Harmonic velocity noise: Non-Markovian features of noise-driven systems at long times

Jing-Dong Bao,[1,*] Yan-Li Song,[1] Qing Ji,[2] and Yi-Zhong Zhuo[3,4]
[1]Department of Physics, Beijing Normal University, Beijing 100875, China
[2]Department of Physics, Hebei University of Technology, Tianjin 300130, China
[3]China Institute of Atomic Energy, P. O. Box 275, Beijing 102413, China
[4]Institute of Theoretical Physics, Chinese Academy of Science, Beijing 100080, China
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We propose a harmonic velocity noise with a broadband feature, which is the time derivative of the harmonic noise. If this noise is regarded as a thermal one, the system has a vanishing effective friction and it should induce ballistic diffusion of a free particle at long times. The effective temperature of the system coupled to such a structured heat bath represented by the harmonic velocity noise is introduced. This means that any initial preparation will approach asymptotically a preparation-dependent variance and mean value for velocity variable. Thus the fluctuation-dissipation theorem does not hold as there is no unique stationary state being connected with a breakdown of ergodicity. This noise can show greenness when it is taken as an external noise source to drive a correlation ratchet.

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I. INTRODUCTION

During the last few years substantial progress has been achieved towards an understanding of anomalous diffusive behavior of a system in disordered media. Most of the distributions of these systems in the stationary state are non-Gaussian, for instance, Lévy flights and Tsallis statistics [1]. Whether a thermal Gaussian colored noise produced by a linear stochastic differential equation subjected to a white noise will induce abnormal diffusion? This question needs to be addressed and investigated. There exists a very nice introductory and comprehensive review on colored noise phenomena [2]. As one knows that the limit of superdiffusion is ballistic diffusion, namely, the mean square displacement of a free particle increases with the square of time. The physical realistic Brownian motion for the mean square displacement exhibits a ballistic behavior at short times by expanding the exponential for short times [2]. Very recently, the effect of bacterial motion on micron-scale beads in a freely suspended soap film was reported [3], where the mean-square displacement of the particle shows ballistic diffusion for short time, however, normal diffusion is recovered in a long time limit. This is a result of transient formations of coherent structures in the bacterial bath. Ballistic diffusion has also been theoretically studied in Refs. [4–9]. One of the dynamical origins of anomalous diffusion is nonlocality in time and thus the velocity of the particle shows a memory effect, which results in a generalized Langevin equation (GLE) [10], however, the behaviors of the system at long times remain open.

The simple structured noise is the zero-mean Gaussian quasimonochromatic noise [also called the harmonic noise (HN)], which has a power spectrum having a narrow Lorentzian peak centered, not at zero frequency, but at a finite frequency and has widely been used [11–13]. It is the solution of a second-order stochastic differential equation driven by a Gaussian white noise. However, the time derivative of HN, i.e., the harmonic velocity noise (HVN), used as a random driving force is seldom considered and it can induce, as we will show in this paper, some interesting phenomena in the long time limit. We demonstrate in this work that the spectrum of HVN differs very much from that of HN. The former leads to ballistic diffusion when $t\to\infty$ and the terminal velocity of the particle does not vanish even without external force, while the latter just leads to the ones of normal diffusion. The physical situations for GLE with a HVN as an internal noise can be found for a vortex transport in the presence of magnetic field, and a particle interacting, via dipole coupling, with a blackbody radiation field [14] as well as in the presence of the velocity-dependent coupling [15]. Besides, a breakdown of ergodicity for HVN driven a Gaussian non-Markovian process will be discussed.

This paper is organized as follows. In Sec. II we introduce the harmonic velocity noise and write down its correlation function and spectral density. In Sec. III we report the theoretical and numerical results for a thermal HVN-driven system. In Sec. IV, we study a correlation ratchet driven by the HVN, which acts as an external noise, and a colored Brownian motion. Finally, we summarize the main results and give a brief conclusion in Sec. V.

II. THE HARMONIC VELOCITY NOISE

For our considered Gaussian noise processes the statistics is completely characterized by the power spectrum and the first moment [16]. In order to obtain these quantities, a Gaussian white noise $\xi(t)$ is applied, for example, as an input voltage to a RLC electric circuit, the electrical property is described by the following Langevin equation:

$$\dot{y}=z(t), \quad \dot{z}=-\Gamma z-\Omega_0^2 y+\xi(t), \tag{1}$$

where $\xi(t)$ has zero centered and the correlation function $\langle\xi(t)\xi(t')\rangle=2\alpha\delta(t-t')$ with α being the intensity of white

*Electronic mail: jdbao@bnu.edu.cn

noise, Γ and Ω_0 are the damping and frequency parameters, respectively. This describes clearly the Brownian motion of a simple harmonic oscillator of the thermal noise in a (R,L,C) circuit.

