Convective instability of magnetic fluids under alternating magnetic fields

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A theoretical investigation of the convective instability problem in the thin horizontal layer of a magnetic fluid heated from below and under alternating magnetic fields is carried out. Both the quasistationary model and the model with internal rotation with vortex viscosity are considered. Floquet theory is used for discussing the existence and stability boundaries of the differential equations with periodic coefficients. The Chebyshev pseudospectral method is employed to discretize the partial differential equation, and QZ algorithm is used for solving the eigenvalue problem. For quasistationary model, both free-free and rigid-rigid boundary cases are considered, whereas for the model with internal rotation only rigid-rigid boundary condition is studied. The effect of frequency variations on the stability are considered in all the cases.

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I. INTRODUCTION

Magnetic fluids are colloidal suspensions of fine stable nanoscale magnetic particles in nonconducting liquids. The particles are coated with a surfactant, whose dielectric property matches with the carrier fluid, to prevent particles agglomeration and coagulation. This coating also allows magnetic fluids to maintain fluidity even at very high magnetic fields. Brownian motion, on the other hand, keeps the nanoparticles from settling under gravity or in an external field. These fluids have found numerous applications (e.g., in loudspeakers, rotatory and exclusion seals, bearings, dampers, shock absorbers, medicine drug targeting, and in many thermal transport applications [1–3].

Based on the consideration of particle-particle interaction and the nature of magnetization relaxation, several forms of basic equations, for magnetic fluids, have been proposed in the literature. We feel that these can, roughly, be classified into three categories. The earliest and foremost form is known as quasistationary theory (also called ferrohydrodynamics) [4], which has a wide range of applications [1–3] and has, in some sense, guided further research in the area of magnetic fluids. This theory, however, assumes that the magnetization is collinear with the applied field at all times. The second category theory provides the allowance for the internal rotation of the nanoparticles and does not assume the collinearity between magnetization and the magnetic field. Instead it assumes the particles to relax following Brownian relaxation and thus proposes a magnetization relaxation equation [1,5,6]. Apart from some applications, similar to those of [4], this second theory has been successful in explaining the magnetoviscous effects in magnetic fluids. Thus it has been successful in theoretically describing the experimental results of rotational viscosity (increase in fluid viscosity with the increase of magnetic field) observed by McTague [7]. This theory also has explained the negative viscosity phenomenon (reduction of viscosity with increase in frequency in an alternation magnetic field) [8,9] again observed and verified experimentally [10,11]. The theory, since it is based on single-particle calculations and does not consider interaction between particles, is thus considered to be useful for dilute magnetic fluids. The third-category equations, considering particle-particle interactions, magnetization relaxation, internal rotation, and asymmetric stress, have also been suggested by several authors. Although there have been attempts by various writers [1,12,13] to put forth suitable equations, Rosensweig [14] has recently developed a complete set of basic equations derived on the basis of dynamic balance laws with a dissipation function determined from thermodynamic considerations. This set of equations is quite general and includes almost all other continuum theories previously published in all the categories.

In the present paper, we consider the convective instability problem under an alternating magnetic field. This work is thus complementary to our previous study [15]. In the present case, however, since the differential equations are with periodic coefficients, we employ FLOQUET theory to provide stability boundaries. This leads to finding the eigenvalues of a momodromy matrix. To solve the eigenvalue problem we employ the Chebyshev pseudospectral method. We remark that convective stability problem in magnetic fluids have been studied in a variety of situations and with different degrees of success. In [15], we have provided some references that mostly deal with dc magnetic fields, whether the magnetic fluid is considered as single-component fluid or a binary mixture. Other related references may also be found in [1–3].

Here we consider both the quasistationary model and the model of the magnetic fluid with internal rotation. In the case of the quasistationary model we employ both free-free and rigid-rigid boundary conditions. A qualitative analysis, in the case of free-free boundary predicts the possibility of the phenomenon of parametric resonance. In the model with internal rotation we only consider the rigid-rigid boundary condition. Here we find a sudden increase in the Rayleigh number values after an increase of the frequency over a certain value. The effect on magnetic Rayleigh number is also significant, and although the buoyancy Rayleigh number remains bounded with further increase of frequency, the magnetic Rayleigh number continues to grow as the frequency is increased.

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II. BASIC EQUATIONS

We employ the equations governing the flow of an incompressible magnetic fluid, as given in [1,5,6]

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1}$$

$$\rho \frac{D\boldsymbol{u}}{Dt} = -\nabla P' + (\eta + \xi_r)\nabla^2 \boldsymbol{u} + \mu_0(\boldsymbol{M}\cdot\nabla)\boldsymbol{H} + 2\xi_r(\nabla \times \boldsymbol{\omega}) - \rho g \boldsymbol{k}, \tag{2}$$

$$\rho_0 I \frac{D\boldsymbol{\omega}}{Dt} = 2\xi_r(\nabla \times \boldsymbol{u} - 2\boldsymbol{\omega}) + \mu_0 \boldsymbol{M}\times\boldsymbol{H}, \tag{3}$$

$$\frac{D\boldsymbol{M}}{Dt} = \boldsymbol{\omega}\times\boldsymbol{M} - \frac{1}{\tau_m}(\boldsymbol{M} - \boldsymbol{M}_{eq}) \tag{4}$$

and

$$\left[\rho C_{V,H} - \mu_0 \boldsymbol{H}\cdot\frac{\partial \boldsymbol{M}}{\partial T}\right]\frac{DT}{Dt} + \mu_0 T\left(\frac{\partial \boldsymbol{M}}{\partial T}\right)_{V,H}\cdot\frac{D\boldsymbol{H}}{Dt} = K_t\nabla^2 T + \Phi. \tag{5}$$

Here $\boldsymbol{u}=(u,v,w)$ is the velocity, $D/Dt=\partial/\partial t+\boldsymbol{u}\cdot\nabla$, ρ the density, $P'=P+\frac{1}{2}\mu_0 H^2$, $\boldsymbol{\omega}$ is the average spin velocity of colloidal particles, η is the viscosity of carrier fluid, ξ_r is the vortex (rotational) viscosity, $\boldsymbol{H}$ is the magnetic field, $\boldsymbol{M}$ is the magnetization, $\boldsymbol{M}_{eq}$ is equilibrium magnetization, μ_0 is magnetic permeability (in free space $\mu_0=4\pi\times10^{-7}$ H/m), ρI is the average moment of inertia of the colloidal particles per unit volume, τ_m is the Brownian relaxation time, T is the temperature, $C_{V,H}$ is the specific heat capacity at constant volume and magnetic field, K_t is the thermal conductivity, and Φ is the viscous dissipation. Maxwell's equations in the magnetostatic limit are

$$\nabla\cdot\boldsymbol{B}=0, \quad \nabla\times\boldsymbol{H}=\boldsymbol{0}, \quad \boldsymbol{B}=\mu_0(\boldsymbol{M}+\boldsymbol{H}). \tag{6}$$

The density variation, on assuming Boussinesq approximation, is given as

$$\rho g = \rho_0 g[1 - \alpha(T - T_a)], \tag{7}$$

where α is the thermal expansion coefficient and T_a is average temperature. On neglecting the inertia of the colloidal suspended particles, we can write (3) as

$$\boldsymbol{\omega} = \frac{\mu_0}{4\xi_r}\boldsymbol{M}\times\boldsymbol{H} + \frac{1}{2}(\nabla\times\boldsymbol{u}). \tag{8}$$

On substituting (7) and (8) into (2) and (4), we get

$$\rho\frac{D\boldsymbol{u}}{Dt} = -\nabla P' + \eta\nabla^2\boldsymbol{u} + \mu_0(\boldsymbol{M}\cdot\nabla)\boldsymbol{H} + \frac{\mu_0}{2}\nabla\times(\boldsymbol{M}\times\boldsymbol{H}) - \rho_0 g[1-\alpha(T-T_a)]\boldsymbol{k}, \tag{9}$$

$$\frac{D\boldsymbol{M}}{Dt} = \frac{1}{2}(\nabla\times\boldsymbol{u})\times\boldsymbol{M} - \frac{1}{\tau_m}(\boldsymbol{M}-\boldsymbol{M}_{eq}) - \frac{\mu_0}{4\xi_r}\boldsymbol{M}\times(\boldsymbol{M}\times\boldsymbol{H}). \tag{10}$$

We consider a horizontal layer of an incompressible magnetic fluid heated from below. A Cartesian coordinate system (x,y,z) is used with the z axis normal to the layer that is confined between horizontal plates $z=-\frac{1}{2}d$ and $z=\frac{1}{2}d$. A constant temperature gradient is maintained between the plates. The temperature boundary conditions, thus, are

$$T=T_0 \text{ at } z=\tfrac{1}{2}d, \quad T=T_1 \text{ at } z=-\tfrac{1}{2}d, \quad T_a=\tfrac{1}{2}(T_0+T_1). \tag{11}$$

For velocity we employ no slip boundary conditions, $\boldsymbol{u}=0$ on the rigid plates. An alternating uniform magnetic field is applied normal to the plates. The magnetic boundary conditions are that the tangential component of the magnetic field and normal component of magnetic induction are continuous across the boundary. The latter gives

$$H_3 + M_3 = H^{\text{ext}}\cos(\Omega_a t), \tag{12}$$

where the right-hand side of Eq. (12) represents the external field.

