

Nonequilibrium fluctuations in a resistor

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In small systems where relevant energies are comparable to thermal agitation, fluctuations are of the order of average values. In systems in thermodynamical equilibrium, the variance of these fluctuations can be related to the dissipation constant in the system, exploiting the fluctuation-dissipation theorem. In nonequilibrium steady systems, fluctuation theorems (FT) additionally describe symmetry properties of the probability density functions (PDFs) of the fluctuations of injected and dissipated energies. We experimentally probe a model system: an electrical dipole driven out of equilibrium by a small constant current I , and show that FT are experimentally accessible and valid. Furthermore, we stress that FT can be used to measure the dissipated power $\bar{P} = RI^2$ in the system by just studying the PDFs' symmetries.

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INTRODUCTION

In the last decade, fluctuation theorems (FT) [1,2] appeared in nonequilibrium statistical physics. These theorems relate the asymmetry of fluctuations of energies (or powers) with the dissipated power required to maintain a nonequilibrium steady state of a system. Thus FT give a measure of the distance from equilibrium. Indeed, FT can be extremely useful to probe nonequilibrium states in nanophysics and biophysics where the energies involved are typically small and thermal agitation cannot be neglected: in those systems, the variance of fluctuations is of the order of the average values. Moreover, standard fluctuation-dissipation theorem (FDT) [3–5] is derived for equilibrium systems only and so it may fail to describe states far from equilibrium. FT are a generalization of FDT out of—possibly far from—equilibrium. In order to safely apply FT in complex systems, e.g., in biophysics or nanotechnologies, it is important to test them in all simple cases where theoretical predictions can be accurately checked. In spite of the large theoretical effort, just a few experiments have been conducted on this topic [6,7]. For this reason, we test in this communication their use on a simple electrical system.

Electrical systems are particularly interesting, because they are one of the first where FDT was formalized: Johnson [8] and Nyquist [9] related equilibrium fluctuations of voltage U across a dipole with the resistive part of this dipole. Moreover, all parameters in the setup can easily be controlled. Thus, out of equilibrium, electrical circuits are a key example to stress the differences between FT and FDT, and to verify the validity of FT.

Our system is an electrical dipole constituted of a resistance R in parallel with a capacitor C (Fig. 1). We drive it out of equilibrium by making a constant current I flow in it. Noting k_B the Boltzmann constant and T the absolute temperature, the injected power is typically of some $k_B T$ per second, of the same order as in biophysics or nanoscale physics experiments. This represents the fundamental case of

a system in contact with two different (electrons) reservoirs, one of the simplest and most fundamental problems of nonequilibrium physics [10]. Using a powerful analogy with a forced Langevin equation [11], a precise formulation of FT was recently given [12] in this case. Nyquist's FDT as well as FT are checked experimentally, by looking at the injected power and the dissipated heat in the system.

EXPERIMENTAL SETUP

The circuit we use is composed of a resistor in parallel with a capacitor, as depicted in Fig. 1. The resistance is a standard metallic one of nominal value $R=9.52\text{ M}\Omega$. In parallel, we have an equivalent capacitor of value $C=280\text{ pF}$. This accounts for the capacitance of the set of coaxial connectors and cables that we used. The time constant of the circuit is $\tau_0 \equiv RC=2.67\text{ ms}$. Using a $50\text{-G}\Omega$ resistance, we inject in the circuit a constant current I ranging from 0 to $6 \times 10^{-13}\text{ A}$. This current corresponds to an injected power $I^2 R$ ranging from 0 to $1000\text{ }k_B T/\text{s}$. Experiments are conducted at room temperature $T=300\text{ K}$. The typical values of the injected energy for, e.g., $I=1.4 \times 10^{-13}\text{ A}$ and $\tau = 10\tau_0$ are of order of a few hundreds of $k_B T$, which is small enough to ensure that the resistance is not heating; expected changes of temperature are estimated to be less than 10^{-14} K over a 1-h experiment. Moreover, the resistance is thermo-

