Criticality in a Vlasov-Poisson system: A fermioniclike universality class

A. V. Ivanov,* S. V. Vladimirov, and P. A. Robinson

School of Physics, The University of Sydney, NSW 2006, Sydney, Australia

(Received 17 September 2004; revised manuscript received 25 March 2005; published 26 May 2005)

A model Vlasov-Poisson system is simulated close to the point of marginal stability, thus assuming only the wave-particle resonant interactions are responsible for saturation, and shown to obey the power-law scaling of a second-order phase transition. The set of critical exponents analogous to those of the Ising universality class is calculated and shown to obey the Widom and Rushbrooke scaling and Josephson's hyperscaling relations at the formal dimensionality $d=5$ below the critical point at nonzero order parameter. However, the two-point correlation function does not correspond to the propagator of Euclidean quantum field theory, which is the Gaussian model for the Ising universality class. Instead, it corresponds to the propagator for the fermionic vector field and to the upper critical dimensionality $d_c=2$. This suggests criticality of collisionless Vlasov-Poisson systems corresponds to a universality class analogous to that of critical phenomena of a fermionic quantum field description.

DOI: 10.1103/PhysRevE.71.056406 PACS number(s): 52.35.Mw, 05.30.Fk, 05.70.Fh, 64.60.Fr

I. INTRODUCTION

The remarkable property of critical phenomena is the universal scaling appearing in a vast variety of systems; e.g., magnets and gases follow simple power laws for the order parameter, specific heat capacity, susceptibility, compressibility, etc. [1]. In thermodynamic systems, phase transitions take place at a critical temperature T_{cr} when the coefficients that characterize the linear response of the system to external perturbations *diverge*. Then, long-range order appears, causing a transition to a new phase due to collective behavior of the entire system [2].

The condition for nonlinear saturation in the test case of the bump-on-tail instability in plasmas [3] is

$$\omega_b \approx 3.2\gamma_L, \tag{1}$$

[4,5], where γ_L is the linear growth rate of a weakly unstable Langmuir wave according to the Landau theory [6], and $\omega_b=(eEk/m)^{1/2}$ is the frequency of oscillations for particles trapped by the wave. These trapped particles generate a long-range order of the wavelength k, the saturated amplitude E can be considered as the order parameter, and the condition of saturation can be rewritten as a power law, typical for the second-order phase transitions, $E\sim\gamma_L^{\beta}$.

However, unlike thermodynamics this scaling contains the nonthermal control parameter γ_L, which is determined by the slope of the distribution function $\partial f_0/\partial v$ at the phase speed $v_r=\omega_{pe}/k$ of the perturbation near the electron plasma frequency ω_{pe}. [For thermodynamic systems like magnets, for which the magnetization M below the Curie point T_{cr} is the order parameter, the scaling is $M\propto\epsilon^{\beta}$, where $\epsilon=(T_{cr}-T)/T_{cr}$ at $T<T_{cr}$.] Another difference is the critical exponent itself—relation (1) predicts the very unusual exponent $\beta=2$; in contrast, the hydrodynamic Hopf bifurcation [7], also described by the same scaling between the saturated amplitude and the growth rate, has the mean-field critical exponent $\beta=1/2$.

An analysis [8] assuming thermalization in a Vlasov-Poisson plasma or gravitating system leads to a critical exponent $\beta<1$, and the exponent $\beta=1/2$ has also been hypothesized for the bump-on-tail instability in Ref. [9]. However, detailed center-manifold analysis, which establishes the normal form for a weakly unstable perturbation in a one-component collisionless Vlasov-Poisson system, confirms $\beta=2$ [10]. The exponent $\beta=2$ is also confirmed numerically [11,12].

The striking discrepancy between these exponents can be better understood if we consider the structure of the phase space corresponding to these cases. The exponents $\beta=1/2$ [7] and $\beta<1$ [8] correspond either to saturation of a strongly *dissipative* instability or to a thermalized system. In both these cases the distribution function can be factorized as $f(q,p)=y(q)g(p)$, where $g(p)$ can be assumed to be Gaussian, and the system is described by its momenta. The exponent $\beta=2$ corresponds to saturation due to nonlinear wave-particle interactions in a weakly unstable *collisionless* system, where correlations between coordinates and impulses are not destroyed by dissipative processes, so the description cannot be reduced to moments of the distribution.

More formally, a dissipative and/or thermalized system is represented by a *discrete* set of momenta of the distribution function $f(x,v,t)$, which depends only on the coordinate x, but not on the velocity v, $M=\{\rho,\overline{v},T\}$, where ρ, $\overline{v}$, and T are the local density, the velocity, and the temperature, respectively. The evolution is a flow g_t which maps M onto itself, g_t: $M\rightarrow M$. In fact, in a neighborhood of the threshold $\gamma_L=0$ the evolution g_t: $M\rightarrow M$ can be reduced to a normal form, which maps only the order parameter, g_t': $n\rightarrow n$, where $n=\mathbb{R}^0$ (or $\mathbb{C}^0$, where $\mathbb{C}$ is the set of complex numbers), and therefore the evolution is a *trajectory* $n=n(t)$, or in other words the set $Y=\mathbb{R}^{+}\times\mathbb{R}^0$. The phase space of a one-dimensional collisionless system is a *continuous* set $H=\mathbb{R}\times\mathbb{R}$ and evolution can be represented as the flow w_t:

*Electronic address: ivanov@physics.usyd.edu.au

$H \to H$. (For periodic boundary conditions the phase space is isomorphic to a cylinder $C = \mathbb{T} \times \mathbb{R}$, where $\mathbb{T}$ is isomorphic to a circle.) The sets n and H (or C) have different *dimensionality*, and therefore renormalization of collisionless system in a vicinity of threshold—i.e., transformation of the set H *and* the mapping w_t—involves one or more additional dimension. Further, it is shown below that scaling transformations close to the threshold are inter-related with the additional velocity coordinate, which disappears in hydrodynamic description because of integration of the distribution function $f(x,v,t)$ over v.

From the theory of critical phenomena it is known that dimensionality d is an inseparable part of the threshold description—along with the critical exponents (e.g., Ref. [13]). Besides β, other critical exponents: α, γ, δ, ν, and η, describe the following scalings of the Ising universality class:

(i) the specific heat capacity scales as

$$C = \frac{\delta Q}{dT} \propto |\epsilon|^{-\alpha}; \tag{2}$$

(ii) the susceptibility as

$$\chi = \left(\frac{\partial M}{\partial B}\right)_{B\to 0} \propto |\epsilon|^{-\gamma}; \tag{3}$$

(iii) the response M at $\epsilon = 0$ as

$$M \propto B^{1/\delta}; \tag{4}$$

(iv) the correlation length as

$$\xi \sim |\epsilon|^{-\nu}; \text{ and} \tag{5}$$

(v) the two-point correlation function as

$$G(r) \sim \frac{e^{-r/\xi}}{r^{d-2+\eta}}. \tag{6}$$

These exponents are not independent, but are inter-related via scaling laws, e.g., the Widom equality

$$\gamma = \beta(\delta - 1) \tag{7}$$

[14]. These scaling laws also include *hyperscaling* laws such as Josephson's law

$$\nu d = 2 - \alpha \tag{8}$$

[15], which involves the dimensionality d along with the exponents.