III. QUASISTATIONARY MODEL

The convective instability problem in a steady field using this model has been studied by Finlayson [16] and Stiles and Kagan [17]. In the absence of (ξ_r, τ_m), from Eq. (4), we obtain

$$\boldsymbol{M} = \boldsymbol{M}_{eq} = \frac{\boldsymbol{H}}{H}M_s L(\alpha_L) = \frac{\boldsymbol{H}}{H}M_{eq}(H,T), \tag{13}$$

$$L(\alpha_L) = \coth(\alpha_L) - \frac{1}{\alpha_L}, \quad \alpha_L = \frac{\mu_0 m H}{k_b T}. \tag{14}$$

In the limit of low magnetic field (i.e., $\alpha_L \ll 1$), χ, the susceptibility, is constant and is given as

$$\chi = \frac{\mu_0 m M_s}{3k_B T_a} = \frac{\mu_0 m^2 N}{3k_B T_a}. \tag{15}$$

Here m is the magnetic moment of the single particle, $M_s=mN$, is the saturation magnetization, α_L is Langevin parameter, N is the number of magnetic dipoles per unit volume, $k_B=1.38\times10^{-23}$ J/K^{-1} is the Boltzman constant.

In the quiescent state, following Finlayson [16], we express

$$M_{eq} = M_a + \chi(H - H_a) - K_m(T - T_a), \tag{16}$$

where $\chi=(\partial M/\partial H)_{H_a,T_a}$ is tangent magnetic susceptibility and $K_m=-(\partial M/\partial T)_{H_a,T_a}$ is called the pyromagnetic coefficient. In our case, $H_1 \ll H_3$, $H_2 \ll H_3$, and $H_3 \approx H_a = H_0\cos(\Omega_a t)$, and simplifying Eq. (16), it gives

$$M_{eq1} = \chi H_1,$$

$$M_{eq2} = \chi H_2,$$

$$M_{eq3} = \chi H_0\cos(\Omega_a t) + \chi[H_3 - H_0\cos(\Omega_a t)] - \frac{\chi}{T_a}H_0\cos(\Omega_a t)(T - T_a), \tag{17}$$

The quiescent state solution of the basic equations is given to be

$$\boldsymbol{u}^s = \boldsymbol{0}, \quad T^s = T_a - \beta z, \tag{18}$$

$$\boldsymbol{H}^s = \boldsymbol{k}\left(1 - \frac{\chi\beta z}{T_a(1+\chi)}\right)H_0\cos(\Omega_a t), \tag{19}$$

$$\boldsymbol{M}^s = \boldsymbol{k}\left(1 + \frac{\beta z}{T_a(1+\chi)}\right)\chi H_0\cos(\Omega_a t),$$

$$H^s + M^s = (1+\chi)H_0\cos(\Omega_a t) = H_0^{\rm ext}\cos(\Omega_a t), \tag{20}$$

$$P^s = -\rho g z - \frac{1}{2}\rho g\alpha\beta z^2 - \frac{\mu_0\beta\chi^2H_0^2}{2T_a(1+\chi)}[1+\cos(2\Omega_a t)]z - \frac{\mu_0\beta^2\chi^2H_0^2}{4T_a^2(1+\chi^2)^2}[1+\cos(2\Omega_a t)]z^2. \tag{21}$$

To study the linear stability of the above solution, we now perturb the variables appearing in the above equations. On denoting the perturbation variables by primes, we write

$$[u,v,w,M_x,M_y,M_z,H_x,H_y,H_z,P,\theta]^T = [0,0,0,0,0,M_3^s,0,0,H_3^s,P^s,T^s]^T + [u',v',w',M_x',M_y',M_z',H_x',H_y',H_z',P',\theta']^T. \tag{22}$$

At this point, it is convenient to introduce the following dimensionless quantities:

$$\boldsymbol{x}^* = \frac{\boldsymbol{x}}{d}, \quad t^* = \frac{K_t}{\rho C_{V,H}d^2}t, \quad \theta^* = \frac{1}{\beta d}\theta', \quad P^* = \frac{d^2}{\eta K_\theta}P',$$

$$\boldsymbol{u}^* = \frac{d}{K_\theta}\boldsymbol{u}', \quad \boldsymbol{M}^* = \frac{1}{\chi H_0}\boldsymbol{M}', \quad \boldsymbol{H}^* = \frac{1}{H_0}\boldsymbol{H}', \tag{23}$$

where the parameters Pr, Rg, M_1, M_2, M_3, M_4, χ, ξ, and τ are defined as

$$\Pr = \frac{\eta C_{V,H}}{K_t}, \quad \mathrm{Rg} = \frac{\rho^2 g\alpha\beta C_{V,H}d^4}{\eta K_t}, \quad \tau = \frac{K_t}{\rho C_{V,H}d^2}\tau_m,$$

$$\xi = \frac{\xi_r}{\eta}, \quad \Omega = \frac{\rho C_{V,H}d^2}{K_t}\Omega_a,$$

$$M_1 = \frac{\mu_0\beta\chi^2H_0^2}{\rho g\alpha(1+\chi)T_a^2}, \quad M_2 = \frac{\mu_0\chi^2H_0^2}{\rho C_{V,H}(1+\chi)T_a},$$

$$M_3 = \frac{\mu_0\chi H_0^2}{\rho g\alpha d T_a}, \quad N = \mathrm{Rg}M_1. \tag{24}$$

Here Rg is the viscous Rayleigh number, N is magnetic Rayleigh number, and two other related parameters M_4 and M_5 are denoted by

$$M_4 = \frac{(1+\chi)M_1}{\chi M_3} = \frac{\beta d}{T_a}, \quad M_5 = \frac{M_3\mathrm{Rg}}{(1+\chi)M_1} = \frac{\rho^2 g\alpha C_{V,H}T_a d^3}{\chi\eta K_t}.$$

The parameter M_2 turns out to be very small for all kind of magnetic fluids [15,16] and hence is omitted in subsequent calculations.

On substituting (22) into Eqs. (9), (5), (6), and (1), and using dimensionless parameters (23) and (24), and writing $\boldsymbol{M}' = \chi\boldsymbol{H}'$ and $\boldsymbol{H}' = \boldsymbol{\nabla}\phi$, after dropping the prime and asterisk, we obtain the linearized equations as

$$\frac{1}{\Pr}\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \nabla^2 u + M_5\cos(\Omega t)(\chi M_3 + M_1 z)\frac{\partial^2\phi}{\partial z\partial x}, \tag{25}$$

$$\frac{1}{\Pr}\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \nabla^2 v + M_5\cos(\Omega t)(\chi M_3 + M_1 z)\frac{\partial^2\phi}{\partial z\partial y}, \tag{26}$$

$$\frac{1}{\Pr}\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \nabla^2 w + M_5\cos(\Omega t)(\chi M_3 + M_1 z)\frac{\partial^2\phi}{\partial z^2} - \chi M_1 M_5\frac{\partial\phi}{\partial z} - [1 + M_1\cos^2(\Omega t)]\mathrm{Rg}\theta, \tag{27}$$

$$\frac{\partial\theta}{\partial t} = \nabla^2\theta + w, \tag{28}$$

$$(1+\chi)\nabla^2\phi - \chi M_4\cos(\Omega t)\frac{\partial\theta}{\partial z} = 0, \tag{29}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{30}$$

On taking curlcurl of Eqs. (25)–(27), the vertical component of the resulting equation gives

$$\frac{1}{\Pr}\frac{\partial\nabla^2 w}{\partial t} = \nabla^4 w + \mathrm{Rg}[1 + M_1\cos^2(\Omega t)]\nabla_1^2\theta - M_3\mathrm{Rg}\cos(\Omega t)\frac{\partial\nabla_1^2\phi}{\partial z} \tag{31}$$

where $\nabla_1^2 = [(\partial^2/\partial x^2) + (\partial^2/\partial y^2)]$. The boundary conditions for velocity and temperature, for the two cases are [18]

$$w = \frac{\partial^2 w}{\partial z^2} = \theta = 0, \quad z = \pm\frac{1}{2}, \quad \text{(free-free boundary)} \tag{32}$$

$$w = \frac{\partial w}{\partial z} = \theta = 0, \quad z = \pm\frac{1}{2}, \quad \text{(rigid-rigid boundary)} \tag{33}$$

The magnetic boundary conditions are that the normal component of magnetic induction and tangential component of magnetic field are continuous across the boundary