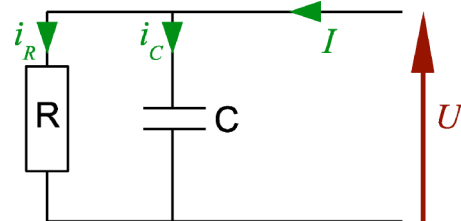


FIG. 1. (Color online) Model circuit: an electrical dipole is composed of a resistive part R and a capacitive part C . Due to thermal fluctuations of charges positions, a fluctuating voltage U is observed. We drive the system away from equilibrium by imposing a constant flux of electrons, via a constant current I .

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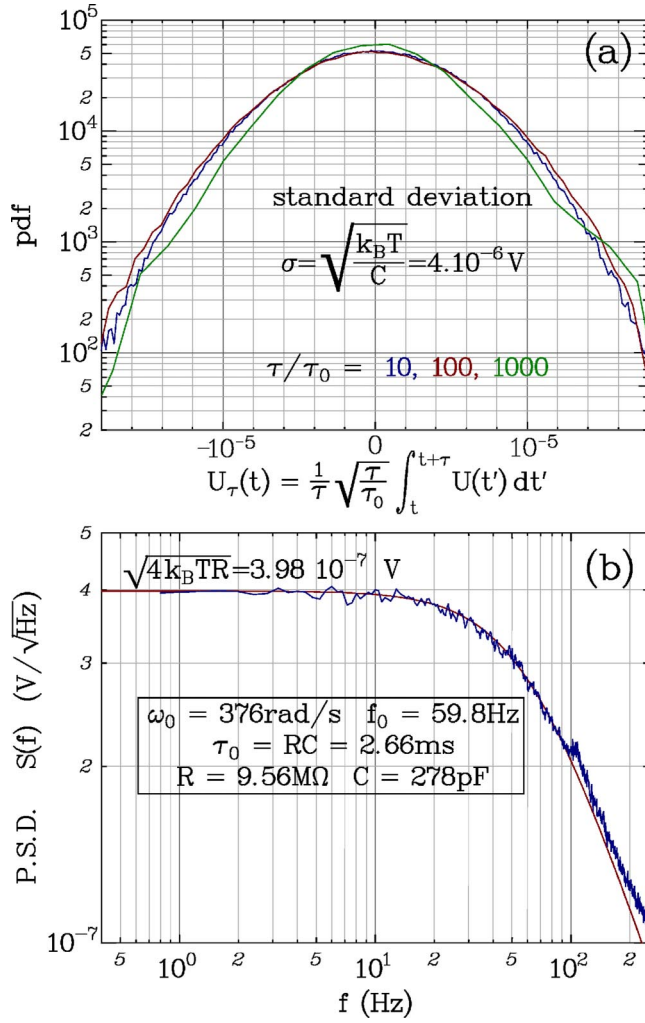


FIG. 2. (Color online) (a) For $I=0$ A, Johnson-Nyquist noise has a Gaussian distribution. Relation (2) is verified. (b) The noise is white up to the cutoff frequency f_0 of the RC dipole. We use FDT to extract from the energy of the noise the value of the resistive, dissipative part of the circuit R : for low frequency, the noise level spectral density is constant, equal to $\sqrt{4k_B TR}$ in a band $[f; f+df]$. A Lorentzian fit of the spectrum (thin line) additionally gives $\tau_0 = RC$.

stated: all the heat dissipated by the joule effect is absorbed by the thermal bath.

The fluctuating voltage U across the dipole is measured with a resolution of 10^{-11} V sampled at 819.2 Hz. This is achieved by first amplifying the signal by 10^4 , using a field-effect transistor (FET) amplifier, with a 4-G Ω input impedance, a voltage noise level of 5 nV/ $\sqrt{\text{Hz}}$, and a current noise of 10^{-15} A/ $\sqrt{\text{Hz}}$. The signal is then digitized with a 24-bit data acquisition card at frequency 8192 Hz, and decimated by averaging ten consecutive points.