In Eq. (1), the voltage-variable $y(t)$ is the well-known Gaussian HN [11,12] with the following correlation function:

$$\langle y(t)y(t')\rangle = \frac{\alpha}{\mu_1^2-\mu_2^2}\left(\frac{1}{\mu_1}\exp(\mu_1|t-t'|) - \frac{1}{\mu_2}\exp(\mu_2|t-t'|)\right), \tag{2}$$

and the spectrum

$$S_{yy}(\omega) = \frac{2\alpha}{(\omega^2-\Omega_0^2)^2+\Gamma^2\omega^2}, \tag{3}$$

where μ_1 and μ_2 are two roots of the equation $\mu^2+\Gamma\mu+\Omega_0^2=0$.

Also in Eq. (1), the electrical current-variable $z(t)$ should be regarded as an output signal with a Gaussian distribution, which is called the harmonic velocity noise here. The correlation function for HVN with a stationary behavior (i.e., time-translation invariance at any time) is given by

$$\langle z(t)z(t')\rangle = \frac{\alpha}{\mu_1^2-\mu_2^2}[-\mu_1\exp(\mu_1|t-t'|) + \mu_2\exp(\mu_2|t-t'|)]. \tag{4}$$

At $t=0$, both y and z are assumed to obey two independent Gaussian distributions with $\langle z^2(0)\rangle=\alpha\Gamma^{-1}$, $\langle y^2(0)\rangle=\alpha\Gamma^{-1}\Omega_0^{-2}$, and $\langle z(0)y(0)\rangle=0$ (see Appendix A). The spectral density of HVN is the Fourier transform of the correlation function of the noise which is expressed as

$$S_{zz}(\omega) = \frac{2\alpha\omega^2}{(\omega^2-\Omega_0^2)^2+\Gamma^2\omega^2}, \tag{5}$$

where α will be defined in Appendix A. Note that now the spectral density of HVN is in proportion to the square of frequency.

When $\Omega_0^2\to 0$, $z(t)$ becomes the Ornstein-Uhlenbeck noise (OUN) with the correlation time $\tau=\Gamma^{-1}$ if we choose $\alpha=D\Gamma^2$ where D is the thermal diffusion coefficient; when $\Gamma\to\infty$, the above correlation function reduces further to a δ function and $z(t)$ becomes a white noise. Note that the low-frequency part of this noise has been removed $[S_{zz}(0)=0]$ and the high-frequency part decays $[S_{zz}(\omega\to\infty)\to 0]$, thus the spectrum (5) shows a broadband behavior. For the HN, however, its low-frequency part does not vanish [11,12], i.e., there is no ω dependent term appearing in the numerator of its spectral density $S_{yy}(\omega)$ $[S_{yy}(0)\neq 0]$. Due to this different spectrum from the one of the usual HN, it will lead to much different dynamical behaviors of the system in a long time limit.

All quantities plotted here and below are dimensionless. In Fig. 1, we plot the spectral densities of HN and HVN. The width of spectral density $S(\omega)$ is defined as

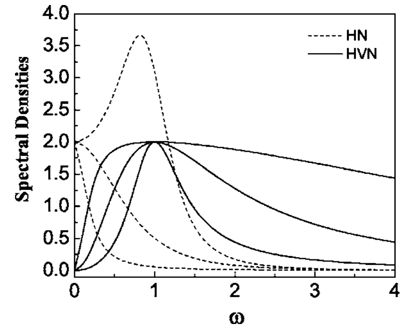

FIG. 1. The spectral densities of HN and HVN as functions of frequency. The parameters used are $\beta_w k_B T=1.0$, $\Omega_0^2=1.0$, and $\Gamma=0.8$, 2.0, 6.0 from top to bottom.

$$\Delta\omega := \frac{\int_0^\infty S(\omega)d\omega}{S(\omega_m)}, \tag{6}$$

where ω_m is the center frequency of the spectral density. For the HN, $\Delta\omega=\Omega_0^2\pi/(2\Gamma)$ when $2\Omega_0^2<\Gamma^2$ $(\omega_m=0)$ and $\Delta\omega=(4\Omega_0^2-\Gamma^2)\Gamma\pi/(8\Omega_0^2)$ when $2\Omega_0^2>\Gamma^2$ $(\omega_m=\sqrt{\Omega_0^2-\Gamma^2/2})$. For the HVN, $\Delta\omega=\Gamma\pi/2$ and $\omega_m=\Omega_0$, so the width spectral density of HVN increases with the increase of the damping parameter of the noise and which reduces approximately to the OUN when $\Gamma\to\infty$.