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial x},\quad \frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial y},\quad M_z+\frac{\partial\phi}{\partial z}=\frac{\partial\psi}{\partial z},\quad z=\pm\frac{1}{2}, \tag{34}$$

where ψ is the magnetic potential outside of fluid, and it satisfies

$$\nabla^2\psi=0,\quad |z|\geqslant\tfrac{1}{2}. \tag{35}$$

In view of the linear relation between magnetization and magnetic field, the above conditions reduce to [16,17]

$$(1+\chi)\frac{\partial\phi}{\partial z}\pm\phi=0,\quad \text{at } z=\pm\frac{1}{2}. \tag{36}$$

A. Free-free boundary

We first consider the ideal case of free-free boundary. These conditions would be feasible for free perfectly heat-conducting boundaries made of a superconductor. Since magnetic field perturbations do not penetrate into such a wall, conditions (36) reduce to

$$\frac{\partial\phi}{\partial z}=0,\quad \text{at } z=\pm\frac{1}{2}. \tag{37}$$

Moreover, in order to reduce the relevant equations in the same form as to those of Finlayson [16], we nondimensionalize the magnetic potential ϕ as $\phi=(\beta d/T_a)[\chi/(1+\chi)]\phi'$, and omitting the dashes, Eqs. (31), (29), and (28) now become

$$\frac{1}{\Pr}\frac{\partial\nabla^2 w}{\partial t}=\nabla^4 w+\mathrm{Rg}[1+M_1\cos^2(\Omega t)]\nabla_1^2\theta - \mathrm{Rg}M_1\cos(\Omega t)\frac{\partial\nabla_1^2\phi}{\partial z} \tag{38}$$

$$\nabla^2\phi-\cos(\Omega t)\frac{\partial\theta}{\partial z}=0, \tag{39}$$

$$\frac{\partial\theta}{\partial t}-\nabla^2\theta=w. \tag{40}$$

On assuming a periodic dependence of the variables on horizontal coordinates, we take w, θ, and ϕ in (38)–(40) of the form

$$(w,\theta,\phi)=\{w(z,t),\theta(z,t),\phi(z,t)\}e^{i(k_x x+k_y y)}. \tag{41}$$

On substituting (41) into (38)–(40), we obtain

$$\frac{1}{\Pr}(\boldsymbol{D}^2-k^2)\dot{w}-(\boldsymbol{D}^2-k^2)^2 w=-k^2[\mathrm{Rg}+N\cos^2(\Omega t)]\theta + Nk^2\cos(\Omega t)\boldsymbol{D}\phi, \tag{42}$$

$$(\boldsymbol{D}^2-k^2)\phi=\cos(\Omega t)\boldsymbol{D}\theta, \tag{43}$$

$$\dot{\theta}-(\boldsymbol{D}^2-k^2)\theta=w, \tag{44}$$

where $\boldsymbol{D}=\partial/\partial z$ and $k^2=k_x^2+k_y^2$.

In order to satisfy boundary conditions (32) and (37), we select

$$w=a(t)\cos(\pi z),\quad \theta=b(t)\cos(\pi z),\quad \phi=c(t)\sin(\pi z). \tag{45}$$

Substitution of (45) into (42)–(44) leads to

$$\Pr^{-1}(\pi^2+k^2)\dot{a}+(\pi^2+k^2)^2 a = k^2[\mathrm{Rg}+N\cos^2(\Omega t)]b-\pi k^2 N\cos(\Omega t)c,$$

$$\dot{b}+(\pi^2+k^2)b=a,\quad (\pi^2+k^2)c=\pi\cos(\Omega t)b, \tag{46}$$

where the dots denote derivative with respect to time t. On eliminating a and c in (46), we arrive at

$$\Pr^{-1}(\pi^2+k^2)^2\ddot{b}+(\Pr^{-1}+1)(\pi^2+k^2)^3\dot{b}+(\pi^2+k^2)^4 b = k^2[(\pi^2+k^2)\mathrm{Rg}+k^2N\cos(\Omega t)]b. \tag{47}$$

On making the following substitutions in (47):

$$b=\frac{f}{(\pi^2+k^2)\sqrt{\Pr}},\quad t=\frac{\tau}{(\pi^2+k^2)\sqrt{\Pr}},\quad \gamma=\frac{\Omega}{(\pi^2+k^2)\sqrt{\Pr}}, \tag{48}$$

we finally obtain

$$\frac{d^2f}{d\tau^2}+2\epsilon\frac{df}{d\tau}+[1-Rg-\bar{N}-\bar{N}\cos(2\gamma\tau)]f=0, \tag{49}$$

where

$$\bar{R}g=\frac{\mathrm{Rg}}{R_0},\quad 2\bar{N}=\frac{N}{N_0},\quad R_0=\frac{(\pi^2+k^2)^3}{k^2},$$

$$N_0=\frac{(\pi^2+k^2)^4}{k^4},\quad 2\epsilon=\left(\sqrt{\Pr}+\frac{1}{\sqrt{\Pr}}\right)$$

In the above equation, we note that, R_0 is the critical Rayleigh number in the absence of a magnetic field and N_0 is the critical magnetic Rayleigh number in a stationary field in the absence of gravity. If we now further let

$$f=\sigma e^{-\epsilon\tau},\quad \beta=\gamma\tau$$

in Eq. (49), it reduces to the standard form of the Mathieu equation

$$\frac{d^2\sigma}{d\beta^2}+[p-q\cos(2\beta)]\sigma=0, \tag{50}$$

where

$$p=\left[1-\frac{\mathrm{Rg}k^2}{(\pi^2+k^2)^3}-\frac{Nk^4}{(\pi^2+k^2)^4}-\epsilon^2\right]\frac{(\pi^2+k^2)\Pr}{\Omega^2},$$

$$q=\frac{Nk^4\Pr}{2(\pi^2+k^2)^4\Omega^2} \tag{51}$$

and where Rg and N are defined in (24).

The solution to the Mathieu equation (50) is of the form [19]

$$\sigma(\beta) = Ae^{\mu\beta}\psi(\beta) + Be^{-\mu\beta}\psi(-\beta) \tag{52}$$

where A and B are constants and $\psi(\beta)$ is the periodic function with period π or 2π and μ is the characteristic or FLOQUET exponent, which is a function of p and q. Thus, depending on the nature of μ, the solution may be unstable, stable and periodic, and stable and nonperiodic. The determination of μ, however, requires extensive calculations of Hill's determinants, etc. We follow the alternate method to study the qualitative behavior of the Mathieu equation by using the stability diagram, including the effects of damping. The stability diagram Fig. 1(a) is reproduced from the data of McLachlan [20].

Figure 1(a) shows the plot of p against q and the exact location of the boundaries between stable and unstable solutions. The plane is divided up into discrete regions. The diagram is symmetrical about p axis. It can be seen that if the point (p,q) lies in any of the shaded regions in Fig. 1(a), the solutions of the Mathieu equations are unstable. The solutions are oscillatory but with exponentially increasing amplitude. However, if the point (p,q) lies in any one of the unshaded regions, the solutions are stable, again being oscillatory, though not regularly periodic. In the case of unstable solutions, there is exact periodicity of π or 2π in β. Moreover, it can be seen from the figure that parametric resonance between forced and free oscillations occurs when q is small and p is close to an integer value.

With regard to the solution in our case, we note, from (51) that q is always positive, but p could be positive or negative depending on the parameters Rg, N, ϵ, Pr, Ω, and k. The stability nature thus depends, as expected, on these parameters. We also note from the figure that, for the point (p,q) to lie in the stable region, p has to be positive and possibly large and q has to be small. In order for p to be positive, we require

$$1 > \left[\frac{k^2}{(\pi^2+k^2)^3}\left(\mathrm{Rg} + \frac{Nk^2}{(\pi^2+k^2)}\right) + \epsilon^2\right]. \tag{53}$$

Inequality (53) can be satisfied in only very special circumstances. In the case of the plane layer heated from below, both Rg and N are expected to be positive and ϵ^2 is always positive. In fact $\epsilon^2=1$ if $\sqrt{\mathrm{Pr}}=1$ and $\epsilon^2>1$ if $\sqrt{\mathrm{Pr}}>1$, and $\epsilon^2<1$ if $\sqrt{\mathrm{Pr}}<1$. Thus the likelihood of p being negative is favorable. The other possibility of p being large is when Ω is small. However, when Ω is small then q becomes large. Thus it appears that the point (p,q) in our case is likely to lie in the shaded region; that is, the phenomenon of parametric resonance will be observed. We note that as $\Omega \to \infty$, $p \to 0$ and $q \to 0$, and thus the stability character reduces to those of the basic state.

