Synchronization of spectral components and its regularities in chaotic dynamical systems

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The chaotic synchronization regime in coupled dynamical systems is considered. It has been shown that the onset of a synchronous regime is based on the appearance of a phase relation between the interacting chaotic oscillator frequency components of Fourier spectra. The criterion of synchronization of spectral components as well as the measure of synchronization has been discussed. The universal power law has been described. The main results are illustrated by coupled Rössler systems, Van der Pol and Van der Pol–Duffing oscillators.

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I. INTRODUCTION

Chaotic synchronization is one of the fundamental phenomena actively studied recently [1], having both important theoretical and applied significances (e.g., for information transmission by means of deterministic chaotic signals [2,3], in biological [4] and physiological [5] tasks, etc.). Several different types of chaotic synchronization of coupled oscillators—i.e., generalized synchronization [6], phase synchronization [1], lag synchronization [7], and complete synchronization [8]—are traditionally distinguished. There are also attempts to find a unifying framework for the chaotic synchronization of coupled dynamical systems [9–11].

In our works [12,13] it was shown that phase, generalized, lag, and complete synchronization are closely connected with each other and, as a matter of fact, they are different manifestations of one type of synchronous oscillation behavior of coupled chaotic oscillators called time-scale synchronization. The synchronous regime character (phase, generalized, lag, or complete synchronization) is defined by the presence of synchronous time scales s, introduced by means of continuous wavelet transform [14–16] with a Morlet mother wavelet function. Each time scale can be characterized by the phase $\phi_s(t)=\arg W(s,t)$, where $W(s,t)$ is the complex wavelet surface. In this case, the phenomenon of the chaotic synchronization of coupled systems is manifested by a synchronous behavior of the phases of coupled chaotic oscillators $\phi_{s1,2}(t)$ observed on a certain synchronized time-scale range $s_m<s<s_b$, for time scales s from which the phase-locking condition

$$|\phi_{s1}(t)-\phi_{s2}(t)|<\text{const} \tag{1}$$

is satisfied, and the part of the wavelet spectrum energy falling in this range does not equal zero (see [12,17] for details). The range of synchronized time scales $[s_m;s_b]$ expands when the coupling parameter between systems increases. If the coupling type between oscillators is defined in such a way that the lag synchronization appearance is possible, then all time scales become synchronized with further coupling parameter increasing, while the coinciding states of interacting oscillators are shifted in time relative to each other: $\mathbf{x}_1(t-\tau)\simeq\mathbf{x}_2(t)$. A further coupling parameter increase leads to a decrease of the time shift τ. The oscillators tend to the regime of complete synchronization, $\mathbf{x}_1(t)\simeq\mathbf{x}_2(t)$, and the phase difference $[\phi_{s1}(t)-\phi_{s2}(t)]$ tends to be zero on all time scales.

The time scale s introduced into consideration by means of a continuous wavelet transform can be considered as a quantity which is inversely proportional to the frequency f defined with the help of a Fourier transformation. For the Morlet mother wavelet function [16] with parameter $\Omega=2\pi$ the relationship between the frequency f and the time scale is quite simple: $s=1/f$. Therefore, time-scale synchronization should also manifest in the appearance of the phase relation between frequency components f of corresponding Fourier spectra $S(f)$ of interacting oscillators.

In this paper we consider the synchronization of spectral components of the Fourier spectra of coupled oscillators. We discuss the mechanism of the chaotic synchronization regime manifestation in coupled dynamical systems based on the appearance of the phase relation between frequency components of the Fourier spectra of interacting chaotic oscillators (see also [18]). One can also consider the obtained results as a criterion of the existence (or, otherwise, the impossibility of the existence) of a lag synchronization regime in coupled dynamical systems.

The structure of this paper is the following. In Sec. II we discuss the synchronization of spectral components of Fourier spectra. We illustrate our approach with the help of two coupled Rössler systems in Sec. III. The quantitative measure of synchronization is described in Sec. IV. The universal power law taking place in the presence of the time scale synchronization regime is discussed in Secs. V and VI. The final conclusion is presented in Sec. VII.