III. NON-MARKOVIAN FEATURES AND EFFECTIVE TEMPERATURE

The GLE describing the motion of a particle can be derived by using the Zwanzig-Mori projection method [17,18] and the system-plus-reservoir method [4,19,20]. The microscopic derivation of the equation for generalized Brownian motion in this context has been presented in Ref. [21], where the friction is generally state-dependent and becomes stationary only with an average over the initial probability of starting values. For the exception of a state independent friction in the GLE [18], a discussion of possible pitfalls and open problems is given in Ref. [22].

Now we show that physical systems might exhibit an internal broadband noise within the framework of GLE. The blackbody radiation field [14] and the magnetic force [15] are two examples. Here we focus on dynamical results of the equation with frequency-dependent but state-independent friction. We write the following GLE including a thermal HVN $z(t)$ for the system, i.e.,

$$\dot{x}(t) = v,$$

$$m\dot{v}(t) = -m\int_0^t \beta(t-t')v(t')dt' - U'(x) + z(t), \tag{7}$$

where m is the mass of the particle, $\beta(t)$ is the damping kernel due to HVN. In the Langevin formalism, the random force $z(t)$ is assumed to be uncorrelated to the initial velocity and must satisfy Kubo second fluctuation-dissipation theorem [23,24] expressed as $\langle z(t)z(0)\rangle=mk_BT\beta(t)$, where k_B is

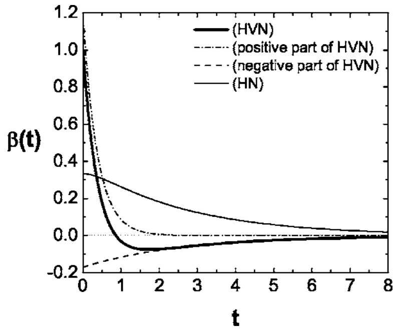

FIG. 2. The damping kernel functions vs time. The parameters used are $\Gamma=3.0$, $\Omega_0=1.0$, $\beta_0=3.0$, and $\beta_w=1.0$.

the Boltzmann constant and T is the temperature of the heat bath. In particular, the FDT is necessary, but not sufficient to yield the consistent thermal Brownian motion [25].

It is well known that the correlation function of HN has a decaying oscillation which becomes negative at some times in the underdamped case; elsewhere, which is positive in the overdamped case. However, the integration of damping kernel over time is always positive in both cases. In Fig. 2, we plot the damping kernel functions of HVN and HN in the overdamped case. For thermal HN, its damping kernel is positive at any time and the effective Markovian friction is defined by $\beta_0 \equiv \int_0^\infty \beta(t)dt = \alpha/(mk_BT\Omega_0^4)$ [12]; however, in sharp contrast to HN, this quantity is equal to zero for HVN. Besides, it is seen that the damping kernel function of thermal HVN starts out positive, crosses zero towards negative values, and assumes in the asymptotic long time limit zero from below.

In a case where the potential is absent, the solution of Eq. (7) can be obtained by the Laplace transform [26],

$$x(t) = x_0 + v_0 H(t) + \frac{1}{m}\int_0^t H(t-t')z(t')dt', \tag{8}$$

$$v(t) = v_0 \dot{H}_t(t) + \frac{1}{m}\int_0^t \dot{H}_t(t-t')z(t')dt', \tag{9}$$

where x_0 and v_0 are the initial position and velocity of the particle. The response function $H(t)$ is the inverse Laplace transform $\hat{H}(s)=[s^2+s\hat{\beta}(s)]^{-1}$, where $\hat{\beta}(s) = \int_0^\infty \beta(t)\exp(-st)dt$ is the Laplace transform of the damping memory kernel. Applying the residue theorem, we have the response function as

$$H(t) = \beta_w \Gamma^2 \Omega_0^{-4} b^2 + bt + \Xi(t) \tag{10}$$

with $b=\Omega_0^2/(\Omega_0^2+\beta_w\Gamma)$ and $\Xi(t)=(\nu_1-\nu_2)^{-1}[\sum_{j=1}^2(-1)^j\nu_j^{-2}\beta_w\Gamma\exp(\nu_j t)]$, where ν_1 and ν_2 are two roots of the equation $\nu^2+\Gamma\nu+\Omega_0^2+\beta_w\Gamma=0$ and the real parts of both two roots are negative. Here we have chosen $D=\beta_w k_B T$, where β_w is the friction coefficient of the corresponding thermal white noise used to drive HVN.