Returning to the solution for f or $b(t)$, we note that $f = \sigma e^{-\epsilon\tau}$ and the damping term $e^{-\epsilon\tau}$ has a stabilizing influence on the solution. In relation to the characteristic exponent μ, the exponential factor now becomes $\left[\mu\Omega - \frac{1}{2}(\mathrm{Pr}+1)(\pi^2+k^2)\right]t$. The stability criterion in this case is $(\mathrm{Pr}+1)(\pi^2+k^2) > 2\mu\Omega$. Thus for low values Ω, and depending on the values of Pr, μ, and k, this inequality may be satisfied. For higher values of Ω, it has good possibility of being violated.

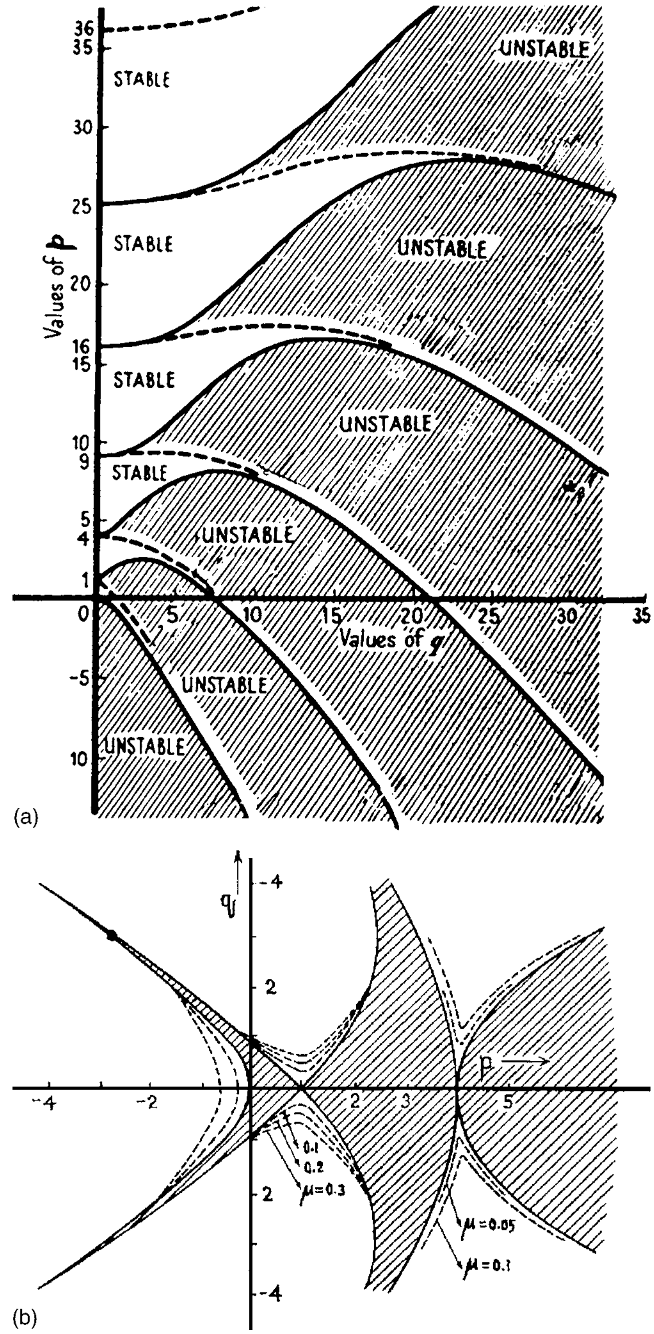

FIG. 1. (a) Stability diagram for the solutions of Mathieu's equation [20]. (b) Stability diagram for Mathieu equations including the effects of damping. The lined area is the stable region.

The damping effect is exhibited in Fig. 1(b). This figure is drawn for small values of (p,q); for higher values of (p,q), predictions are evident. Also the shaded region in this figure represents the stable region. The damping-effect region has been marked with broken lines for different values of μ. It can be seen from the figure that the effect of damping is to reduce instability regions at lower values of q. For higher values of q, the effect is considerably less appreciable. Thus we remark that although, with damping effect, the stable regions cover a greater area of stability in the diagram, the unstable regions are not completely eliminated. There is, thus, always the possibility of resonance phenomenon.

B. Rigid-rigid boundary

In the case of the rigid-rigid boundary condition, it is not possible to obtain an analytical solution and we have to revent to numerical methods.

In order to match the domain of Chebyshev pseudospectral-QZ method, we first reset the present domain from $[-\frac{1}{2},\frac{1}{2}]$ to $[-1,1]$ with coordinate transformation of z to $2z$ in Eqs. (31), (28), and (29), and in the boundary conditions. We perform the standard normal mode analysis and look for the solution of variables w, θ, and ϕ in the form (41). The form of boundary conditions for magnetic field in the new domain are

$$M_z + 2\frac{\partial\phi}{\partial z} \pm k\phi = 0, \quad \text{at } z = \pm 1. \tag{54}$$

On substituting (41) into Eqs. (31), (28), and (29), in the new domain, we obtain a set of differential equation in $w(z,t)$, $\theta(z,t)$, and $\phi(z,t)$. The discretized form of these equations along with the new boundary conditions are given in the following section, where we present the numerical solution of the resulting equations.

IV. MODEL WITH INTERNAL ROTATION

For the nonzero vortex viscosity model, both the coefficients (ξ,τ) should be considered. The magnetization equation (10) and the rotational viscosity should be included in the analysis.

The quiescent state solution of the basic equations with corresponding rigid boundary conditions in the low-magnetization limit is given to be

$$\boldsymbol{u}^s = \boldsymbol{0}, \quad T^s = T_a - \beta z, \tag{55}$$

$$\boldsymbol{H}^s = \boldsymbol{k}\left\{(1+\chi)H_0\cos(\Omega_a t) - \frac{\chi H_0[(1+\chi)T_0+\beta z]}{T_0[(1+\chi)^2+(\tau_0\Omega_a)^2]} \times[(1+\chi)\cos(\Omega_a t)+\tau_0\Omega_a\sin(\Omega_a t)]\right\}, \tag{56}$$

$$\boldsymbol{M}^s = \boldsymbol{k}\frac{\chi H_0[(1+\chi)T_0+\beta z]}{T_0[(1+\chi)^2+(\tau_0\Omega_a)^2]}[(1+\chi)\cos(\Omega_a t) + \tau_0\Omega_a\sin(\Omega_a t)], \tag{57}$$

$$H^s + M^s = \boldsymbol{k}(1+\chi)H_0\cos(\Omega_a t) = \boldsymbol{k}H_0^{\text{ext}}\cos(\Omega_a t), \tag{58}$$

$$P^s = -\rho g z - \frac{1}{2}\rho g\alpha\beta z^2 - \frac{\mu_0\beta\chi^2H_0^2[2(1+\chi)T_0+\beta z]z}{4T_a^2[(1+\chi^2)^2+(\tau_0\Omega_a)^2]} - \frac{\mu_0\beta\chi^2H_0^2[2(1+\chi)T_0+\beta z]z}{4T_a^2[(1+\chi^2)^2+(\tau_0\Omega_a)^2]^2}\{[(1+\chi^2)^2 - (\tau_0\Omega_a)^2]\cos(2\Omega_a t) + 2(1+\chi)\tau_0\Omega_a\sin(2\Omega_a t)\}. \tag{59}$$

As in Sec. III, we now perturb the variables appearing in the above equations. On denoting the perturbation variables by primes, we write

$$[u,v,w,M_x,M_y,M_z,H_x,H_y,H_z,P,\theta]^T = [0,0,0,0,0,M_3^s,0,0,H_3^s,P^s,T^s]^T + [u',v',w',M_x',M_y',M_z',H_x',H_y',H_z',P',\theta']^T. \tag{60}$$

On using dimensionless equations (23) and (24), and writing $\boldsymbol{H}'=\boldsymbol{\nabla}\phi$, after dropping the prime and asterisk, the linearized equations are given

$$\frac{1}{\text{Pr}}\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \nabla^2 u - \frac{1}{2}\psi_1\text{Rg}M_3\frac{\partial\phi}{\partial x} + \frac{1}{2}\psi_1 M_3[\text{Rg}z + M_5\chi \times(1+\chi)]\frac{\partial^2\phi}{\partial z\partial x} - \frac{1}{2}\psi_1\chi\text{Rg}M_3M_x - \frac{1}{2}[\psi_1\chi\text{Rg}M_3 z - \psi_2\chi(1+\chi)M_3M_5]\frac{\partial M_x}{\partial z}, \tag{61}$$

$$\frac{1}{\text{Pr}}\frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \nabla^2 v - \frac{1}{2}\psi_1\text{Rg}M_3\frac{\partial\phi}{\partial y} + \frac{1}{2}\psi_1 M_3[\text{Rg}z + M_5\chi \times(1+\chi)]\frac{\partial^2\phi}{\partial z\partial y} - \frac{1}{2}\psi_1\chi\text{Rg}M_3M_y - \frac{1}{2}[\psi_1\chi\text{Rg}M_3 z - \psi_2\chi(1+\chi)M_3M_5]\frac{\partial M_y}{\partial z}, \tag{62}$$