For thermodynamics the mean-field exponents are of the Landau-Weiss set, $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, $\delta = 3$, $\nu = 1/2$, and $\eta = 0$, and the scaling laws hold at the formal dimensionality $d = 4$. However, the possibilities of critical phenomena are not exhausted by the Ising universality class—the percolation critical exponents [16], which describe another vast class of critical phenomena, are different from those in thermodynamics, and scaling laws hold at a different dimensionality. In particular, for the Bethe lattice (or Cayley tree) [17] Josephson's law holds at dimensionality $d = 6$. Despite the description being the same, this difference separates the cases into different *universality classes* with different *upper critical dimensions*: $d_c = 4$ for the Ising universality class [18] and $d = 6$ for percolation.

For a collisionless gravitating system, where the saturation mechanism is the same as for the bump-on-tail instability in plasmas, the critical exponent $\beta = 1.907 \pm 0.006$, and the critical exponents $\gamma = 1.075 \pm 0.05$, $\delta = 1.544 \pm 0.002$ can be determined analogously to thermodynamics and calculated from the response to an external pump [19]. These exponents are very different from the thermodynamic set, but nevertheless satisfy the Widom equality, thus suggesting the validity of scaling laws. Josephson's law also holds, but at a rather surprising dimensionality which is the fractal one, $d \approx 4.68$ [19]. At the same time, the processes resulting in $\beta \approx 1.9$ differ *qualitatively* from those resulting in $\beta = 2$, similar to thermodynamics where spatial fluctuations of the order parameter, neglected in mean-field theories, result in $\beta \approx 0.33$, therefore suggesting other universality classes were not completely ruled out. These could be the wave-wave interactions, responsible for the strong turbulence in plasma [20]), which are next in dynamical importance and have fewer degrees of freedom [21].

In this paper, we use numerical simulations to study the threshold scalings in a weakly unstable collisionless Vlasov-Poisson system. Depending on the sign of the Poisson equation, this set of equations describes either a plasma system or a gravitating system. The saturation mechanism in a collisionless gravitating system is the same as for the bump-on-tail instability in plasmas, and threshold corresponds to the condition $\gamma_L = 0$ in both cases. We show in Sec. II that the eigenfrequency contains only an imaginary part, and therefore is the simplest model to study the threshold. Section III describes the results of computations of the critical exponents and demonstrates that the scaling laws describing saturation are the same for plasma and gravitation. Section IV addresses the scaling transformations of the phase space and the scaling law, which appears as a result of this symmetry. The exponent which describes correlations are obtained in Sec. V, where Fisher's equality $\gamma = \nu(2 - \eta)$ is also proved. In Sec. VI we show that the criticality in the system is described by the Dirac propagator for a fermionic field. We obtain hyperscaling laws and calculate upper critical dimensionalities in Sec. VII.

II. BASIC EQUATIONS

The eigenfrequencies and eigenvectors of oscillations in a Vlasov-Poisson system are given by the dispersion relation

$$\varepsilon[\omega(\mathbf{k}),\mathbf{k}] = 0, \tag{9}$$

where ε is the permittivity (dielectric permittivity in the plasma case). The boundary between stable and unstable cases is determined by the condition

$$\mathrm{Im}[\varepsilon(\omega,\mathbf{k})] = 0 \tag{10}$$

[22]. For the bump-on-tail instability condition (10) simplifies to $\gamma_L \equiv \mathrm{Im}(\omega) = 0$, and criticality is related to the zero of the imaginary part of the eigenfrequency. Therefore, we can employ a model which does not contain the real part; i.e., $\mathrm{Re}(\omega) = 0$. The simplest is the one-dimensional self-

gravitating Vlasov-Poisson model, which is described by the equations

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v} = 0, \tag{11}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = \int_{-\infty}^{\infty} f(x,v,t)dv - 1, \tag{12}$$

where $f(x,v,t)$ is the distribution function, and Φ is the gravitational potential. Boundary conditions are assumed to be periodic in the x direction and all quantities are dimensionless.

Equations (11) and (12) describe the mean-field evolution of a self-gravitating model immersed in a homogeneous repulsive background. Analogous repulsion terms appear in some cosmological simulations [23] as a result of the expansion of the universe. Models of rotating stellar systems like disk galaxies provide other examples—"repulsion" appears as a result of the centrifugal force in the local rest frame (e.g., Ref. [24]). The system (11) and (12) could thus be considered as an idealized model of the galactic corotation region where a nonaxisymmetric perturbation is responsible for visible spiral structure, and stars orbit with the same angular speed (e.g., Ref. [25]), and only angular motions are allowed.

For the eigenfunctions

$$\mathbf{X} = \sum_{m=-\infty}^{\infty} \mathbf{X}_m \exp(ik_m x), \tag{13}$$

where

$$\mathbf{X} = [f(x,v,t),\Phi(x,t)]^T, \tag{14}$$

$$\mathbf{X}_m = [f_m(v,t),\Phi_m(t)]^T, \tag{15}$$

are the spatial Fourier components, the superscript T stands for transpose, $k_m = 2\pi m/L$ is the wave vector, and L is the system length, we find

$$\dot{f}_m + ik_m v f_m + i\sum_{m=m'+m''}\frac{1}{k_{m'}}\int_{-\infty}^{\infty} f_{m'}dv\frac{\partial f_{m''}}{\partial v} = 0, \tag{16}$$

or, explicitly for the components $m=\{0,1,2\}$ and for $L=2\pi$

$$\dot{f}_0 + i\frac{\partial}{\partial v}(\rho_1 f_{-1} - \rho_{-1} f_1) = 0, \tag{17}$$

$$\dot{f}_1 + ivf_1 + i\frac{\partial}{\partial v}\left(\rho_1 f_0 + \frac{1}{2}\rho_2 f_{-1} - \rho_{-1} f_2\right) = 0, \tag{18}$$

$$\dot{f}_2 + i2vf_2 + i\frac{\partial}{\partial v}\left(\frac{1}{2}\rho_2 f_0 + \rho_1 f_1\right) = 0, \tag{19}$$

where

$$\rho_m(t) = \int_{-\infty}^{\infty} f_m(v,t)dv, \tag{20}$$

is the Fourier component of density, and $f_{-1}=f_1^*$.