When the driving noise is internal, one can get a more convenient form [27] of the mean-square displacement of the free particle, which is given by

$$\langle x^2(t)\rangle = [x_0 + v_0H(t)]^2 + \frac{k_BT}{m}\left(2\int_0^t H(t')dt' - H^2(t)\right). \tag{11}$$

If the particle starts from the origin of coordinate $(x_0=0)$, we have

$$\begin{aligned}\{\langle x^2(t)\rangle\} = {} & \left[\frac{k_BT}{m}b + \left(\{v_0^2\} - \frac{k_BT}{m}\right)b^2\right]t^2 + \frac{k_BT}{m}\left(2\frac{\beta_w\Gamma^2}{\Omega_0^4}b^2 t \right. \\ & \left. + 2\int_0^t \Xi(t')dt'\right) + \left(\{v_0^2\} - \frac{k_BT}{m}\right)\left[\left(\frac{\beta_w\Gamma^2}{\Omega_0^4}b^2\right)^2 \right. \\ & \left. + 2\frac{\beta_w\Gamma^2}{\Omega_0^4}b^3 t + 2b\left(\frac{\beta_w\Gamma^2}{\Omega_0^4}b + t\right)\Xi(t) + \Xi^2(t)\right].\end{aligned} \tag{12}$$

Here $\{\cdots\}$ denotes the average with respect to the initial states of the particle and might differ from the ensemble average $\langle\cdots\rangle$.

When $t\to\infty$, $\{\langle x^2(t)\rangle\} \propto \{(k_BT/m)b + [\{v_0^2\}-(k_BT/m)]b^2\}t^2$, so that the proposed HVN can induce ballistic diffusion in a long time limit. This motion is also called a fast superdiffusion [9], furthermore, we call here the process in the presence of a potential as a fast non-Markovian one. Note that when $\Gamma\Omega_0^{-2}\to\infty$ and then $b\to 0$, the term $2\beta_w\Gamma^2\Omega_0^{-4}b^2t \to 2\beta_w^{-1}t$ in Eq. (12), the present process reduces to the normal diffusive one. Indeed, the ballistic diffusion is the limit of superdiffusion and is also an intermediate result between the internal Gaussian white noise case and the external white noise without friction case. The asymptotical mean-square displacement of a free particle is proportional to t for the former and t^3 for the latter.

The other non-Markovian features for HVN-driven particle in long time limit are found as the following:

$$\langle v(t\to\infty)\rangle = v_0 b \tag{13}$$

and

$$\{\langle v^2(t\to\infty)\rangle\} = \frac{k_BT}{m} + b^2\left(\{v_0^2\} - \frac{k_BT}{m}\right). \tag{14}$$

It is seen from Eq. (13) that the asymptotic mean velocity does not vanish and from Eq. (14) that an additional term appears, however, which is absent in normal Brownian motion. Only when the initial state of the particle is also chosen to be a thermal equilibrium state, i.e., $\{v_0^2\}=k_BT/m$ [24], namely, the initial velocity of the particle obeys a Gaussian distribution with zero mean and variance k_BT/m. In this case, the second moment of the velocity can arrive at its equilibrium value. This means that the breakdown of the FDT being connected with the breakdown of ergodicity as there is no unique stationary state for this ballistic diffusion. Here the parameter b measures the nonergodicity strength, it is the source of the issue of violation of ergodicity.

When the initial distribution of the particle is not in the assumed equilibrium state, an effective temperature for the system needs to be introduced

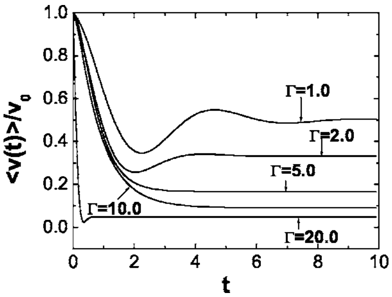

FIG. 3. Time-dependent average velocity of the free particle related to its initial velocity for various Γ in the absence of external driving force. The parameters used are $\beta_w=1.0$ and $\Omega_0=1.0$.

$$\frac{T_{\mathrm{eff}}}{T}=1+b^2\left(\frac{T_0}{T}-1\right), \qquad (15)$$

where $T_{\mathrm{eff}}=\langle v^2(t\to\infty)\rangle m/k_B$ and $T_0=\{v_0^2\}m/k_B$. From Eq. (15) it is seen obviously that $T_{\mathrm{eff}}<T$ when $T_0<T$; $T_{\mathrm{eff}}>T$ when $T_0>T$; and $T_{\mathrm{eff}}=T$ when $T_0=T$. Moreover, if $\Gamma\gg\Omega_0$, $T_{\mathrm{eff}}=T$, which is just the result of a normal thermal noise-driven system with a finite low-frequency Markovian friction.

In Fig. 3, the average velocity $\langle v(t)\rangle$ of the particle related to its fixed initial value v_0 is plotted as a function of time for various values of the damping parameter Γ of HVN. It is shown that the asymptotic velocity of the particle does not relax towards zero and decreases as Γ increases. At the beginning stage, the average velocity of the particle decreases partly due to a positive friction and then keeps a finite value when the friction becomes negative. The initial velocity of the particle is dissipated partly by anomalous memory friction. This means that the dissipative dynamics in the presence of ballistic diffusion differs from normal Brownian motion.