$$\frac{1}{\text{Pr}}\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \nabla^2 w - \frac{1+\chi}{2}M_3M_5\cos(\Omega_a t)\nabla^2\phi + \frac{1}{2}\psi_1 M_3[\text{Rg}z + M_5\chi(1+\chi)]\frac{\partial^2\phi}{\partial z^2} - \frac{1}{2}\psi_1\chi\text{Rg}M_3M_z - \frac{1}{2}[\psi_1\chi\text{Rg}M_3 z - \psi_2\chi(1+\chi)M_3M_5]\frac{\partial M_z}{\partial z} + \text{Rg}\theta, \tag{63}$$

$$\frac{\partial M_x}{\partial t} = \left\{\psi_3 - \frac{1}{\tau}\right\}M_x + \frac{1}{2}\psi_1(M_4 z + 1 + \chi)\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \left\{\psi_4 + \frac{1}{\tau}\right\}\frac{\partial\phi}{\partial x}, \tag{64}$$

$$\frac{\partial M_y}{\partial t} = \left\{\psi_3 - \frac{1}{\tau}\right\}M_y + \frac{1}{2}\psi_1(M_4 z + 1 + \chi)\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + \left\{\psi_4 + \frac{1}{\tau}\right\}\frac{\partial\phi}{\partial y}, \tag{65}$$

$$\frac{\partial M_z}{\partial t} = -\frac{1}{\tau}M_z + \frac{1}{\tau}\frac{\partial\phi}{\partial z} - \frac{1}{\tau}M_4\cos(\Omega t)\theta - \psi_1 M_4 w, \tag{66}$$

$$\nabla^2\phi + \chi\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} + \frac{\partial M_z}{\partial z}\right) = 0, \tag{67}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{68}$$

$$\frac{\partial \theta}{\partial t}=\nabla^2\theta+w, \tag{69}$$

where

$$\psi_1=\frac{(1+\chi)\cos(\Omega t)+\Omega\tau\sin(\Omega t)}{(1+\chi)^2+(\Omega\tau)^2}, \tag{70}$$

$$\psi_2=\frac{[1+\chi+(\tau\Omega)^2]\cos(\Omega t)-\Omega\tau\chi\sin(\Omega t)}{(1+\chi)^2+(\Omega\tau)^2}, \tag{71}$$

$$\psi_3=\frac{(1+\chi)M_1\mathrm{Rg}}{4\xi}\psi_1^2z^2+\frac{(1+\chi)^2M_1M_5}{4\xi}(\chi\psi_1^2-\psi_1\psi_2)z-\frac{\chi(1+\chi)^2M_3M_5}{4\xi}\psi_1\psi_2, \tag{72}$$

$$\psi_4=\frac{(1+\chi)M_1\mathrm{Rg}}{4\xi\chi}\psi_1^2z^2+\frac{(1+\chi)^2M_1M_5}{2\xi}\psi_1^2z+\frac{\chi(1+\chi)^2M_3M_5}{4\xi}\psi_1^2. \tag{73}$$

On taking curlcurl of Eqs. (61)–(63), the vertical component of the resulting equation gives

$$\frac{1}{\mathrm{Pr}}\frac{\partial\nabla^2 w}{\partial t}=\nabla^4 w+\mathrm{Rg}\nabla_1^2\theta-\chi M_3\mathrm{Rg}\psi_1\nabla^2M_z+\frac{1}{2}\chi M_3\times[(1+\chi)M_5\psi_2-\mathrm{Rg}\psi_1 z]\frac{\partial\nabla^2M_z}{\partial z}-M_3\mathrm{Rg}\psi_1\times\frac{\partial\nabla^2\phi}{\partial z}-\frac{1}{2}M_3\psi_1[\chi(1+\chi)M_5+\mathrm{Rg}z]\frac{\partial^2\nabla^2\phi}{\partial z^2}+\frac{1+\chi}{2}M_3M_5\cos(\Omega t)\nabla^4\phi \tag{74}$$

where $\nabla^2=(\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2)$, and $\nabla_1^2=(\partial^2/\partial x^2+\partial^2/\partial y^2)$.

Taking divergence of magnetization equations (64)–(66), and using (67), we get

$$\frac{\partial\nabla^2\phi}{\partial t}=\chi\psi_3\frac{\partial M_z}{\partial z}+\frac{\chi}{\tau}M_4\cos(\Omega t)\frac{\partial\theta}{\partial z}-(1+\chi)\times\left[\frac{1}{\tau}+\frac{\chi}{4\xi}(1+\chi+M_4z)\psi_1M_3M_5\cos(\Omega t)\right]\nabla^2\phi+\frac{\chi^2}{4\xi}(1+\chi+M_4z)^2\psi_1^2M_3M_5\frac{\partial^2\phi}{\partial z^2}+\frac{\chi}{2}(1+\chi+M_4z)\psi_1\nabla^2w+\chi M_4\psi_1\frac{\partial w}{\partial z}. \tag{75}$$

We have four equations (66), (69), (74), and (75) for four variables w, M_z, ϕ, and θ.

We perform the standard normal mode analysis and look for the solution of variables w, M_z, ϕ, and θ in the form

$$\begin{bmatrix} w \\ M_z \\ \phi \\ \theta \end{bmatrix}=\begin{bmatrix} w(z,t) \\ M_z(z,t) \\ \phi(z,t) \\ \theta(z,t) \end{bmatrix}\exp[(k_x x+k_y y)i]. \tag{76}$$

On substituting (76) into Eqs. (66), (69), (74), and (75), and transferring them in the new domain, we obtain a set of differential equations in $w(z,t)$, $M_z(z,t)$, $\theta(z,t)$, and $\phi(z,t)$. The discretized form of these equations along with the new boundary conditions are given in Sec. V.

V. NUMERICAL RESULTS

In our previous paper [15] a short account of the Chebyshev pseudospectral Tau method was given. Briefly it involves the expansion of unknown variables in Chebyshev polynomials in which the integration in determining the coefficients is replaced by a numerical discrete integral on collocation points.

After discretizing by the Chebyshev pseudospectral method, equations of the quasistationary model can be written in the matrix form as

$$\boldsymbol{B}\frac{\partial \boldsymbol{X}}{\partial t}=[\boldsymbol{A_0}+\cos(\Omega t)\boldsymbol{A_1}+\cos(2\Omega t)\boldsymbol{A_2}]\boldsymbol{X}, \tag{77}$$

where variables vector $\boldsymbol{X}=[w_0,w_1,\ldots,\theta_0,\theta_1,\ldots,\phi_0,\phi_2,\ldots]$, $(w_i,\theta_i,\phi_i,i=0,\ldots,N$ are values of w, θ, and ϕ on discrete collocation points), and matrices $\boldsymbol{B}$, $\boldsymbol{A}_0$, $\boldsymbol{A}_1$, and $\boldsymbol{A}_2$ can be expressed as

$$\boldsymbol{B}=\begin{bmatrix} \frac{1}{\mathrm{Pr}}(4\boldsymbol{D}^2-k^2\boldsymbol{I}) & 0 & 0 \\ 0 & \boldsymbol{I} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{78}$$

$$\boldsymbol{A_0}=\begin{bmatrix} (16\boldsymbol{D}^4-8k^2\boldsymbol{D}^2+k^4\boldsymbol{I}) & -\frac{1}{2}(2+M_1)\mathrm{Rg}k^2\boldsymbol{I} & 0 \\ \boldsymbol{I} & 4\boldsymbol{D}^2-k^2\boldsymbol{I} & 0 \\ 0 & 0 & -(1+\chi)(4\boldsymbol{D}^2-k^2\boldsymbol{I}) \end{bmatrix}, \tag{79}$$

$$\boldsymbol{A_1}=\begin{bmatrix}0 & 0 & 2M_3\mathrm{Rg}k^2\boldsymbol{D}\\ 0 & 0 & 0\\ 0 & 2\chi M_4\boldsymbol{D} & 0\end{bmatrix}, \tag{80}$$

$$\boldsymbol{A_2}=\begin{bmatrix}0 & -\frac{1}{2}M_1\mathrm{Rg}k^2\boldsymbol{I} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \tag{81}$$

where $k^2=k_x^2+k_y^2$, matrix $\boldsymbol{D}$ is differentiation matrix introduced in [15], $\boldsymbol{I}$ is an identity matrix.