FLUCTUATION-DISSIPATION THEOREM

The electrical dipole of Fig. 1 is a pure resistance R in parallel with a capacitor C ; its complex impedance reads $Z(f) = 1/(1/R + i2\pi RCf)$, where $i^2 = -1$ and f is the fre-

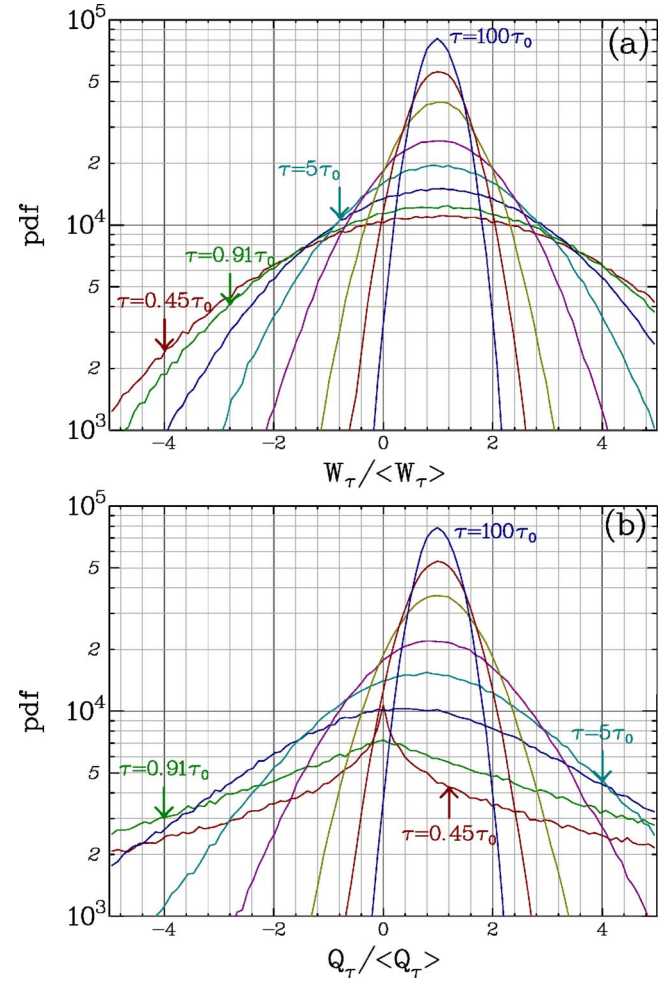


FIG. 3. (Color online) Histograms of W_τ and Q_τ when a current $I = 1.4 \times 10^{-13}$ A is flowing in the dipole. This corresponds to $\langle W_\tau \rangle = \langle Q_\tau \rangle = \tau \bar{P}$ with $\bar{P} = RI^2 = 45$ k $_B T/s$.

quency. The effective dissipative part is the real part $\text{Re}(Z)$ of Z . It was experimentally observed by Johnson [8], and then demonstrated by Nyquist [9], that the potential difference U across the dipole fluctuates with a stationary power spectral density $S(f)df$ such that

$$S(f)df = 4k_B T \text{Re}(Z)df. \quad (1)$$

At equilibrium $I=0$, in average no current is flowing in the circuit, and mean U is zero. Integrating over all positive frequencies, one gets the variance of U ,

$$\langle U^2 \rangle = \frac{k_B TR}{\tau_0} = \frac{k_B T}{C}. \quad (2)$$

Equations (1) and (2) are the expressions of the FDT for electrical circuits.