For a Maxwellian distribution

$$f_0(v) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{v^2}{2\sigma^2}\right), \tag{21}$$

the dispersion relation is

$$1 + \frac{1}{2}\frac{\sigma_J^2(m)}{\sigma^2}\left.\frac{dZ(\zeta)}{d\zeta}\right|_{\zeta=\zeta_m} = 0, \tag{22}$$

where Z is the plasma dispersion function [26], $\zeta_m = z_m/\sqrt{2}$, $z_m = \omega_m/(k_m\sigma)$, and σ is velocity dispersion. In Eq. (22)

$$\sigma_J^2(m) = \frac{1}{k_m^2} \equiv \frac{1}{m^2}, \tag{23}$$

is the critical (Jeans) velocity dispersion for the mode m, and ρ_0 is the background density. For small $|z| \ll 1$ [i.e., $\sigma^2/\sigma_J^2(m) \ll 1$], the dispersion relation reads

$$\varepsilon(\omega_m,k_m) = 1 - \frac{\sigma_m^2}{\sigma^2}\left(1 + i\sqrt{\frac{\pi}{2}}\frac{\omega_m}{k_m\sigma}\right) = 0, \tag{24}$$

or

$$\omega_m = -i\sqrt{\frac{2}{\pi}}k_m\sigma\left[\frac{\sigma^2 - \sigma_J^2(m)}{\sigma_J^2(m)}\right], \tag{25}$$

which is remarkably simpler than in the plasma case, where the bump-on-tail instability and Landau damping appear due to the wave-particle resonance at the phase velocity of the wave $v_{ph} = \omega_{pe}/k$. The frequency spectrum in the case of gravitation does not contain a real part, so the resonance occurs at $v=0$; i.e., in the main body of the particle distribution. For all m, $\sigma_J^2(m) > \sigma_J^2(m+1)$; thus, if we write $\sigma_J^2(1) \equiv \sigma_{cr}^2$, the distance from the instability threshold is

$$\theta = \frac{\sigma^2 - \sigma_{cr}^2}{\sigma_{cr}^2}, \tag{26}$$

analogously to $\epsilon = (T - T_{cr})/T_{cr}$. Using (25) and (26), the linear damping or growth rate γ_L can be written as

$$\gamma_L \equiv \mathrm{Im}(\omega_1) = -\sqrt{\frac{2}{\pi}}\theta. \tag{27}$$

Time is measured in units of the inverse of $|\gamma_L|$, $t' = t|\gamma_L|$.

III. RESULTS

Dispersion relation (25) shows that there are no unstable modes above the threshold σ_{cr}^2, $\sigma^2 > \sigma_{cr}^2$, and therefore the system remains invariant with respect to translations $x' \to x + \tau$, where τ is any number. Below the threshold the mode $m=1$ becomes unstable, and therefore the *continuous* symmetry breaks and reduces to a lower *discrete* one with respect to translations $x' \to x + L$. Therefore, σ_{cr}^2 can be considered as the critical point of a second-order phase transition, and the amplitude of the mode $m=1$ as the order parameter—following the definition of Landau [2].

A. Order parameter scaling

Equations (11) and (12) with initial distribution

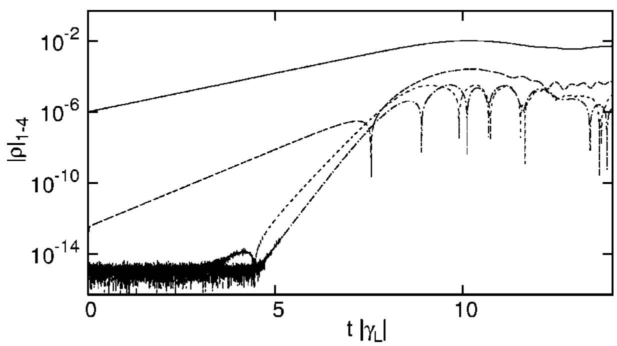

FIG. 1. Amplitudes of the first four harmonics, $m=1,2,3,4$, at $\theta=-0.04$. Quantities are dimensionless.

$$f(x,v,0) = f_0(v)[1 + A_0 \cos(k_1 x)], \tag{28}$$

were integrated numerically using the Cheng-Knorr method [27]. The amplitudes $A_m(t)=|\rho_m(t)|$ for $m=1,2,3,4$ are shown in Fig. 1. The perturbation $m=1$ grows exponentially with the growth rate predicted by dispersion relation (25). Then, the growth saturates at some moment $t=t_{sat}$ at the amplitude $A_{sat}=A_1(t_{sat})$.

Figure 1 also shows the (exponential) growth of perturbations with $m>1$, while (25) predicts exponential damping for these modes. This growth occurs because of nonlinear coupling between modes in Eq. (16). For instance, since the term $\rho_1 f_1$ dominates over the term $\rho_2 f_0$ in Eq. (19) initially, when $f_2 \ll f_1$ and therefore for the mode $m=2$, one has $\gamma_2 = 2\omega_1$ for the growth rate γ_2.

Figure 2 shows that A_{sat} is independent of A_0 for small A_0, but there exists some threshold value of the initial perturbation A_0 when it becomes dependent on A_0. This threshold amplitude A_{thr} corresponds to the trapping frequency $\omega_b = \sqrt{A_{thr}} \approx \omega_1$. At $\omega_b \approx \omega_1$ the processes due to trapping become as important as the resonance between wave and particles responsible for the linear Landau damping (or growth) in collisionless media. Therefore, to rule out the influence of trapping processes on linear growth the amplitude A_0 must be small to provide

$$\omega_b \ll \omega_1. \tag{29}$$

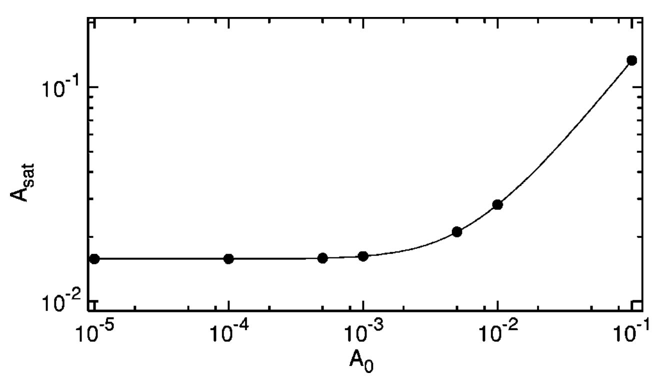

FIG. 2. The saturated amplitude A_{sat} vs A_0 for $\theta=-0.05$. Circles are calculated values, the curve is a spline approximation. Quantities are dimensionless.