Boundary conditions in discrete form are

$$w_0=w_N=\sum_{j=1}^{N-1}D_{0,j}w_j=\sum_{j=1}^{N-1}D_{N,j}w_j=\theta_0=\theta_N=0,$$

$$2(1+\chi)\sum_{j=0}^{N}D_{0,j}\phi_j-k\phi_0=2(1+\chi)\sum_{j=0}^{N}D_{N,j}\phi_j+k\phi_N=0. \tag{82}$$

After discretizing with the Chebyshev pseudospectral method, equations with the internal rotation model can be written in the matrix form as

$$\boldsymbol{B}\frac{\partial \boldsymbol{X}}{\partial t}=[\boldsymbol{A_0}+\cos(\Omega t)\boldsymbol{A_1}+\cos(2\Omega t)\boldsymbol{A_2}+\sin(\Omega t)\boldsymbol{A_3}+\sin(2\Omega t)\boldsymbol{A_4}]\boldsymbol{X}, \tag{83}$$

where variables vector $\boldsymbol{X}=[w_0,w_1,\cdots,M_{z0},M_{z1},\cdots,\phi_0,\phi_2,\cdots,\theta_0,\theta_1,\cdots]$. In this case the matrices $\boldsymbol{B}$, $\boldsymbol{A}_0$, $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$, and $\boldsymbol{A}_4$ can be expressed

$$\boldsymbol{B}=\begin{bmatrix}\frac{1}{\mathrm{Pr}}(4\boldsymbol{D}^2-k^2\boldsymbol{I}) & 0 & 0 & 0\\ 0 & \boldsymbol{I} & 0 & 0\\ 0 & 0 & 4\boldsymbol{D}^2-k^2\boldsymbol{I} & 0\\ 0 & 0 & 0 & \boldsymbol{I}\end{bmatrix}, \tag{84}$$

$$\boldsymbol{A}_0=\begin{bmatrix}16\boldsymbol{D}^4-8k^2\boldsymbol{D}^2+k^4\boldsymbol{I} & 0 & 0 & -\mathrm{Rg}k^2\boldsymbol{I}\\ 0 & -\frac{1}{\tau}\boldsymbol{I} & \frac{2}{\tau}\boldsymbol{D} & 0\\ 0 & \boldsymbol{A_{0}}_{(3,2)} & \boldsymbol{A_{0}}_{(3,3)} & 0\\ \boldsymbol{I} & 0 & 0 & 4\boldsymbol{D}^2-k^2\boldsymbol{I}\end{bmatrix}, \tag{85}$$

$$\boldsymbol{A_1}=\begin{bmatrix}0 & \boldsymbol{A_1}_{(1,2)} & \boldsymbol{A_1}_{(1,3)} & 0\\ -\frac{M_4(1+\chi)}{(1+\chi)^2+(\Omega\tau)^2}\boldsymbol{I} & 0 & 0 & -\frac{M_4}{\tau}\boldsymbol{I}\\ \boldsymbol{A_1}_{(3,1)} & 0 & 0 & \frac{2\chi M_4}{\tau}\boldsymbol{D}\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{86}$$

$$\boldsymbol{A_2}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & \boldsymbol{A_2}_{(3,2)} & \boldsymbol{A_2}_{(3,3)} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{87}$$

$$\boldsymbol{A_3}=\begin{bmatrix}0 & \boldsymbol{A_3}_{(1,2)} & \boldsymbol{A_3}_{(1,3)} & 0\\ -\frac{M_4\Omega\tau}{(1+\chi)^2+(\Omega\tau)^2}\boldsymbol{I} & 0 & 0 & 0\\ \boldsymbol{A_3}_{(3,1)} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{88}$$

$$\boldsymbol{A_4}=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & \boldsymbol{A_4}_{(3,2)} & \boldsymbol{A_4}_{(3,3)} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}, \tag{89}$$

where

$$\boldsymbol{A_0}_{(3,2)}=\frac{\chi^2M_3M_5}{16\xi[(1+\chi)^2+(\Omega\tau)^2]}\{\chi M_4^2\boldsymbol{Z}^2-2(1-\chi^2)M_4\boldsymbol{Z}-4(1+\chi)^2\boldsymbol{I}\}\boldsymbol{D},$$

$$\boldsymbol{A_0}_{(3,3)}=\frac{\chi M_3M_5}{16\xi[(1+\chi)^2+(\Omega\tau)^2]}\{[2\chi M_4^2\boldsymbol{Z}^2-4(1-\chi^2)M_4\boldsymbol{Z}-8(1+\chi)^2\boldsymbol{I}]\boldsymbol{D}^2+(1+\chi)^2M_4k^2\boldsymbol{Z}+2(1+\chi)^3k^2\boldsymbol{I}\}-\frac{(1+\chi)}{\tau}(4\boldsymbol{D}^2-k^2\boldsymbol{I}),$$

$$A_{1(1,2)} = \frac{\chi(1+\chi)M_3}{(1+\chi)^2+(\Omega\tau)^2}\Big(\{-2\mathrm{Rg}\boldsymbol{Z}+4[1+\chi+(\Omega\tau)^2]M_5\boldsymbol{I}\}\boldsymbol{D}^3-4\mathrm{Rg}\boldsymbol{D}^2+\tfrac{1}{2}k^2\{\mathrm{Rg}\boldsymbol{Z}-2[1+\chi+(\Omega\tau)^2]M_5\boldsymbol{I}\}\boldsymbol{D}+\mathrm{Rg}k^2\boldsymbol{I}\Big),$$

$$A_{1(1,3)} = \frac{(1+\chi)M_3}{(1+\chi)^2+(\Omega\tau)^2}\Big(\{-4\mathrm{Rg}\boldsymbol{Z}+8[1+\chi+(\Omega\tau)^2]M_5\boldsymbol{I}\}\boldsymbol{D}^4-8\mathrm{Rg}\boldsymbol{D}^3+k^2[\mathrm{Rg}\boldsymbol{Z}-2[\chi^2+3\chi+2+2(\Omega\tau)^2]M_5\boldsymbol{I}\}\boldsymbol{D}^2+2k^2\mathrm{Rg}\boldsymbol{D}+\tfrac{1}{2}M_5k^4[(1+\chi)^2+(\Omega\tau)^2]\boldsymbol{I}\Big),$$

$$A_{1(3,1)} = \frac{\chi(1+\chi)}{(1+\chi)^2+(\Omega\tau)^2}\left\{[M_4\boldsymbol{Z}+2(1+\chi)\boldsymbol{I}]\boldsymbol{D}^2+2M_4\boldsymbol{D}-\frac{1}{4}k^2[M_4\boldsymbol{Z}+2(1+\chi)\boldsymbol{I}]\right\},$$

$$A_{2(3,2)} = \frac{\chi^2M_3M_5}{16\xi[(1+\chi)^2+(\Omega\tau)^2]^2}\{\chi[(1+\chi)^2-(\Omega\tau)^2]M_4^2\boldsymbol{Z}^2-2(1+\chi)[(1+\chi)(1-\chi^2)+(1+3\chi)(\Omega\tau)^2]M_4\boldsymbol{Z}-4(1+\chi)^2[(1+\chi)^2+(1+2\chi)(\Omega\tau)^2]\boldsymbol{I}\}\boldsymbol{D},$$

$$A_{2(3,3)} = \frac{\chi M_3M_5}{16\xi[(1+\chi)^2+(\Omega\tau)^2]^2}(\{2\chi[(1+\chi)^2-(\Omega\tau)^2]M_4^2\boldsymbol{Z}^2-4(1+\chi)[(1+\chi)(1-\chi^2)+(1+3\chi)(\Omega\tau)^2]M_4\boldsymbol{Z}-8(1+\chi)^2[(1+\chi)^2+(1+2\chi)(\Omega\tau)^2]M_4\boldsymbol{I}\}\boldsymbol{D}^2+(1+\chi)^2[(1+\chi)^2+(\Omega\tau)^2]k^2[M_4\boldsymbol{Z}+2(1+\chi)\boldsymbol{I}]),$$

$$A_{3(1,2)} = \frac{\chi M_3\Omega\tau}{(1+\chi)^2+(\Omega\tau)^2}\left\{[-2\mathrm{Rg}\boldsymbol{Z}-4\chi(1+\chi)M_5\boldsymbol{I}]\boldsymbol{D}^3-4\mathrm{Rg}\boldsymbol{D}^2+\frac{1}{2}k^2[\mathrm{Rg}\boldsymbol{Z}+2\chi(1+\chi)M_5\boldsymbol{I}]\boldsymbol{D}+\mathrm{Rg}k^2\boldsymbol{I}\right\},$$

$$A_{3(1,3)} = \frac{M_3\Omega\tau}{(1+\chi)^2+(\Omega\tau)^2}\{[-4\mathrm{Rg}\boldsymbol{Z}-8\chi(1+\chi)M_5\boldsymbol{I}]\boldsymbol{D}^4-8\mathrm{Rg}\boldsymbol{D}^3+k^2[\mathrm{Rg}\boldsymbol{Z}+2\chi(1+\chi)M_5\boldsymbol{I}]\boldsymbol{D}^2+2\mathrm{Rg}k^2\boldsymbol{D}\},$$

$$A_{3(3,1)} = \frac{\chi\Omega\tau}{(1+\chi)^2+(\Omega\tau)^2}\left\{[M_4\boldsymbol{Z}+2(1+\chi)\boldsymbol{I}]\boldsymbol{D}^2+2M_4\boldsymbol{D}-\frac{1}{4}k^2[M_4\boldsymbol{Z}+2(1+\chi)\boldsymbol{I}]\right\},$$

$$A_{4(3,2)} = \frac{\chi^2(1+\chi)M_3M_5\Omega\tau}{8\xi[(1+\chi)^2+(\Omega\tau)^2]^2}\{\chi M_4^2\boldsymbol{Z}^2-[1-2\chi-3\chi^2+(\Omega\tau)^2]M_4\boldsymbol{Z}-2(1+\chi)[1-\chi^2+(\Omega\tau)^2]\boldsymbol{I}\}\boldsymbol{D},$$

$$A_{4(3,3)} = \frac{\chi(1+\chi)M_3M_5\Omega\tau}{16\xi[(1+\chi)^2+(\Omega\tau)^2]^2}(\{4\chi M_4^2\boldsymbol{Z}^2-4M_4[1-2\chi-3\chi^2+(\Omega\tau)^2]\boldsymbol{Z}-8(1+\chi)[1-\chi^2+(\Omega\tau)^2]\boldsymbol{I}\}\boldsymbol{D}^2+k^2[(1+\chi)^2+(\Omega\tau)^2][M_4\boldsymbol{Z}+2(1+\chi)\boldsymbol{I}]). \quad (90)$$