II. SYNCHRONIZATION OF SPECTRAL COMPONENTS OF FOURIER SPECTRA

It should be noted that the continuous wavelet transform is characterized by a frequency resolution lower than the Fourier transformation (see [15,16]). The continuous wavelet transform appears as a smoothing of the Fourier spectrum, whereby the dynamics on a time scale s is determined not only by the spectral component $f=1/s$ of the Fourier spec-

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trum. This dynamics is also influenced by the neighboring components as well; the degree of this influence depends both on their positions in the Fourier spectrum and on their intensities. Thus, the fact that coupled chaotic oscillators exhibit synchronization on a time scale s of the wavelet spectrum by no means implies that the corresponding components $f=1/s$ of the Fourier spectrum of these systems are also synchronized.

Let $x_1(t)$ and $x_2(t)$ be the time series generated by the first and second coupled chaotic oscillators, respectively. The corresponding Fourier spectra are determined by the relations

$$S_{1,2}(f) = \int_{-\infty}^{+\infty} x_{1,2}(t) e^{-i2\pi ft} dt. \quad (2)$$

Accordingly, each spectral component f of the Fourier spectrum $S(f)$ can be characterized by an instantaneous phase $\phi_f(t)=\phi_{f0}+2\pi ft$, where $\phi_{f0}=\arg S(f)$. However, since the phase $\phi_f(t)$ corresponding to the frequency f of the Fourier spectrum $S(f)$ increases with time linearly, the phase difference of the interacting oscillators at this frequency $\phi_{f1}(t)-\phi_{f2}(t)=\phi_{f01}-\phi_{f02}$ is always bounded and, hence, the traditional condition of phase entrainment (used for detection of the phase synchronization regime),

$$|\phi_1(t) - \phi_2(t)| < \text{const}, \quad (3)$$

is useless. Apparently, a different criterion should be used to detect the synchronization of coupled oscillators at a given frequency f.

In the regime of lag synchronization, the behavior of coupled oscillators is synchronized on all time scales s of the wavelet transform (see [12]). Therefore, one can expect that all frequency components of the Fourier spectra of the systems under consideration should be synchronized as well. In this case, $x_1(t-\tau)\simeq x_2(t)$ and, hence, taking into account Eq. (2) one has to obtain

$$S_2(f) \simeq S_1(f) e^{i2\pi\tau f}. \quad (4)$$

Thus, in the case of coupled chaotic oscillators occurring in the regime of lag synchronization their instantaneous phases corresponding to the spectral component f of the Fourier spectra $S_{1,2}(f)$ will be related to each other as $\phi_{f2}(t)\simeq\phi_{f1}(t)+2\pi\tau f$ and, hence, the phase difference $\phi_{f2}(t)-\phi_{f1}(t)$ of coupled oscillators on the frequency f must obey the relation

$$\Delta\phi_f = \phi_{f1}(t) - \phi_{f2}(t) = \phi_{f01} - \phi_{f02} = 2\pi\tau f. \quad (5)$$

Accordingly, the points corresponding to the phase difference $\Delta\varphi_f$ of the spectral components of chaotic oscillators in the regime of lag synchronization on the $(f,\Delta\phi_f)$ plane must fit a straight line with slope $k=2\pi\tau$. In the case of the complete synchronization of two coupled identical oscillators the slope of this line, k, is equal to zero (see also [19]).