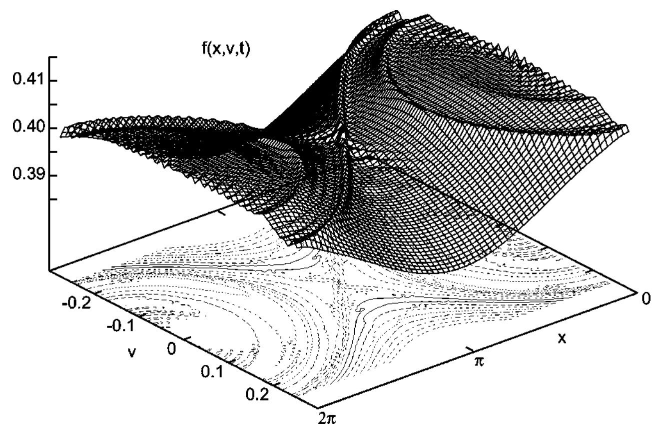

FIG. 3. Distribution function (surface) and its isocontours (lines in the x-v plane) in configuration space at the moment of saturation $t=t_{sat}$. Quantities are dimensionless.

The distribution function $f(x,v,t)$ is plotted in Fig. 3 as a surface at the moment $t=t_{sat}$. Note that the distribution function $f(x,v,t)$ becomes flat in the part of the x-v domain

$$\frac{v^2}{2} + A_{sat} \cos(k_1 x) \leqslant A_{sat} \tag{30}$$

separatrix, as predicted for the bump-on-tail instability [28,29]. Outside this area the Fourier component $f_1(v,t)$ remains modulated by the background Maxwellian distribution as assumed at $t=0$, and the components with $m>1$ remain negligible [19], so the dynamically important area lies at $v \leqslant |v_{sep}|$, where $v_{sep} = \pm 2\sqrt{A_{sat}}$. The width of the dynamically important area must be small compared to σ at maximum amplitude (i.e., A_{sat}, $v_{sep} \ll \sigma$), otherwise the background distribution will be altered by evolution.

Assuming the above two criteria, A_{sat} is calculated as a function of θ and plotted in Fig. 4. From Fig. 4 we see that this dependence can be approximated by the power law

$$A_{sat} \propto (-\theta)^{\beta}, \tag{31}$$

while $\beta = 1.9950 \pm 0.0034$ for $\theta \leqslant 0$. Rewritten in terms of the bounce frequency ω_b and the linear growth rate γ_L, the power law (31) becomes

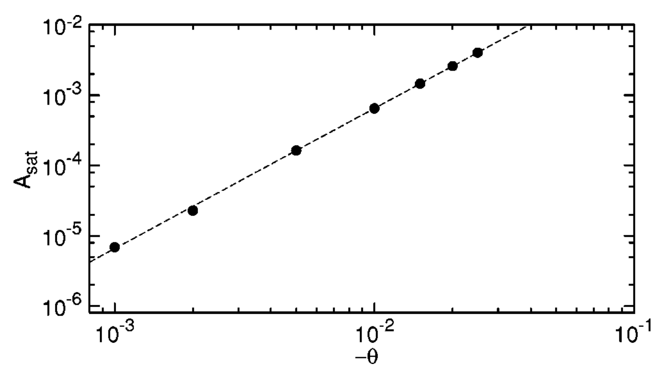

FIG. 4. Amplitude A_{sat} as a function of θ. Circles represent calculated data; the dashed line is the power-law best fit. Quantities are dimensionless.

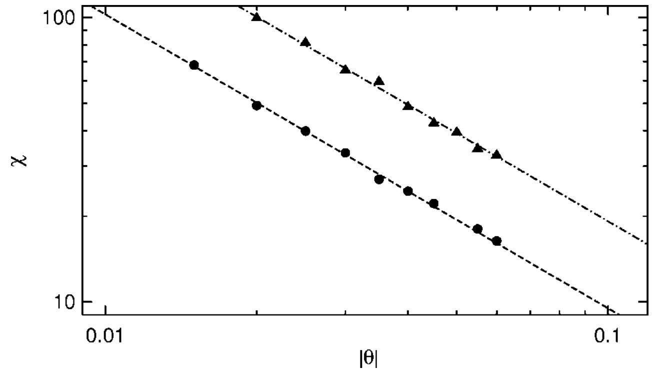

FIG. 5. Plot of χ vs $|\theta|$. Circles $(\theta>0)$ and triangles $(\theta<0)$ represent numerical data; the dashed lines are the power-law best fits. Quantities are dimensionless.

$$\omega_b = c\gamma_L, \tag{32}$$

and the coefficient $c=3.22\pm0.01$. These values are almost identical to $\beta=2$, and to the coefficient 3.2 in relation (1) as predicted and calculated for the bump-on-tail instability [10,12,29].

B. Response scaling

Subjecting the system to an external pump of the form $F(x)=F_m\cos(k_m x+\varphi)$ allows one to calculate the other two critical exponents, γ and δ, which describe the response properties. The index γ describes the divergence of the susceptibility, which can be written as

$$\chi(\theta) = \left.\frac{\partial A_{sat}(\theta)}{\partial F_1}\right|_{F_1\to 0}, \tag{33}$$

for $m=1$. The results are shown by triangles $(\theta<0)$ and circles $(\theta>0)$ in Fig. 5.

Computation of χ at some θ requires at least five values of A_{sat} corresponding to the given F_1. At the same time, F_1 must be small enough to avoid the effects of A_{sat} depending nonlinearly on F_1. Again, it requires extensive calculation of all quantities to high accuracy. In both cases $\chi(\theta)$ is approximated by

$$\chi_\pm \propto |\theta|^{-\gamma_\pm}, \tag{34}$$

and $\gamma_-=1.028\pm0.025$ for $\theta<0$, $\gamma_+=1.033\pm0.016$ for $\theta>0$, giving $\gamma_-\approx\gamma_+=\gamma\approx1$. These exponents are very close to the corresponding results for Ref. [19], because χ is the only linear coefficient, and this is common to both wave-particle and wave-wave interactions.

The exponent γ is the same as for the mean-field thermodynamic models but, opposite to thermodynamics, the response is *stronger* at $\theta<0$ than at $\theta>0$, as Fig. 5 shows. The susceptibilities are

$$\chi_- \approx 2\chi_+. \tag{35}$$

This difference, as well as the appearance of scaling (1) and (32) with $\beta=2$ instead of $\beta=1/2$, can be explained if one takes into account the difference between the Landau-Ginzburg Hamiltonian

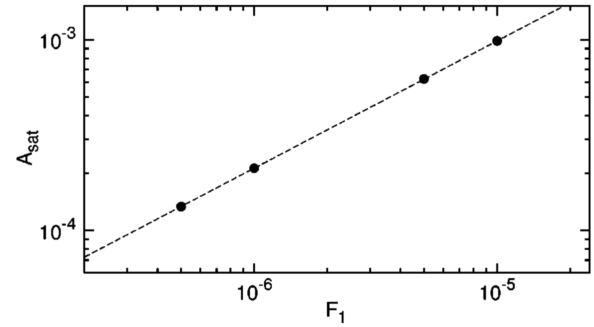

FIG. 6. A_{sat} as a function of F_1. Circles represent numerical data; the dashed lines are the power-law best fit. Quantities are dimensionless.

$$\mathcal{H}_{LG} = \frac{\kappa^2}{2}|\nabla\phi|^2 + \frac{\mu^2}{2}|\phi|^2 + \frac{\lambda}{4!}(|\phi|^2)^2, \tag{36}$$

[where ϕ is the order parameter and $\mu^2\sim(T-T_{cr})/T_{cr}$], which describes the Ising universality class, and the equation

$$\dot{y} = y\left[\gamma_L - \frac{1}{4\gamma_L^3}y^2 + O(y^4)\right], \tag{37}$$

which describes the amplitude of a weakly unstable perturbation in a one-species Vlasov-Poisson system [10]. According to (37) the maximum amplitude y_{sat} at $\dot{y}=0$ scales with γ_L as $y_{sat}=\gamma_L^2$.