The exact value of our capacitance was determined by fitting the power-density spectrum of equilibrium fluctuations (at imposed $I=0$ A) by a Lorentzian low-pass transfer function [Eq. (1)], as illustrated in Fig. 2(b). Application of FDT leads with a very good accuracy to the determination of R , in perfect accordance with the measured nominal value

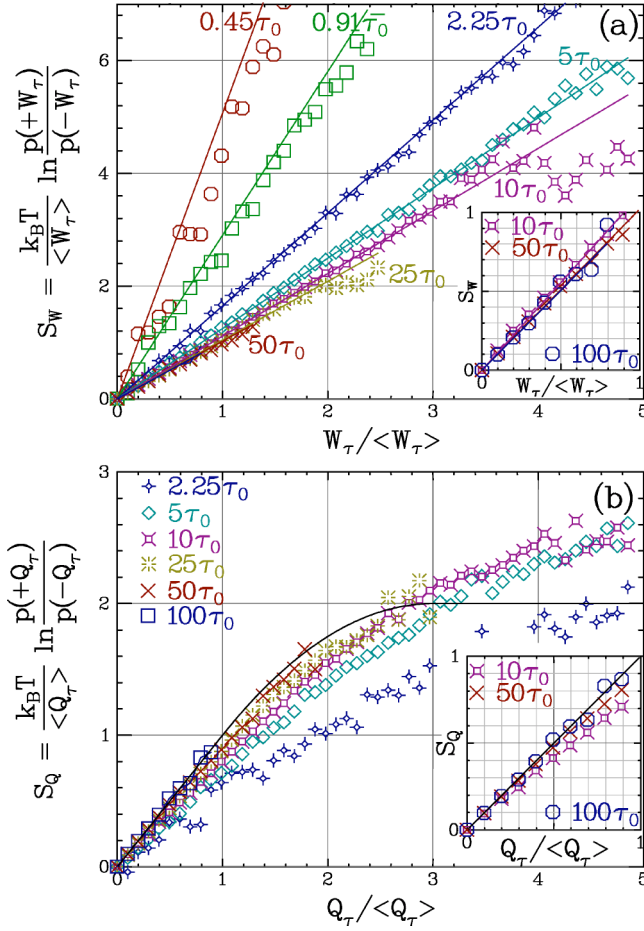


FIG. 4. (Color online) Normalized symmetry functions S_W and S_Q for W_τ and Q_τ when $I=1.4 \times 10^{-13}$ A (same conditions as Fig. 3). (a) For any τ , S_W is a linear function of $a=W_\tau/\langle W_\tau \rangle$. Straight lines are theoretical predictions from [12]. For $\tau \rightarrow \infty$, $S_W(a)$ tends to have a slope 1. (b) On the contrary, $S_Q(a)$ tends to a limit function (continuous black curve) which is a straight line of slope 1 for $a < 1$ only.

(Fig. 2). When $I=1.4 \times 10^{-13}$ A, we found the same power spectral density for U , and performing the same treatments gave the same estimates of R and C . We therefore conclude that FDT is still holding in our system driven out of equilibrium.

FLUCTUATION THEOREMS

The power injected in the circuit is $\mathcal{P}_{\text{in}}=UI$, but only the resistive part dissipates, so the dissipated power is $\mathcal{P}_{\text{diss}}=Ui_R$, where i_R is the current flowing in the resistor (Fig. 1). As already noted [13], in average, one expects $\langle \mathcal{P}_{\text{in}} \rangle = \langle \mathcal{P}_{\text{diss}} \rangle \equiv \bar{\mathcal{P}}$, where the brackets stand for time average over sufficiently long times compared to τ_0 . This is very well checked in our experiment. \mathcal{P}_{in} and $\mathcal{P}_{\text{diss}}$ fluctuate in time because U itself is fluctuating. If one assumes that fluctuations of U have a Gaussian distribution, which is the case at equilibrium when $I=0$, then \mathcal{P}_{in} has also a Gaussian distribution, because I is constant. On the contrary, i_R fluctuates, as we see from Kirchoff's laws,