It is possible to reformulate Eq. (7) into a set of Markovian Langevin equations through introducing variable transformations [28] (see Appendix B). We have a set of Markovian Langevin equations written as

$$\dot{x}=v,$$

$$m\dot{v}=-U'(x)+w,$$

$$\dot{w}=-\Gamma w-\beta_w\Gamma v-\Omega_0^2 y-u+\xi(t),$$

$$\dot{u}=\Omega_0^2(w-z),$$

$$\dot{y}=z,$$

$$\dot{z}=-\Gamma z-\Omega_0^2 y+\xi(t), \qquad (16)$$

where $\xi(t)$ is a zero-mean Gaussian white noise with $\langle\xi(t)\xi(s)\rangle=2\beta_w\Gamma^2 k_B T\delta(t-s)$. The above equations with $U(x)=0$ define our embedding, thus the process $x(t)$ is completely decoupled and can be ignored now. For a Gauss-Markov process in five dimensions $X(t)=(v(t),w(t),u(t),y(t),z(t))$ is discussed in detail in Appendix B.

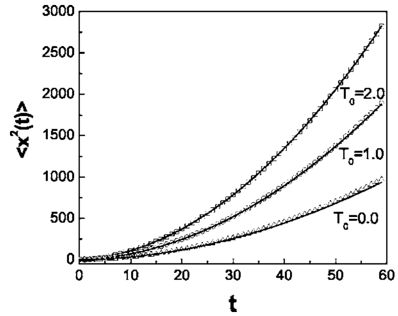

FIG. 4. Calculated mean square displacement of a free particle for various initial temperatures T_0. The solid lines and open points are analytical and numerical results, respectively. The parameters used are $T=1.0$, $\beta_w=0.2$, $\Gamma=5.0$, and $\Omega_0=1.0$.

We can simulate a set of Markovian Langevin equations (16) by using the Monte Carlo method. In order to produce a stationary process for the present thermal colored noise, the distributions of all noise variables at initial time are chosen to be independent Gaussian functions with zero mean and variances $\{y_0^2\}=\beta_w\Gamma k_B T/\Omega_0^2$, $\{z_0^2\}=\beta_w\Gamma k_B T$, $w_0=z_0$, and a δ function for $u_0=0$.

In Fig. 4, we show the calculated result for the mean-square displacement of a free particle with the initial velocity obeying a Gaussian distribution, $P(v_0)=(2\pi\{v_0^2\})^{-1/2}\exp[-v_0^2/(2\{v_0^2\})]$ where $\{v_0^2\}=k_B T_0/m$. The theoretical result is also plotted in the same figure by using the expression (12). It is observed that the mean-square displacement of the particle is proportional to t^2 and shows a parabolic behavior, namely, the ballistic diffusion appears in the long time limit. The theoretical results are in agreement with the numerical simulation. Moreover, the asymptotical result for a thermal HVN-driven system is sensitive to the initial condition. This implies that non-Markovian effect with a vanishing effective friction influences not only on the transient process but also on the long-time result for a system of this kind.

The explicit velocity correlation function (VCF) of a free particle is derived and given by

$$\begin{aligned}\{\langle v(t)v(s)\rangle\}=&\{v_0^2\}b^2+\frac{k_BT}{m}\eta\Gamma\left(\frac{b^2}{\Omega_0^2}+\frac{B}{\nu_2}\exp(\nu_2|t-s|)\right.\\&\left.-\frac{B}{\nu_1}\exp(\nu_1|t-s|)\right)+\left(\{v_0^2\}-\frac{k_BT}{m}\right)\\&\times B(b\{C[\exp(\nu_1 t)+\exp(\nu_1 s)]-D[\exp(\nu_2 t)\\&+\exp(\nu_2 s)]\}+BC^2\exp[\nu_1(t+s)]\\&+BD^2\exp[\nu_2(t+s)]-BCD[\exp(\nu_1 t+\nu_2 s)\\&+\exp(\nu_2 t+\nu_1 s)]),\end{aligned} \qquad (17)$$

where $B=(\nu_1-\nu_2)^{-1}$, $C=(\nu_1^2+\Gamma\nu_1+\Omega_0^2)/\nu_1$, and $D=(\nu_2^2+\Gamma\nu_2+\Omega_0^2)/\nu_2$. It can be seen that the distribution of the velocity does not become stationary at any time, if the second moment of initial velocity of the particle is not chosen to be the value at the equilibrium state. As soon as $\{v^2(0)\}=k_BT/m$, all the aging terms in Eq. (17) vanish, the VCF