On the assumption that the system responds linearly to an external pump ∂F, one can obtain the response ∂y_{sat} to ∂F as

$$\gamma_L\partial y_{sat} - \frac{3y_{sat}^2}{4\gamma_L^3}\partial y_{sat} + \partial F = 0. \tag{38}$$

However, Eq. (37) is not valid at $\gamma_L<0$ since it predicts unlimited growth instead of damping in this case. At small initial perturbation y_0 the correct evolution is given by the linear equation $\dot{y}=\gamma_L y$, and the susceptibility

$$\chi(\gamma_L) = \frac{\partial y_{sat}}{\partial F}, \tag{39}$$

is

$$\chi_+ = (-\gamma_L)^{-1}, \tag{40}$$

and at $\gamma_L>0$

$$\chi_- = 2\gamma_L^{-1}. \tag{41}$$

At the critical point $\theta=0$ (or $\gamma_L=0$) the response is described by another critical exponent δ

$$A_{sat} \propto F_1^{1/\delta}. \tag{42}$$

The results of simulation are plotted in Fig. 6, giving $\delta=1.503\pm0.005$. This exponent cannot be obtained by the previous simple assumption from (37) because of its singularity at $\gamma_L=0$.

IV. SCALING LAWS AND SYMMETRIES OF THE MODEL

The remarkable property of the critical exponents γ, β, and δ is that they satisfy the Widom equality (7) [14] with high accuracy. In thermodynamics the Widom equality is a consequence of the scaling of the Gibbs free energy under the transformation

$$\mathfrak{G}(\lambda^{a_\epsilon}\epsilon,\lambda^{a_B}B) = \lambda\mathfrak{G}(\epsilon,B), \tag{43}$$

from which it can be derived straightforwardly [1]. The functions which comply with the condition (43) are called *generalized homogeneous functions*, and the condition itself is termed a *homogeneity* condition.

The nature of this scaling for the marginally stable Vlasov-Poisson system is clear from Fig. 3, where the distribution function $f(x,v,t)$ is plotted at the moment $t=t_{sat}$. The remarkable property of the critical dynamics is the topological equivalence of the phase portraits for different θ: at the moments of saturation t_1 and t_2 corresponding to θ_1 and θ_2, we can write

$$f(x,\lambda^{a_v}v,\theta_1,t_1) = \lambda f(x,v,\theta_2,t_2), \tag{44}$$

or

$$f_m(\lambda^{a_v}v,\theta_1,t_1) = \lambda f_m(v,\theta_2,t_2), \tag{45}$$

for the Fourier component. Transformation between t_1, θ_1 and t_2, θ_2 can be written as $t' \to \lambda^{a_t}t$, $\theta' \to \lambda^{a_\theta}\theta$, so

$$f_m(\lambda^{a_t}t,\lambda^{a_v}v,\lambda^{a_\theta}\theta) = \lambda f_m(t,v,\theta). \tag{46}$$

A weak external pump in the form $F(x)=F_1\cos(k_1x+\varphi)$ creates a similar topology in the phase space because of the same mechanism of the saturation, and adds an additional variable to the distribution function. The transformation can be written as $F_1' \to \lambda^{a_{F_1}}F_1$. Finally, for $f_m=f_m(t,v,\theta,F_1)$ we can write the homogeneity condition as

$$f_m(\lambda^{a_t}t,\lambda^{a_v}v,\lambda^{a_\theta}\theta,\lambda^{a_{F_1}}F_1) = \lambda f_m(t,v,\theta,F_1). \tag{47}$$

The critical exponents β, γ, and δ can be expressed via the scaling exponents a_v, a_θ, and a_{F_1} from which the Widom equality for the Vlasov-Poisson system can be proved directly (Appendix A). They also provide a deep insight into symmetry properties of the system. According to expression (A4)

$$\beta = \frac{1+a_v}{a_\theta},$$

rescaling the parameter of θ (or growth rate) also rescales the distribution function $f(t,v,x)$ in the v direction. This situation differs significantly from thermodynamics, where

$$\beta = \frac{1-a_B}{a_\epsilon}. \tag{48}$$

This expression rescales the normalized distance from the critical point with external field B.

Substituting A_{sat} according to the power law (31) for the order parameter to $v_{sep}^2=4A_{sat}\propto(-\theta)^\beta$, one can obtain (assuming $a_v=1$)

$$v_{sep} \propto (-\theta)^{1/a_\theta}, \tag{49}$$

from which the scaling exponent is $a_\theta=1$ for $\beta=2$ ($a_\theta=4$ for $\beta=1/2$). Remarkably, the two different processes—the linear growth of an unstable perturbation due to the resonant wave-particle interaction and the subsequent nonlinear saturation of this process due to particle trapping—are inter-related.

While there is no thermodynamic equilibrium in the collisionless system considered here, one can define the quantity which describes the response of the system to external thermal perturbation, just as the specific heat capacity describes the response of a thermodynamic system to heat transfer, $C=\delta Q/dT$. For the case considered here

$$C = \frac{\delta Q}{d\theta} \equiv \frac{dV}{d\theta}, \tag{50}$$

where V is the potential energy of the system. To calculate the specific heat capacity, V_{sat} corresponding to A_{sat} is used. The critical exponent α can be calculated straightforwardly from (50) and (31). Because perturbations $m>1$ are negligible for $|\theta|\ll 1$, $V_{sat}\propto A_{sat}\Phi_{sat}$, where $\Phi_{sat}=-A_{sat}$; i.e., $V_{sat}\propto -(-\theta)^{2\beta}$, $\theta<0$, and the heat capacity is given by

$$C \propto -(-\theta)^{-\alpha}, \tag{51}$$

where

$$\alpha = -(2\beta-1). \tag{52}$$

The scaling law (52) can be proven using the homogeneity condition (47) (Appendix A). The critical exponent α does not depend on the sign of Poisson's equation, and the result is the same for the plasma case.