Here $\boldsymbol{Z}$ is the diagonal coordinate matrix [15]. Boundary conditions in discrete form can be expressed as

$$w_0=w_N=\sum_{j=1}^{N-1}D_{0,j}w_j=\sum_{j=1}^{N-1}D_{N,j}w_j=\theta_0=\theta_N=0,$$

$$M_{z0}+2\sum_{j=0}^{N}D_{0,j}\phi_j-k\phi_0=M_{zN}+2\sum_{j=0}^{N}D_{N,j}\phi_j+k\phi_N=0. \quad (91)$$

The matrix $\boldsymbol{B}$ in (78) for quasistationary model (77) is not a full-rank matrix. We have $\cos(\Omega t)2\chi M_4\boldsymbol{D}\boldsymbol{\theta}-(1+\chi)(4\boldsymbol{D}^2-k^2\boldsymbol{I})\boldsymbol{\phi}=0$. Substituting variable vector $\boldsymbol{\phi}$ into (77) and implementing the boundary condition (82), Eq. (77) can be written as

$$\begin{bmatrix}\boldsymbol{B}_{1,1} & \boldsymbol{B}_{1,2}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{X}}_1\\ \dot{\boldsymbol{X}}_2\end{bmatrix}=\begin{bmatrix}\boldsymbol{A}_{1,1}(t) & \boldsymbol{A}_{1,2}(t)\\ \boldsymbol{Bc}_{2,1} & \boldsymbol{Bc}_{2,2}\end{bmatrix}\begin{bmatrix}\boldsymbol{X}_1\\ \boldsymbol{X}_2\end{bmatrix}. \quad (92)$$

Variable vector $\boldsymbol{X}_2$ are the boundary points and the outermost internal points for Chebyshev pseudospectral method. If matrix $\boldsymbol{Bc}_{2,2}$ is not singular, the variable vector $\boldsymbol{X}_2$ can be condensed and Eq. (92) becomes

$$\dot{\boldsymbol{X}}_1=[\boldsymbol{B}_{1,1}-\boldsymbol{B}_{1,2}\boldsymbol{Bc}_{2,2}^{-1}\boldsymbol{Bc}_{2,1}]^{-1}[\boldsymbol{A}(t)_{1,1}-\boldsymbol{A}(t)_{1,2}\boldsymbol{Bc}_{2,2}^{-1}\boldsymbol{Bc}_{2,1}]\boldsymbol{X}_1. \quad (93)$$

Proceeding exactly in the same manner, Eq. (83) can be expressed similar to the Eq. (93). Thus after condensing $\boldsymbol{X}_2$, Eqs. (77) and (83) each has the form

$$\dot{\boldsymbol{X}}=\boldsymbol{A}(t)\boldsymbol{X}. \quad (94)$$

Here $\boldsymbol{A}(t)$ is periodic in time with period $T=2\pi/\Omega_a$. According to FLOQUET theory [19] any solution of Eqs. (94) has following form:

$$\boldsymbol{X}(t)=\gamma\boldsymbol{X}_0(t)=e^{\lambda t}\boldsymbol{X}_0(t) \quad (95)$$

with $\boldsymbol{X}(t)$ and $\boldsymbol{X}_0(t)$ as vector functions, $\boldsymbol{X}_0(t)$ is periodic, and γ is the FLOQUET multiplier. From FLOQUET theory, monodromy matrix can be computed numerically as described in [21].

Let $\boldsymbol{Y}(t)=[\boldsymbol{X}_1(t),\boldsymbol{X}_2(t),\cdots,\boldsymbol{X}_n(t)]$ be a set of linearly independent solutions of Eq. (94), then $\boldsymbol{Y}(0)\boldsymbol{M}=\boldsymbol{Y}(T)$, and $\boldsymbol{M}$ is called the monodromy matrix.

For every $\boldsymbol{X}_i(0)$ as the initial condition, using numerical integration to integrate Eq. (94) for a period T, we have the vector $\boldsymbol{X}_i(T)$. Repeating from $i=1$ to $i=n$, we have monodrony matrix $M=\boldsymbol{Y}(0)^{-1}\boldsymbol{Y}(T)$. We choose $\boldsymbol{Y}(0)$, an identity matrix, and have $M=\boldsymbol{Y}(T)$. The eigenvalues of monodromy matrix are FLOQUET multipliers γ_k. Equation (94) is stable when the modulus of any multiplier $|\gamma_k|$ does not exceed unity. If multipliers are ordered, $|\gamma_1|\geqslant|\gamma_2|\geqslant\cdots\geqslant|\gamma_k|$, then

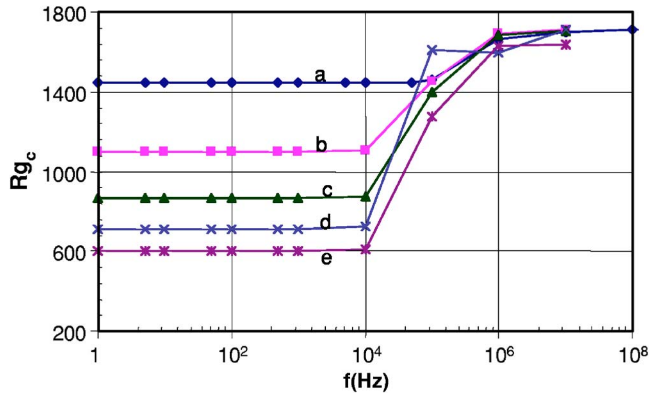

FIG. 2. Plot of critical Rayleigh number Rg_c against frequency for water-based magnetic fluid for different αL (model with internal rotation, thickness=1 mm): (a) $\alpha_L=0.2$, (b) $\alpha_L=0.4$, (c) $\alpha_L=0.6$, (d) $\alpha_L=0.8$, and (e) $\alpha_L=1.0$.

$|\gamma_1|=1$ corresponds the instability boundary and determines critical parameters.

In this work, the operations of equations are carried by MAPLE program and numerical calculation by MATLAB program. The procedure and algorithms of numerical analysis is as follows. For a given β, wave number k and H_0 with other physical parameters, we build monodromy matrix by integrating Eq. (94) in the time domain for a period T using the ODE15S function of the MATLAB program, which uses backward differentiation formulas.

The QZ algorithm, EIG function in MATLAB, was used to compute the eigenvalues of monodromy matrix. We find the maximum modulus of eigenvalue $|\gamma_1|$ for corresponding wave number k. Adjusting β by the secant method, we get the temperature gradient β when the modulus $|\gamma_1|=1$.

We then follow the algorithm for determining neutral stability curves [22]. From the neutral stability curves (β,k), the critical temperature gradient β with critical wave number k_c can be defined as

$$\beta_c = \min_k \beta(\mathrm{Pr}, H_0, \cdots). \tag{96}$$

The minimization of Eq. (96) is carried out by the function FMINBND of MATLAB, which is a combination searching of

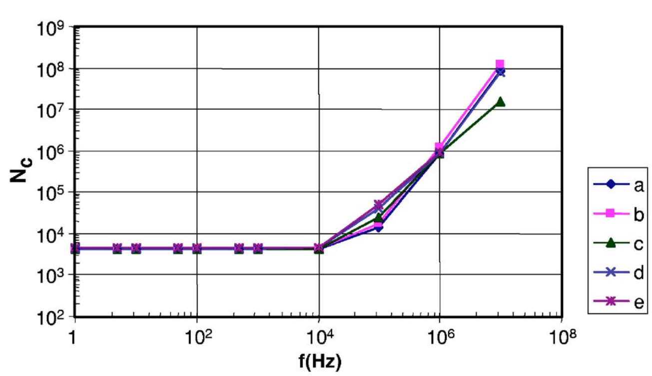

FIG. 3. Plot of critical magnetic Rayleigh number N_c against frequency (model with internal rotation, thickness=1 mm): (a) $\alpha_L=0.2$, (b) $\alpha_L=0.4$, (c) $\alpha_L=0.6$, (d) $\alpha_L=0.8$, and (e) $\alpha_L=1.0$.