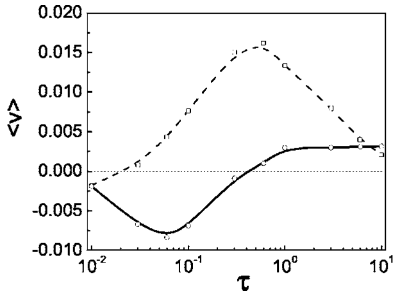

FIG. 5. The steady mean velocity of the particle versus the correlation time of noise τ. The parameters used are $D=0.1$, $Q=0.25$, and $\Gamma=20$. The solid and dashed lines are the results of HVN and HN [29], respectively.

becomes stationary, so that the VCF depends on the time difference only. In this case, the time derivative of the VCF at $t=0$ is equal indeed to zero, $(d/dt)[\langle v(t)v(0)\rangle]_{t=0}=0$.

Especially for the harmonic potential $U(x)=\frac{1}{2}m\omega_0^2x^2$ with $m\omega_0^2>0$, the problem is still exactly solvable for Eq. (7), the results for the second moments of coordinate and velocity are shown in Appendix C. It is noticed that the mean square displacement of the particle remains now bounded, but will be independent of the initial conditions. Thus both the energy balance theorem and the ergodicity are obeyed in this case.

IV. HVN DRIVEN CORRELATION RATCHET

For general references for ratchets and Brownian motors see Refs. [29–35]. Here we just discuss the case when the proposed HVN $z(t)$ is used as an external noise with intensity Q to drive a correlation ratchet [29], i.e.,

$$\dot{x}(t) = -U'(x) + \sqrt{2D}\xi_1(t) + z(t), \tag{18}$$

where $U(x)$ is a ratchet potential and chosen to be

$$U(x) = -\frac{1}{2\pi}[\sin(2\pi x) + 0.25\sin(4\pi x)]. \tag{19}$$

In Eq. (18), $\xi_1(t)$ is a white noise with $\langle\xi_1(t)\rangle=0$ and $\langle\xi_1(t)\xi_1(s)\rangle=\delta(t-s)$, which is independent of the white noise $\xi(t)$ in $z(t)$. In the case of small noise intensity, noise-induced steady average velocity is determined by

$$\langle v\rangle = A\{\exp[-F''(0)c^{+}/D] - \exp[-F''(0)c^{-}/D]\} \tag{20}$$

with $F(\omega)=[1+RS_{zz}(\omega)]^{-1}$, and $c^{\pm}=\int_{x_0}^{x\pm}dxU'(x)[U''(x)]^2$, where A is the Arrhenius-Kramers factor, $R=Q/D$, and x_0 is the minimum of the ratchet potential with the left (right) maxima $x^{-}(x^{+})$. The quantities $c^{\pm}$ are positive and $c^{-}>c^{+}$ for a forward ratchet potential [29], thus $\langle v\rangle>0$ or $\langle v\rangle<0$ when $F''(0)>0$ or $F''(0)<0$. For HN, $F''(0)=4R\Omega_0^{-2}(\Gamma^2/\Omega_0^2-2)/(1+2R)^2$; for HVN, $F''(0)=-4R\Gamma^2\Omega_0^{-4}<0$. It is found that HVN-induced flow is always negative, which is opposite to that of OUN. The former shows the greenness [30,31] and the latter shows the redness.

In Fig. 5, we calculate numerically the steady average velocity of the particle via Monte Carlo simulation for Eq. (18). It is compared with the result induced by a Gaussian white noise plus a harmonic noise $y(t)$ [29]. The steady average velocity $\langle v\rangle$ of the particle is plotted as a function of the noise parameter $\tau=\Gamma/\Omega_0^2$. For the HN case, the direction of motion of the particle occurs reversal when $\tau<2/\Gamma$ [29]; however, for the HVN case, the particle changes its direction of the motion when τ becomes large, i.e., Ω_0^2 becomes small, because the harmonic velocity noise $z(t)$ reduces to the Ornstein-Uhlenbeck noise in Eq. (18). It is concluded that $y(t)$ shows redness and $z(t)$ can show greenness in the case of small-to-medium τ.

Note that in the case of inertia ratchet with coexisting regular attractors which means nonergodicity in phase space, the diffusion behavior emerges also as being ballistic with a second moment that grows proportional to t^2 [36]. These inertial ratchet trajectories thus seems to mimic the behavior of free, nonergodic Brownian motion behavior in the absence of a potential.