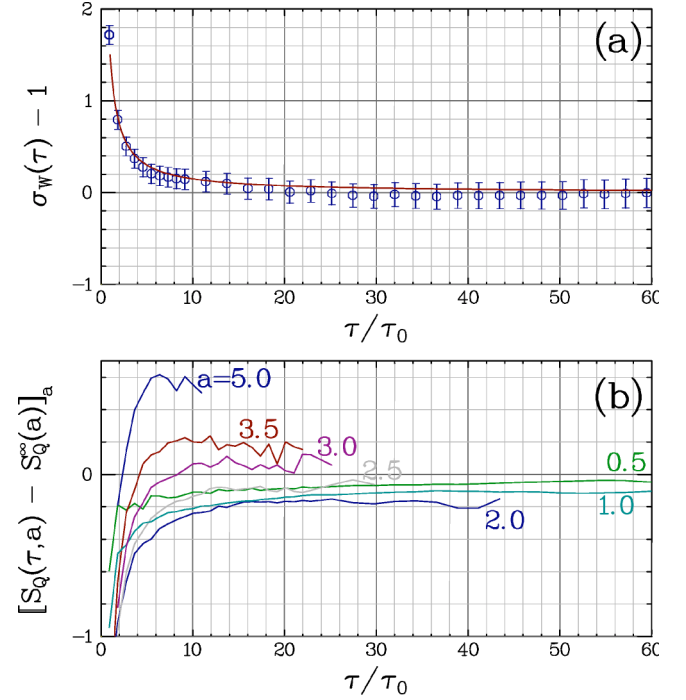


FIG. 5. (Color online) (a) Dependence on τ of the slope of S_W in Fig. 4. Slope is converging from above. (b) Distance between $S_Q(\tau)$ for finite τ and theoretical prediction of S_Q for infinite τ , for several values of $a=Q/\langle Q \rangle$. For $a < 3$, convergence is from below, whereas it is from above for $a > 3$.

$$I = i_R + C \frac{dU}{dt}, \quad \text{so} \quad \mathcal{P}_{\text{in}} = \mathcal{P}_{\text{diss}} + \frac{1}{2} C \frac{dU^2}{dt}, \quad (3)$$

and therefore, the probability distribution of $\mathcal{P}_{\text{diss}}$ is not Gaussian [12]. It is worth noting that for large current I , some orders of magnitude larger than the one we use, \mathcal{P}_{in} and $\mathcal{P}_{\text{diss}}$ will be much larger than the conservative part $(C/2) \times (dU^2/dt)$ and therefore the probability distributions of both the injected and dissipated power will be practically equal and usually Gaussian, as it is expected in macroscopic systems.

We call $\langle g \rangle_\tau(t) = 1/\tau \int_t^{t+\tau} g(t') dt'$ the time-averaged value of a function g over a time τ .

Reasoning with energies instead of powers, we define $W_\tau(t) = \tau \langle \mathcal{P}_{\text{in}} \rangle_\tau(t)$, the energy injected in the circuit during time τ , analogous to the work performed on the system (positive when received by the system). In the same way, we write $Q_\tau(t) = \tau \langle \mathcal{P}_{\text{diss}} \rangle_\tau(t)$, the energy dissipated by the joule effect during time τ , analogous to the heat dissipated by the system (positive when given by the system to the thermostat). We used values of τ spanning from fractions of τ_0 up to hundreds of τ_0 .

For a given value of I , we measure $U(t)$ and compute \mathcal{P}_{in} and $\mathcal{P}_{\text{diss}}$. We then build the probability density functions of cumulated variables W_τ and Q_τ using samples of 10^6 points; their typical distributions are plotted in Fig. 3. As expected, fluctuations of W_τ are Gaussian for any τ , whereas heat fluctuations are not for small values of τ . They are exponential

for small τ : large fluctuations of heat Q_τ are more likely to occur than large fluctuations of work W_τ .