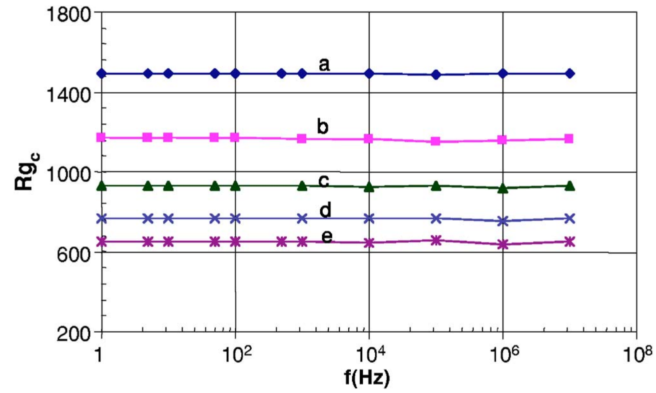

FIG. 4. Plot of Rg_c against frequency fin a water-based magnetic fluid (quasistationary model, thickness=1 mm): (a) $\alpha_L=0.2$, (b) $\alpha_L=0.4$, (c) $\alpha_L=0.6$, (d) $\alpha_L=0.8$, and (e) $\alpha_L=1.0$.

gold section and parabolic method. Rg_c and N_c are calculated by Eq. (24) using β_c.

In our numerical calculations we consider a magnetic fluid with fixed magnetic particles ($M_d=4.46\times10^5$ A/m) and diameters ($d=10$ nm). Table 1, in our previous work [15] (which is taken from [1]), provides the physical properties. In the following calculations, we have computed vortex viscosity ξ_r by the formula $\xi_r=\frac{3}{2}\eta\varphi$ and Brownian relaxation time $\tau_m=3V\eta/K_BT$. Here V is the particle effective hydrodynamic volume, and K_BT is thermal energy.

First, we checked the algorithms with the monodromy matrix. For alternating frequency $f=0$ Hz, i.e., in the absence of oscillating behavior we apply both the above-described algorithms and the algorithms discussed in [15]. We find almost identical results for both Rg_c and N_c. In the following we will consider both cases, i.e., when the convection is driven by both gravity and magnetic field and when it is only driven by magnetic forces. In our discussion we have taken the thin layer thickness $d=0.001$ m and f is defined as $f=\Omega_a/2\pi$.

Figure 2 shows the plot of Rayleigh number Rg_c against the frequency f for different α_L values in the water-based magnetic fluid. Up to the values of $f=1\times10^4$ Hz, we note only the effect of the variation of the magnetic field. Thus,

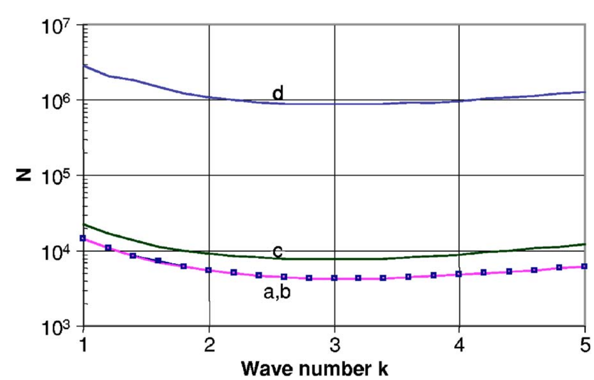

FIG. 5. Variation of magnetic Rayleigh number N with the wave number k for water-based magnetic fluid (model with internal rotation, thickness=1 mm, $\alpha_L=0.4$): (a) $f=10^2$ Hz, (b) $f=10^3$ Hz, (c) $f=5\times10^4$ Hz, and (d) $f=10^6$ Hz.

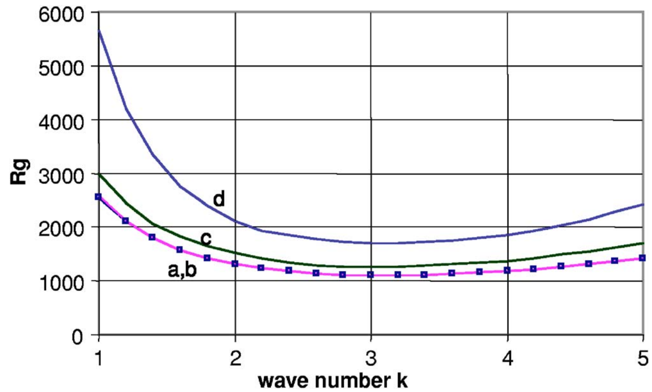

FIG. 6. Variation of Rayleigh number Rg with the wave number k for water-based magnetic fluid (model with internal rotation, thickness=1 mm, α_L=0.4): (a) $f=10^2$ Hz, (b) $f=10^3$ Hz, (c) $f=5\times10^4$ Hz, and (d) $f=10^6$ Hz.

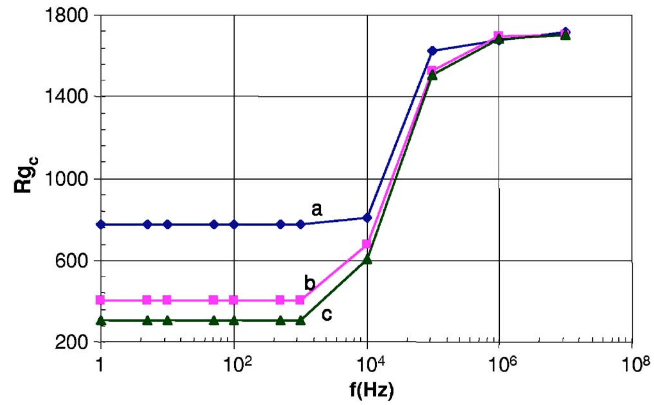

FIG. 7. Variation of critical Rayleigh number Rg_c against frequency f for different ester based magnetic fluid (model with internal rotation, thickness=1 mm, α_L=1.0): (a) ester I, (b) ester II, and (c) ester III.

we find that as the magnetic field increases, Rg_c decreases, indicating that the convection would initiate earlier for higher magnetic field values. Around $f=10^4$ Hz, and greater than this value, there is a sudden increase in Rg_c values for, almost all values of α_L. This increase in Rg_c values continues up to $f=10^6$ Hz and then it becomes steady in almost all cases. With $\tau_m=2.67\times10^6$ s, for water I, we note that as f varies between 10^4 and 10^6, $\Omega_a\tau_m=2\pi f\tau_m$ varies between 0.168 and 16.8. We also note from Fig. 2 that, between $10^5 \leqslant f \leqslant 10^7$, all the curves flatten, indicating the maximin and minimum values of $\Omega_a\tau_m$. The situation, in the magnetic Rayleigh number case (gravity free) is somewhat similar. Here we do not observe, (see Fig. 3) variation in N_c values up to $f=10^4$, but then again there is a sudden increase in N_c values as f increases beyond $f=10^4$. There is, however, one significant difference at these higher values. Although in the case of Rg_c critical, the values beyond $f=10^4$ become steady, (very close to standard Bénard problem Rg_c=1707), in the case of N_c critical these remain unbounded. This means that in the case when buoyancy forces are negligible and if the frequency increases indefinitely, there will be no convection possible. This may be related to the kind of magnetization equation used in our work.

Figure 4 plots the Rg_c against f for a quasistationary model. Here we note that increasing frequency has, essentially, very little effect on Rg_c values. However, increasing α_L (i.e., the magnetic field) has a destabilizing effect. Both frequency change and magnetic field change have virtually no effect when convection is due to magnetic forces only.

Figures 5 and 6 present neutral curves of magnetic instability for several values of the frequency f. In the quasistationary model, there is essentially no effect of frequency variation, and therefore the graphs are not presented. In the particle rotation model, there is some difference with the frequency and wave-number variation. Again minimum occurs at much lower values in Rg, as compared to N, but around the same wave number in each case.

In order to check whether ester-based magnetic fluids will behave similar to water-based fluids or not, we carried out calculations with ester I, II, and III base magnetic fluids. Figure 7 shows the plot of critical Rg_c against f for ester I, II, and III at α_L=1. Here we note that while ester I base behaves much like the water-based magnetic fluid, there is some difference with ester II and ester III based magnetic fluids. It appears that ester I is more stable than the two other magnetic fluids. Again near $f=10^4$ Hz, we find a sudden change in the stability character for all three base fluids. The graphs between N_c and f for different ester-based fluids were also plotted. But since these turn out to be similar to those of Fig. 7, these are not reported.

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