V. SUMMARY

We have presented a solvable non-Markovian Gaussian model for understanding anomalous features of noise-driven systems at long times. A harmonic velocity noise is proposed here; its spectrum differs very much from that of the harmonic noise. The effective Markovian friction of the system is equal to zero due to the spectral density with the vanishing zero frequency. This results in ballistic diffusion and nonvanishing asymptotic velocity of a free particle at long times, the second moment of the velocity can arrive at the equilibrium value only when the initial velocity distribution of the particle is assumed to be the thermal equilibrium state, otherwise, the system exists as an effective temperature. This means in general FDT does not hold as there is no unique stationary state for the embedded process which is connected with a breakdown of ergodicity. When this noise is used as an external noise, which can show greenness in comparison with the Ornstein-Uhlenbeck noise in the cases of small-to-medium correlation times, to drive a correlation ratchet system. Our model can be generalized to a j-order derivative of the solution of a n-order linear stochastic differential equation subjected to a Gaussian white noise, $1\leqslant j\leqslant n$, which is regarded as a thermal noise and will also induce ballistic diffusion. The transport driven by a harmonic velocity noise or a derivative noise might be applied to various problems in the future.

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APPENDIX A: DETERMINATION OF PARAMETERS OF HARMONIC VELOCITY NOISE

We write the "velocity" variable for the solution of Eq. (1) as

$$z(t) = a_{10}\mu_1 e^{\mu_1 t} + a_{20}\mu_2 e^{\mu_2 t} + \frac{1}{\mu_1 - \mu_2}\int_0^t (\mu_1 e^{\mu_1(t-t')} - \mu_2 e^{\mu_2(t-t')})\xi(t')dt', \quad (A1)$$

where the two coefficients are given by

$$y(0) = a_{10} + a_{20}, \quad z(0) = \mu_1 a_{10} + \mu_2 a_{20}. \quad (A2)$$

The correction function of the noise variable $z(t)$ is given by

$$\begin{aligned}\langle z(t)z(s)\rangle &= \frac{\alpha}{\mu_1^2 - \mu_2^2}[-\mu_1 \exp(\mu_1|t-s|) + \mu_2 \exp(\mu_2|t-s|)] \\ &+ \left(\mu_1^2\langle a_{10}^2\rangle + \frac{\mu_1\alpha}{(\mu_1-\mu_2)^2}\right)\exp[\mu_1(t+s)] \\ &+ \left(\mu_2^2\langle a_{20}^2\rangle + \frac{\mu_2\alpha}{(\mu_1-\mu_2)^2}\right)\exp[\mu_2(t+s)] \\ &+ \mu_1\mu_2\left(\langle a_{10}a_{20}\rangle - \frac{2\alpha}{(\mu_1+\mu_2)(\mu_1-\mu_2)^2}\right) \\ &\times (e^{\mu_1 t+\mu_2 s} + e^{\mu_2 t+\mu_1 s}). \end{aligned} \quad (A3)$$

In order to produce the stationary correction function of the noise at any time, namely, it is only a function of $|t-s|$, we let the latter three terms equal zero, so

$$\langle a_{10}^2\rangle + \frac{\alpha}{\mu_1(\mu_1-\mu_2)^2} = 0,$$

$$\langle a_{20}^2\rangle + \frac{\alpha}{\mu_2(\mu_1-\mu_2)^2} = 0,$$

$$\langle a_{10}a_{20}\rangle - \frac{2\alpha}{(\mu_1+\mu_2)(\mu_1-\mu_2)^2} = 0. \quad (A4)$$

Those lead to $\langle y^2(0)\rangle = \alpha/(\Gamma\Omega^2)$, $\langle z^2(0)\rangle = \alpha/\Gamma$, and $\langle y(0)z(0)\rangle = 0$. We assume that the initial distribution of the two noise variables y and z are the Gaussian ones with the above variances.

It is known that $\mu_1 = (-\Gamma + \sqrt{\Gamma^2 - 4\Omega^2})/2$ and $\mu_2 = (-\Gamma - \sqrt{\Gamma^2 - 4\Omega^2})/2$ from the equation below Eq. (3), and thus $\mu_1 = 0$ and $\mu_2 = -\Gamma$ when $\Omega \to 0$, so $\langle z(t)z(s)\rangle = \alpha/\Gamma \exp(-\Gamma|t-s|)$. This means that the harmonic velocity noise can reduce to the OUN in the limit of $\Omega \to 0$, the parameter α needs to obey the condition $\alpha/\Gamma = D\Gamma$, i.e., $\alpha = \Gamma^2 Q$. This noise can reduce further to a white noise $\langle z(t)z(s)\rangle = 2D\delta(t-s)$ when $\Gamma \to \infty$. Therefore, $D = \beta_w k_B T$ for a thermal white noise, where β_w is the friction strength of the system in this case. According to the above idea, $\alpha = \Omega^4 \beta_w k_B T$ for obtaining the harmonic noise [11,12].