Unlike thermodynamics, where the relation between exponents β, γ, and α is given by Rushbrooke's equality, $\alpha+2\beta+\gamma=2$, the scaling law (52) does not contain the critical exponent γ. Nevertheless, the set of critical exponents $\alpha=-2.990\pm0.006$, $\beta=1.995\pm0.003$, and $\gamma=1.031\pm0.021$ satisfy Rushbrooke's equality with high accuracy.

V. CORRELATION EXPONENTS

The correlation function of fluctuations for the field

$$E = -\frac{\partial\Phi}{\partial x}, \tag{53}$$

can be found from the fluctuation-dissipation theorem [31] as

$$\langle E^2\rangle_{\omega\mathbf{k}} = \frac{T}{2\pi\omega}\frac{\mathrm{Im}[\varepsilon(\omega,\mathbf{k})]}{|\varepsilon|^2}, \tag{54}$$

[32], where the permittivity ε is given by (24). Relation (54) can be integrated using the Kramers-Kronig dispersion relations, and in the static limit $\omega\to 0$ (54) becomes

$$\langle E^2\rangle_{k_m} = \frac{4\pi\sigma^2}{m_pk_B}\left[1-\frac{1}{\varepsilon(0,k_m)}\right], \tag{55}$$

where k_B is the Boltzmann constant and m_p is the particle mass. This equation can be rewritten as

$$\langle E^2\rangle_{k_m} = -\frac{4\pi}{m_pk_B}\frac{\sigma^2}{\theta_m}, \tag{56}$$

where

$$\theta_m = \frac{\sigma^2 - \sigma_J^2(m)}{\sigma_J^2(m)}. \quad (57)$$

The susceptibility χ can be written in terms of $\langle E^2\rangle_{k_1}$ as

$$\chi = \langle E^2\rangle_{k_1} \propto \theta^{-\gamma}, \quad (58)$$

$\gamma=1$. The combination of σ and ω_1 gives the characteristic length for the system from the dispersion relation (25)

$$\xi = 2\pi \frac{\sigma}{\omega_1}, \quad (59)$$

or, in terms of θ

$$\xi \propto \theta^{-\nu}, \quad (60)$$

as $\theta \to 0$. Therefore, the critical exponent that characterizes the correlation length is $\nu=1$. The correlation function $\langle E^2\rangle_{k_1}$ can be rewritten in terms of $k_\xi = \xi^{-1}$ as

$$\langle E^2\rangle_{k_1} \propto k_\xi^{2-\eta}, \quad (61)$$

where η is another critical exponent which characterizes the correlation function. On the other hand, using (60) one can rewrite this expression as

$$\langle E^2\rangle_{k_1} \propto \theta^{-\nu(2-\eta)}, \quad (62)$$

and, taking into account (58)

$$\theta^{-\gamma} \sim \theta^{-\nu(2-\eta)}, \quad (63)$$

from which finally we obtain the equality

$$\gamma = \nu(2-\eta). \quad (64)$$

The last equality is known as Fisher's equality and gives the last critical exponent, $\eta=1$.

VI. RELATION WITH OTHER UNIVERSALITY CLASSES

The correlation function (56) looks rather counterintuitive, since at $\theta_m > 0$ (damping waves), one has $\langle E^2\rangle_{k_m} < 0$, and the noise is *imaginary*. Nevertheless, this unusual situation has an analog—for particle-particle annihilation reactions of the type $Y+Y \to 0$ (corresponds to equation $dn/dt = -an^2, a>0$), $Y\to 0$ $(dn/dt = -an)$ the correlation function is also negative because of *anticorrelation* of particles [33]. In the case $\theta_m > 0$ the amplitude $A_m \to 0$ as $t\to 0$. It is also shown that the criticality due to these annihilation processes belongs to a certain universality class which is different from the Ising universality class [33,34] and therefore is not described by the Landau-Ginzburg Hamiltonian (36).

Another unusual quantity is the correlation length ξ and the wave vector $k_\xi = \xi^{-1}$, whose use allows us to establish the validity of Fisher's equality for the collisionless system, studied here. It is not related to the size of the system L but to the *fluctuations* in the system which determine an average path of correlated motion of particle in presence of these fluctuations. As the system approaches the threshold, fluctuations become correlated since the characteristic time of correlations $\omega^{-1} \sim \theta^{-1}$ diverges as $\theta \to 0$. This behavior is analogous to thermodynamic systems where the correlation length is the only relevant scale near the critical point as $\epsilon \to 0$.

To demonstrate that the criticality in the Vlasov-Poisson system belongs to a different class, let us compare the critical exponents corresponding to the Jeans instability in a self-gravitating *hydrodynamical* system [30], using the same approach. The dispersion relation for this system is

$$\omega_m^2 = c_s^2 k_m^2 - 4\pi G \rho_0, \quad (65)$$

or

$$\omega_m^2 = (c_s^2 - c_m^2) k_m^2, \quad (66)$$

where $c_m^2 = 4\pi G\rho / k_m^2$ is the critical velocity of sound, corresponding to $\omega_m^2 = 0$. As for the kinetic case if $c^2 > c_1^2 = c_{cr}^2$, there are no unstable modes, and the correlation length is

$$\xi_h = \frac{2\pi c_s}{\omega_1} \sim \frac{1}{k_1} \theta_f^{-1/2}, \quad (67)$$

where $\theta_f = (c_s^2 - c_{cr}^2)/c_{cr}^2$ is the reduced sound velocity in a fluid. Here, we have the mean-field exponent $\nu_f = 1/2$.

Assuming $m=1$ and dividing both sides of the dispersion relation (66) by c_s^2, one can obtain the correlation function as

$$G_h^{(2)}(k_{\xi_h}, \theta_f) = \left(\frac{k_{\xi_h}^2}{k_1^2} - \theta_f \right)^{-1}. \quad (68)$$

This is the propagator of Euclidean theory or of the scalar boson field [13], from which the Landau mean-field theory follows automatically.

On the other hand, dispersion relation (25) for the collisionless case gives

$$G^{(2)}(k_\xi, \theta) = \left(i\sqrt{\frac{2}{\pi}} \frac{k_\xi}{k_1} - \theta \right)^{-1}. \quad (69)$$

For collisionless systems the propagator thus corresponds to the vector *fermionic* field and describes a *different* class of critical phenomena. In the language of quantum field theory the parameters θ_f and θ are *bare masses*. Since

$$G^{(2)}(k,0) \propto \frac{1}{k^{2-\eta}}, \quad (70)$$

from (68) and (69) one can obtain $\eta=0$ for the case of hydrodynamics and $\eta=1$ for collisionless system.

