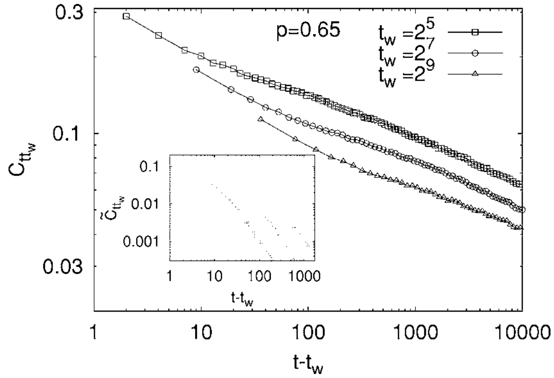

FIG. 4. Log-log plot of the correlation $C(t,t_w)$ as a function of $t-t_w$ for $p=0.65$. Inset: Log-log plot of the connected correlation $\widetilde{C}(t-t_w)$ as a function of $t-t_w$. In the inset, we use the same symbols as in the main figure. Here $m_0=1$.

Here also, one obtains that the corrections to scaling in Eq. (34) vanish for $p=0.8$. We notice that this result is in apparent contradiction with the previous analysis of the nonequilibrium relaxation in this model performed in Ref. [21], where the focus was on the nonconnected susceptibility, a one-time quantity, which instead claimed a "perfect Hamiltonian" for $p \simeq 0.63$. However, the statistical precision of our data does not allow us to make a strong statement about this point, which certainly deserves further investigation.

So far, we have focused on the relaxational dynamics occurring after a quench from a completely disordered initial condition, with zero initial magnetization $m_0=0$, to $T_c(p)$. But it is also interesting to study how these aging properties depend on the initial conditions [35–37]. We have therefore performed numerical simulations where the system is initially prepared in a completely ordered state,

$$S_i(t=0) = +1, \quad \forall \text{ occupied site } i \tag{36}$$

such that the initial magnetization is $m_0=1$. The system is then quenched at $t=0$ to $T_c(p)$ and evolves according to the same aforementioned dynamical rules. We also compute the autocorrelation function $C(t,t_w)$ as defined in Eq. (31). The result of this computation for $p=0.65$ is shown in Fig. 4, where we plot $C(t,t_w)$ as a function of $t-t_w$, for different $t_w=2^5,2^7,2^9$. Here also, one observes a clear dependence on the waiting time t_w, which indicates that the system is aging. Notice, however, that, at variance with the previous situation (Fig. 1), the correlation for a given $t-t_w$ decreases as t_w increases. In addition, at variance with the previous case $m_0=0$, the behaviors of the connected $\widetilde{C}(t,t_w)$ (32) and the nonconnected $C(t,t_w)$ correlations are qualitatively different: this is shown in the inset of Fig. 4, when one observes that $\widetilde{C}(t,t_w)$ decays indeed much faster [38]. This property could be relevant for the computation of the FDR in this situation. The quantitative analysis of the correlation function $C(t,t_w)$ is shown in Fig. 5. Indeed, the curves for different t_w can be plotted on a master curve if one plots, for different t_w, $t_w^{(1+\eta)/z}C_{tt_w}/f(t-t_w)$ as a function of t/t_w, which suggests that also in that case the correlation function can be written under the scaling form as in Eq. (33) with $\widetilde{F}_p(x)=\widetilde{G}_p(x)$. However, as illustrated in Fig. 5, the behavior of $C(t,t_w)$ is strongly affected by the initial condition, the decay being much faster when the system is initially in a random configuration with $m_0=0$. More precisely, as suggested in Fig. 5, our data for $m_0=1$ are compatible with the following scaling form:

$$C_{tt_w} \sim \frac{1}{(t-t_w)^{(1+\eta)/z}}\left(1-\frac{D(p)}{(t-t_w)^b}\right)\left(\frac{t}{t_w}\right)^{(1+\eta)/2z} \sim t^{-\beta/\nu z}, \quad t \gg t_w, \tag{37}$$

where β, ν are the standard equilibrium critical exponents and where we have used the hyperscaling relation $\beta/\nu=(d-2+\eta)/2$. Thus, although we cannot show it analytically for the present problem, we believe that in that case of a fully ordered initial condition ($m_0=1$), although the system displays aging, the exponent λ_c is completely determined by the equilibrium exponents,

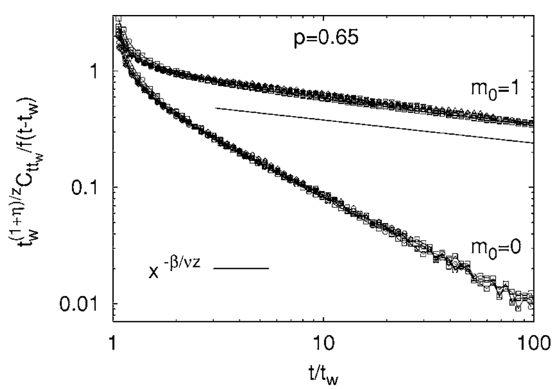

FIG. 5. $t_w^{(1+\eta)/z}C_{tt_w}/f(t-t_w)$ as a function of t/t_w for $p=0.65$ and the two different initial conditions considered here, $m_0=0$ and $m_0=1$. $f(x)$ is defined in the text. The straight line is a guide line for the eyes.

$$\lambda_c = \frac{\beta}{\nu}. \quad (38)$$

This relation (38) can be understood by considering $C(t,0)$. Indeed, for this particular initial condition (36), one has $C(t,0)=M(t)$, where $M(t)$ is the global magnetization at time t. Therefore, at large time, from the standard scaling argument $C(t,0)\sim t^{-\beta/\nu z}$, which thus gives the relation (38). Notice that this relation (38) is also found in the context of pure critical point [36,37].

To sum up, we have performed a rather detailed analysis of the relaxational dynamics up to one loop of the randomly diluted Ising model in dimension $d=4-\epsilon$. The computation of the correlation function $\mathcal{C}^q_{tt_w}$, including its associated scaling function, allows us to show that the fluctuation dissipation ratio reaches, in the large time separation limit, a nontrivial value X^∞, independently of small wave vector q. Although, due to the broad relaxation time spectrum induced by the disorder, the standard argument of Ref. [7] cannot be applied here, we have performed an explicit computation in real space which shows explicitly that the limiting FDR associated with the total magnetization, on the one hand, and the local one, on the other hand, do coincide. And in this respect, it would be interesting to further investigate the FDR associated with other observables, like the energy, for instance [10,11]. These properties could also be tested in numerical simulations.

In addition, we have computed numerically, in $d=3$, the autocorrelation function. It is characterized by a scaling form fully compatible with our one-loop RG calculation in real space. We have, however, shown that this two times quantity is strongly affected by corrections to scaling, which remain to be understood more deeply from an analytical point of view. By taking them properly into account, our data suggest a universal, i.e., p-independent autocorrelation exponent λ_c, which provides an "indirect" measurement of the initial slip exponent θ', which is in reasonably good agreement with the two-loop RG prediction. Finally, we have shown that the critical dynamics following a quench from a completely ordered state ($m_0=1$) displays also aging, but with a quantitatively different behavior, the decaying exponent λ_c being in that case completely determined by the *equilibrium* exponents.

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