The destroying of the lag synchronization regime (e.g., as a result of a decrease of the coupling strength between oscillators) and the transition to the regime of phase synchronization (in the case when the instantaneous phase of the chaotic signal can be introduced correctly [20]) results in a loss of synchronism for a part of the time scales s of the wavelet spectra [12]. Accordingly, one can expect that a part of the spectral components of the Fourier spectra in the phase synchronization regime will also lose synchronism and the points on the $(f,\Delta\phi_f)$ plane will deviate from the straight line (5) mentioned above.[1] It is reasonable to assume that synchronism will be lost primarily for the spectral components f characterized by a small fraction of energy in the Fourier spectra $S_{1,2}(f)$, while the components corresponding to a greater energy fraction will remain synchronized and the corresponding points on the $(f,\Delta\phi_f)$ plane will be located at the straight line as before. As the lag synchronization regime does not occur in the system anymore, the time shift τ can be determined by the delay of the most energetic frequency component f_m in the Fourier spectra $\tau=(\phi_{f_m2}-\phi_{f_m1})/(2\pi f_m)$.

As the coupling parameter decreases further, an increasing part of the spectral components will deviate from synchronism. However, as long as the most "energetic" components remain synchronized, the coupled systems will exhibit the regime of time-scale synchronization. Obviously, for the synchronized spectral component the phase difference $\Delta\varphi_f$ is located after the transient finished independently of initial conditions.

To describe the synchronization of spectral components, let us introduce a quantitative characteristic of a number of spectral components of the Fourier spectra $S_{1,2}(f)$ occurring in the regime of synchronism,

$$\sigma_L = \frac{\int_0^{+\infty} H(|S_1(f)|^2 - L) H(|S_2(f)|^2 - L)(\Delta\phi_f - 2\pi\tau f)^2 df}{\int_0^{+\infty} H(|S_1(f)|^2 - L) H(|S_2(f)|^2 - L) df}, \quad (6)$$

where $H(\xi)$ is the Heaviside function, L is the threshold power level (in dB) above which the spectral components are taken into account, and τ is determined by the time shift of the most energetic frequency component (f_m) in the Fourier spectra, $\tau=(\phi_{f_m2}-\phi_{f_m1})/(2\pi f_m)$. The quantity σ_L tends to be zero in the regimes of complete and lag synchronization. After the destruction of the lag synchronization regime caused by the decrease of the coupling strength the value of σ_L increases with the number of desynchronized spectral components of the Fourier spectra $S_{1,2}(f)$ of coupled oscillators.

Real data are usually represented by a discrete time series of finite length. In such cases, the continuous Fourier transform (2) has to be replaced by its discrete analog (as was done in [19] and the integral (6) by the sum

[1]The same effect will take place if the instantaneous phase of the chaotic signal cannot be introduced correctly due to the noncoherent structure of the chaotic attractor. In this case the phase synchronization cannot be detected, but one can observe the presence of time-scale synchronization.

$$\sigma_L = \frac{1}{N}\sum_{j=1}^{N} (\Delta\phi_{f_j} - 2\pi\tau f_j)^2, \tag{7}$$

taken over all spectral components of the Fourier spectra $S_{1,2}(f)$ with the power above L. In calculating σ_L, it is expedient to perform averaging over a set of time series $x_{1,2}(t)$. The phase shift $\Delta\varphi_f$ can be calculated either as was done in [19] or by means of a cross spectrum [21].

III. TWO MUTUALLY COUPLED RÖSSLER SYSTEM SYNCHRONIZATION

In order to illustrate the approach proposed above, let us consider two coupled Rössler systems

$$\dot{x}_{1,2} = -\omega_{1,2}y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}),$$

$$\dot{y}_{1,2} = \omega_{1,2}x_{1,2} + ay_{1,2} + \varepsilon(y_{2,1} - y_{1,2}),$$

$$\dot{z}_{1,2} = p + z_{1,2}(x_{1,2} - c), \tag{8}$$

where ε is the coupling parameter, $\omega_1=0.98$, and $\omega_2=1.03$. By analogy with the case studied in [22], the values of the control parameters have been selected as follows: $a=0.22$, $p=0.1$, and $c=8.5$. It is known [22] that two coupled Rössler systems with $\varepsilon=0.05$ occur in the regime of phase synchronization, while for $\varepsilon=0.15$ the same systems exhibit lag synchronization.