APPENDIX B: FIVE-DIMENSIONAL GAUSS-MARKOV PROCESS

We introduce two auxiliary variables as follows:

$$w(t) = -m\int_0^t \beta(t-s)v(s)ds + z(t),$$

$$u(t) = \int_0^t [\dot{\beta}(t-s) + \Gamma\beta(t-s)]v(s)ds. \quad (B1)$$

In order to make stochastic variables become stationary, namely, their correlation functions have the behavior of time-translation invariance at any time, we assume that the initial distributions of $y(0)$ and $z(0)$ are Gaussian with the variance determined by Section II, $w(0)$ is the same as $z(0)$ and $u(0) = 0$.

In the absence of potential, a set of equations (16) describe a five-dimensional linear Gauss-Markov process. The stationary solutions of the auxiliary variables and correlations among them with the velocity variable are given by

$$\langle y^2\rangle_{st} = \frac{\beta_w\Gamma}{\Omega_0^2}k_B T, \quad \langle z^2\rangle_{st} = \beta_w\Gamma k_B T,$$

$$\langle yz\rangle_{st} = 0,$$

$$\langle yv\rangle_{st} = 0, \quad \langle zv\rangle_{st} = 0,$$

$$\langle w^2\rangle_{st} = \frac{2m\beta_w^3\Gamma k_B T}{4\Omega_0^2 + \beta_w^2 + 2\beta_w\Gamma},$$

$$\langle u^2\rangle_{st} = m\Omega_0^4\left((1-b^2)k_B T + (1-b)^2 k_B T_0 + \frac{2\beta_w\Gamma}{\Omega_0^2}k_B T\right),$$

$$\langle uv\rangle_{st} = \Omega_0^2[(1-b^2)k_B T - b(1-b)k_B T_0],$$

$$\langle wv\rangle_{st} = 0. \quad (B2)$$

Setting $\Omega_0^2 = 0$, however, which yields an unique distribution of Boltzmann form with a temperature T, because the HVN reduces to the OUN in this case. This process is usually a Markovian one with a behavior of normal diffusion. Indeed, this non-Markovian process with $\beta_w = 0$ (without friction and thermal fluctuation) is the result of a contraction of a nonergodic five-dimensional Gauss-Markov process. This leads to $b = 1$ and then $\langle v^2(t\to\infty)\rangle = \{v_0^2\}$, which results in different stationary states corresponding to different initial states. Thus it is rather natural that any initial preparation will approach asymptotically a preparation-dependent variance and mean value.

APPENDIX C: HARMONIC POTENTIAL CASE

For the harmonic potential $U(x) = \frac{1}{2}m\omega_0^2 x^2$, the solutions of Eq. (7) in terms of the Laplace transform technique become

$$x(t) = \left(1 - \omega_0^2\int_0^t H(s)ds\right)x_0 + H(t)v_0 + \int_0^t H(t-s)z(s)ds, \quad (C1)$$

$$v(t) = v_0 h(t) - \omega_0^2 x_0 H(t) + \int_0^t h(t-s)z(s)ds, \quad (C2)$$

where $H(t)$ and $h(t)$ are the inverse transformations of $\hat{H}(p) = [p^2 + p\hat{\beta}(p) + \omega_0^2]^{-1}$ and $\hat{h}(p) = p[p^2 + p\hat{\beta}(p) + \omega_0^2]^{-1}$, respectively. Thus the second moments of the coordinate and velocity are given by

$$\langle x^2(t)\rangle = x_0^2 + \left(v_0^2 - \frac{k_BT}{m}\right)H^2(t) + 2x_0v_0H(t) \times\left(1-\omega_0^2\int_0^t H(s)ds\right) + \left(\omega_0^2x_0^2 - \frac{k_BT}{m}\right) \times\int_0^t H(s)ds\left(\omega_0^2\int_0^t H(s)ds - 2\right) \quad (C3)$$

and

$$\langle v^2(t)\rangle = \frac{k_BT}{m} - 2x_0v_0\omega_0^2H(t)h(t) + \left(v_0^2 - \frac{k_BT}{m}\right)h^2(t) + \left(\omega_0^2x_0^2 - \frac{k_BT}{m}\right)\omega_0^2H^2(t). \quad (C4)$$

It is known that the distribution of the particle in the harmonic potential is asymmetrically in the long time limit, namely, $\langle x(t\to\infty)\rangle=0$, this leads to $\int_0^\infty H(s)ds=\omega_0^{-2}$. Therefore, the asymptotic results for the second moments of coordinate and velocity of the particle are $\{\langle x^2\rangle\}_{\rm st}=k_BT/(m\omega_0^2)$ and $\{\langle v^2\rangle\}_{\rm st}=k_BT/m$. This implies that the initial conditions do not influence upon the results in the stationary state in the bounded phase space.

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