Then we study the normalized symmetry function

$$S_E(\tau, a) \equiv \frac{k_B T}{\tau \bar{P}} \ln \frac{p(E_\tau = a)}{p(E_\tau = -a)}, \quad (4)$$

where E_τ stands for either $W_\tau/\langle W_\tau \rangle$ or $Q_\tau/\langle Q_\tau \rangle$. If the fluctuation theorem for the work W_τ holds, then one should have [11,12], for large enough τ , the relationship $\lim_{\tau \rightarrow \infty} f_W(\tau, a) = a$. In contrast, if the fluctuation theorem for the heat Q_τ holds, then for $\tau \rightarrow \infty$, the asymptotic values of $S_Q(\tau, a)$ are $S_Q^\infty(a) = a$ for $a \leq 1$, $S_Q^\infty(a) = 2$ for $a \geq 3$, and there is a continuous parabolic connection for $1 \leq a \leq 3$ that has a continuous derivative [11,12]. From histograms of Fig. 3, we compute the symmetry functions $S_W(\tau, a)$ and $S_Q(\tau, a)$ (Fig. 4).

(a) Work fluctuations. First, for any given τ we checked that the symmetry function $S_W(\tau, a)$ is linear in a [Fig. 4(a)]. We measured the corresponding proportionality coefficient $\sigma_W(\tau)$ such that $S_W(\tau, a) = \sigma_W(\tau)a$. This coefficient $\sigma_W(\tau)$ tends to 1 when τ is increased (see Fig. 5).

(b) Heat fluctuations. We found that $S_Q(\tau, a)$ is linear in a only for $a < 1$, as expected [11,12]. Again, as $\tau \rightarrow \infty$, the limit slope of the symmetry function is 1, whereas for $a > 3$, $S_Q(\tau, a)$ tends to 2.

(c) Asymptotic symmetry functions and convergence. In [11,12], expressions for the convergence towards the asymptotic limits $S_W^\infty(a)$ and $S_Q^\infty(a)$ are given in terms of τ . We can check these predictions with our data. The convergence for the work W_τ is very well reproduced by these predictions, as can be seen in Fig. 4(a) continuous straight lines are theoretical predictions for small values of τ , using Eqs. (9) and (10) from [12], with no adjustable parameters. In Fig. 5(a) the slope $\sigma_W(\tau)$ of the experimental symmetry function is plotted. The continuous line is the prediction of [12], which perfectly agrees with our data.

For the heat, we distinguish two regimes for the convergence towards the asymptotic symmetry function. For $a = Q_\tau/\langle Q \rangle < 3$, we find that when τ is increased, symmetry

functions are converging to the asymptotic function from below (Fig. 4), which is the opposite of what is observed for the work. On the contrary, for $a > 3$, convergence to the asymptotic function is from above, thus enhancing the peculiarity of the point $a=3$. In Fig. 5, we have plotted the evolution of $S_Q(\tau, a)$ versus τ for several fixed values of a . The convergence from above for $a \geq 3$ and from below for $a < 3$ is clear. For increasing values of τ , only smaller and smaller values of a are accessible because of the averaging process. Therefore, the accessible values of a are quickly lower than 1, and only the linear part of $S_Q(\tau, a)$ can be experimentally tested. Nevertheless, for intermediate time scales τ we see in Fig. 4 that the data converge towards the theoretical asymptotic nonlinear symmetry function [11,12] [smooth curve in Fig. 4(b)].

We observed that the convergence reproduced in Fig. 5 depends on the injected current I , as pointed out in [12]. Other experiments with a larger current ($\bar{P} = 186 k_B T/s$) give a faster convergence; corresponding results will be reported elsewhere.

CONCLUSIONS

We have shown experimentally that the asymmetry of the probability distribution functions of work and heat in a simple electrical circuit driven out of equilibrium by a fixed current I , is linked to the averaged dissipated power in the system. The recently proposed fluctuation theorems for first-order Langevin systems are then experimentally confirmed. Exploiting formula (4), FT can be used to measure an unknown averaged dissipated power $\bar{P} = \lim_{\tau \rightarrow \infty} Q_\tau/\tau$ by using only the symmetries of the fluctuations, i.e., computing S_W or S_Q and measuring their asymptotic slope.

We operated with energies of order of $k_B T$ in order to have strong fluctuations compared to the averaged values. It is worth noting that as the driving current is increased up to macroscopic values, the fluctuations become more and more negligible, therefore fluctuation theorems become harder and harder to use, and therefore less relevant.

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