VII. HYPERSCALING LAWS

The approach assumed in the previous section allows us to establish the hyperscaling law for the Vlasov-Poisson system which involves the dimensionality d along with critical exponents like Josephson's law (8). Using propagator (69), which is the potential energy, the specific heat capacity C in d-dimensional space at $\theta \to 0$ can be obtained as

$$C \sim \frac{\partial}{\partial \theta} \int d^d k_\xi G^{(2)}(k_\xi, \theta), \quad (71)$$

which gives

$$C \propto \xi^{2-d}. \quad (72)$$

With relation (60) and (72) becomes

$$C \propto \theta^{-\nu(2-d)}. \tag{73}$$

Taking into account the scaling law (51) for the specific heat capacity C, one can obtain the hyperscaling relation which inter-relates the exponents α, ν, and the dimensionality d

$$\alpha = \nu(2-d). \tag{74}$$

The last equality reveals $d=2$ as the *upper critical dimensionality* for the Vlasov-Poisson system since the heat capacity becomes divergent if $d<2$, thus indicating the importance of fluctuations in the critical area. It also shows that the dimensionality corresponding to the critical exponents $\alpha=-3$ and $\nu=1$ is $d=5$, fluctuations at $\theta\approx 0$ are insignificant, and therefore $\alpha=-3$, $\beta=2$, $\gamma=1$, $\nu=1$, and $\eta=1$ are the mean-field exponents.

The use of the scalar field propagator (68) instead of (69) gives

$$\alpha = \nu(4-d), \tag{75}$$

and at $\alpha=0$ the upper critical dimensionality is $d_c=4$, which is the Landau mean-field theory case for the Ising universality class. However, relation (75) is not valid for the Vlasov-Poisson system because of its different propagator. On the contrary to relations (74) and (75) which are valid for specific propagators (68) and (69), Josephson's law (8) is universal for all cases considered. With exponents $\nu=1$ and $\nu_f=1/2$ it gives $d_c=2$ and $d_c=4$ as the upper critical dimensionalities for the collisionless and hydrodynamic cases, respectively, and $d=5$ for the exponents of the Vlasov-Poisson system calculated here. Without going into details here, we note that this universality appears because the fundamental description is given by the same functional integrals in both cases. In particular, for the free-scalar bosonic field (no interactions) the partition function is

$$Z_G = \int \mathcal{D}\phi \exp\left[-\int d^d\mathbf{x}\mathcal{H}_0\right],$$

where $\mathcal{H}_0$ is the Landau-Ginzburg Hamiltonian $\mathcal{H}_{LG}$ (36) without the quadratic term. In the fermionic case the Lagrangian for a Dirac spinor field is used instead of $\mathcal{H}_0$.

VIII. CONCLUSIONS

We have studied numerically and analytically a model Vlasov-Poisson system near the point of a marginal stability. The most important finding is that the criticality of the Vlasov-Poisson model studied here belongs to a universality class described by the propagator corresponding to a *fermionic vector* field. This finding is in striking contrast with the previous critical phenomena studies concerning systems whose criticality belongs to universality classes corresponding to the scalar *bosonic* fields, like the Ising universality class.

This fundamental discrepancy emerges from the *qualitative* difference between objects considered: the Landau-Ginzburg Hamiltonian (36) takes into account spatial variations of the order parameter via the local differential operator ∇, whereas the integro-differential operator for the Vlasov-Poisson model acts on the distribution function containing the additional dimension of velocity.

We have calculated numerically the critical exponents which describe the critical state of the model, and established analytically that these exponents and the dimensionality are inter-related by the scaling and hyperscaling laws like the Widom, Rushbrooke, and Josephson laws at the formal dimensionality $d=5$. The upper critical dimensionality is $d_c=2$ and, since $d>d_c$, the calculated exponents are the mean-field exponents, different from those which one might expect with the Landau-Weiss set of critical exponents corresponding to the Ising mean-field model where $d_c=4$. This is related to the higher dimensionality of the Vlasov-Poisson kinetic problem associated with the velocity space and to the type of the criticality of the Vlasov-Poisson systems, which belongs to a universality class *different* from the Ising universality class.

The critical exponents we have found here are $\alpha=-3$, $\beta=2$, $\gamma=1$, $\delta=1.5$, $\nu=1$, and $\eta=1$. The difference between this set and the set $\alpha\approx-2.814$, $\beta\approx1.907$, $\gamma\approx1$, $\delta\approx1.544$, $\nu=1$, and $\eta=1$ [19] is because A_{sat} is about 50 times larger for the latter case, thus causing *wave-wave* interactions to dominate, thereby yielding a different universality class. More important, the later exponents satisfy scaling laws at *fractal* dimension $d\approx4.68$, indicating *reduced* dimensionality because wave-wave interactions have fewer degrees of freedom than wave-particle ones [21].

ACKNOWLEDGMENTS

This work was supported by the Australian Research Council and a University of Sydney SESQUI grant.

APPENDIX A: RELATION BETWEEN THE SCALING AND CRITICAL EXPONENTS

From the homogeneity condition (47)

$$f_m(\lambda^{a_t}t,\lambda^{a_v}v,\lambda^{a_\theta}\theta,\lambda^{a_{F_1}}F_1) = \lambda f_m(t,v,\theta,F_1), \tag{A1}$$

for ρ_m components by integration over v, one has

$$\lambda^{-a_v}\rho_m(\lambda^{a_t}t,\lambda^{a_\theta}\theta,\lambda^{a_{F_1}}F_1) = \lambda\rho_m(t,\theta,F_1). \tag{A2}$$

For any two $A_{sat}=\rho_1(t_{sat})$ and $A'_{sat}=\rho_1(t'_{sat})$, one can write

$$\lambda^{-a_v}A_{sat}(\lambda^{a_\theta}\theta,\lambda^{a_{F_1}}F_1) = \lambda A_{sat}(\theta,F_1). \tag{A3}$$

Assuming $\lambda=(-1/\theta)^{1/a_\theta}$, the critical exponent β can be rewritten in terms of the scaling exponents a_v and a_θ as

$$\beta = \frac{1+a_v}{a_\theta}. \tag{A4}$$

In the similar way for γ and δ, one can write

$$\gamma = \frac{-a_v-1+a_{F_1}}{a_\theta}, \tag{A5}$$

$$\delta = \frac{a_{F_1}}{1+a_v}, \tag{A6}$$

and the Widom relation follows from (A5) straightforwardly

$$\gamma = \frac{-a_v - 1 + a_{F_1}}{a_\theta} = -\frac{1+a_v}{a_\theta} + \frac{a_{F_1}}{a_\theta} = -\beta + \beta\delta = \beta(\delta - 1). \tag{A7}$$

Equations (A4)–(A6) can be rewritten in matrix form as

$$\mathbf{W}\mathbf{A} = \mathbf{X}, \tag{A8}$$

where $\mathbf{A}=[a_\theta, a_v, a_{F_1}]^T$, $\mathbf{X}=[1,-1,-\delta]^T$, and the matrix $\mathbf{W}$ is

$$\mathbf{W} = \begin{pmatrix} \beta & -1 & 0 \\ \gamma & 1 & -1 \\ 0 & \delta & -1 \end{pmatrix}. \tag{A9}$$

The determinant of $\mathbf{W}$ is

$$\det \mathbf{W} = -\beta + \delta\beta - \gamma \equiv 0. \tag{A10}$$

Using (A6) to eliminate a_v, the system (A8) can be reduced to

$$\beta a_\theta - \frac{1}{\delta} a_{F_1} = 0, \tag{A11}$$

$$\gamma a_\theta + \left(\frac{1}{\delta} - 1\right) a_{F_1} = 0, \tag{A12}$$

for which solution exists only if the Widom equality $\gamma = \beta(\delta - 1)$ holds. Therefore, a_θ and a_{F_1} can be formally considered as the eigenvectors of $\mathbf{W}$ whose eigenvalue is $\lambda=0$. In particular

$$a_\theta = \frac{1}{\beta+\gamma} a_{F_1}, \tag{A13}$$

which indicates that rescaling of the distribution function under an external pump is equivalent to rescaling due to the field which appears for nonzero order parameter.

APPENDIX B: RUSHBROOKE'S LAW FOR VLASOV-POISSON SYSTEM

The heat capacity can be formally defined as

$$C = \frac{\delta Q}{d\theta} \equiv \frac{dV}{d\theta}, \tag{B1}$$

where V is the potential energy of the system. To calculate the specific heat capacity, V_{sat} corresponding to A_{sat} is used.

Because perturbations $m > 1$ are negligible for $|\theta| \ll 1$, $V_{sat} \propto A_{sat}\Phi_{sat}$, where $\Phi_{sat} = -A_{sat}$, and

$$V_{sat} \propto A_{sat}^2. \tag{B2}$$

From (A3) one can obtain

$$\frac{\partial}{\partial\theta}\lambda^{-2a_v} A_{sat}^2(\lambda^{a_\theta}\theta, \lambda^{a_{F_1}}F_1) = \frac{\partial}{\partial\theta}\lambda^2 A_{sat}^2(\theta, F_1), \tag{B3}$$

or

$$\frac{\partial}{\partial\theta}\lambda^{-2a_v-2} A_{sat}^2(\lambda^{a_\theta}\theta, \lambda^{a_{F_1}}F_1) = \frac{\partial}{\partial\theta} A_{sat}^2(\theta, F_1). \tag{B4}$$

Assuming $\lambda = \theta^{-1/a_\theta}$ and $F_1 = 0$, Eq. (B4) can be rewritten as

$$\frac{\partial}{\partial\theta}[\theta^{(2a_v+2)/a_\theta} A_{sat}^2(-1,0)] = \frac{\partial}{\partial\theta} A_{sat}^2(\theta,0), \tag{B5}$$

or

$$\frac{2a_v+2}{a_\theta} A_{sat}^2(-1,0)\,\theta^{(2a_v+2)/a_\theta - 1} = \frac{\partial}{\partial\theta} A_{sat}^2(\theta,0), \tag{B6}$$

or

$$\frac{2a_v+2}{a_\theta} A_{sat}^2(-1,0)\,\theta^{[(2a_v+2)/a_\theta]-1} = C(\theta,0). \tag{B7}$$

Equation (B5) has the form of the power law, $C(\theta,0) \propto \theta^{-\alpha}$, with

$$\alpha = -2\frac{a_v+1}{a_\theta} + 1 = -2\beta + 1. \tag{B8}$$

The last relation corresponds to Rushbrooke's equality $\alpha + 2\beta + \gamma = 2$ at $\gamma = 1$.

[1] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Clarendon, Oxford, 1971).

[2] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, Oxford, 1980).

[3] E. Frieman, S. Bodner, and P. Rutherford, Phys. Fluids **6**, 1298 (1963).

[4] B. C. Fried, C. S. Liu, R. W. Means, and R. Z. Sagdeev, Plasma Physics Group Report PPG-93, University of California, Los Angeles, 1971 (unpublished).

[5] M. B. Levin, M. G. Lyubarsky, I. N. Onishchenko, V. D. Shapiro, and V. I. Shevchenko, Sov. Phys. JETP **35**, 898 (1972).

[6] L. D. Landau, J. Phys. **10**, 25 (1946).

[7] J. E. Marsden and M. McCracken, *The Hopf Bifurcation and Its Applications* (Springer-Verlag, New York, 1976).

[8] V. Latora, A. Rapisarda, and S. Ruffo, Phys. Rev. Lett. **80**, 692 (1998).

[9] A. Simon and M. Rosenbluth, Phys. Fluids **19**, 1567 (1976).

[10] J. D. Crawford, Phys. Plasmas **2**, 97 (1995).

[11] J. Denavit, Phys. Fluids **28**, 2773 (1986).

[12] J. Candy, J. Comput. Phys. **129**, 160 (1996).

[13] D. J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 1984).

[14] B. Widom, J. Chem. Phys. **41**, 1633 (1964).

[15] B. D. Josephson, Proc. Phys. Soc. London **92**, 269, 276 (1967).

[16] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1994).

[17] C. Domb, Adv. Phys. **9**, 45 (1960).

[18] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group* (Addison-Wesley, Reading, MA, 1992).
[19] A. V. Ivanov, Astrophys. J. **550**, 622 (2001).
[20] P. A. Robinson, Rev. Mod. Phys. **69**, 507 (1997).
[21] S. V. Vladimirov, V. N. Tsytovich, S. I. Popel, and F. Kh. Khakimov, *Modulational Interactions in Plasmas* (Kluwer, Dordrecht, 1995).
[22] S. Ichimaru, D. Pines, and N. Rostoker, Phys. Rev. Lett. **8**, 231 (1962).
[23] B. N. Miller and J. L. Rouet, Phys. Rev. E **65**, 056121 (2002).
[24] Y. Watanabe, S. Inagaki, M. T. Nishida, Y. D. Tanaka, and S. Kato, Publ. Astron. Soc. Jpn. **33**, 541 (1981).
[25] J. Binney and S. Tremaine, *Galactic Dynamics* (Princeton University Press, Princeton, 1987).
[26] B. D. Fried and S. D. Conte, *Plasma Dispersion Function: The Hilbert Transform of the Gaussian* (Academic Press, New York, 1961).
[27] C. Z. Cheng and G. Knorr, J. Comput. Phys. **22**, 330 (1976).
[28] R. K. Mazitov, Zh. Prikl. Mekh. Fiz. **1**, 27 (1965).
[29] T. M. O'Neil, J. H. Winfrey, and J. H. Malmberg, Phys. Fluids **14**, 1204 (1971).
[30] J. Jeans, *Astronomy and Cosmogony* (University Press, Cambridge, 1928).
[31] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).
[32] R. J. Kubo, J. Phys. Soc. Jpn. **12**, 570 (1957).
[33] M. J. Howard and U. C. Täuber, J. Phys. A **30**, 7721 (1997).
[34] B. P. Lee, J. Phys. A **27**, 2633 (1994).