Figure 1(a) shows a plot of the value σ_L versus coupling parameter ε. One can see that σ_L tends to be zero when the coupling parameter ε increases, which is evidence of the transition from phase to lag synchronization. Figures 1(b)–1(f) illustrate the increase in the number of synchronized spectral components of the Fourier spectra $S_{1,2}(f)$ of two coupled systems with coupling strength ε increasing. Indeed, Fig. 1(b) corresponds to the asynchronous dynamics of coupled oscillators ($\varepsilon=0.02$). There are no synchronous spectral components for such coupling strength and dots are scattered randomly over the $(f,\Delta\phi_f)$ plane. The weak phase synchronization ($\varepsilon=0.05$) after the regime occurrence is shown in Fig. 1(c). There are a few synchronized spectral components the phase shift $\Delta\varphi_f$ of which satisfies the condition (5). Almost all spectral components are nonsynchronized; therefore, the points corresponding to the phase differences $\Delta\varphi_f$ are spread over the $(f,\Delta\phi_f)$ plane and the value of σ_L is rather large.

Figures 1(d) and 1(e) correspond to the well-pronounced phase synchronization ($\varepsilon=0.08$ and 0.1, respectively). Figure 1(f) shows the state of lag synchronization ($\varepsilon=0.15$), when all spectral components f of the Fourier spectra are synchronized. Accordingly, in this case all points on the $(f,\Delta\phi_f)$ plane are at line (5) with slope $k=2\pi\tau$. With the coupling strength ε increasing, the value of σ_L decreases monotonically, which verifies the assumption that when two coupled chaotic systems undergo a transition from asynchronous dynamics to lag synchronization, more and more spectral components become synchronized. When all spectral components f are synchronized, the phase shift $\Delta\varphi_f$ for them is $2\pi\tau f$; therefore, the points on the $(f,\Delta\phi_f)$ plane lie on the straight line (5) and the value of σ_L is equal to zero.

Another important question is which spectral components of the Fourier spectra of interacting chaotic oscillators are synchronized first and which are last. Figure 2(a) shows a plot of the σ_L value for the coupling strength $\varepsilon=0.05$ (corresponding to the weak phase synchronization) versus power L at which the spectral components f_j of the Fourier spectra $S_{1,2}(f)$ are taken into account in Eq. (7). One can see that the "truncation" of the spectral components with small energy leads to a decrease of the σ_L value. Figures 1(b)–1(e) illustrate the distribution of the phase difference $\Delta\varphi_f$ of the spectral components f with the power exceeding the preset level L. The data in Fig. 2 show that the most "energetic" spectral components are first synchronized upon the onset of time-scale synchronization. On the contrary, the components with low energies are the first to go out from synchronism upon destruction of the lag synchronization regime.

IV. CRITERION AND MEASURE OF SYNCHRONIZATION

Let us briefly discuss a criterion of spectral components synchronization. Obviously, the relation (5) is quite convenient as a criterion of synchronism in the case of lag synchronization destruction in the way considered above. If the type of coupling between systems has been defined in such a manner that the lag synchronization regime cannot appear, relation (5) cannot be the criterion of spectral component synchronization. So as a general criterion of synchronism of identical spectral components f of coupled systems we have to select a different condition rather than Eq. (5). As such a criterion we have chosen the establishment of the phase shift

$$\Delta\varphi_f = \phi_{f01} - \phi_{f02} = \text{const}, \tag{9}$$

which must not depend on initial conditions. To illustrate it let us consider the distribution of the phase difference $\Delta\varphi_f$ obtained from the series of 10^3 experiments for Rössler systems (8). Figure 3(a) corresponds to the asynchronous dynamics of coupled oscillators when the coupling parameter $\varepsilon=0.02$ is below the threshold the appearance of chaotic synchronization [see also Fig. 1(a)]. One can see that the phase difference $\Delta\varphi_f$ for the spectral components f of the Fourier spectra $S_{1,2}(t)$ in this case is distributed randomly over all intervals from $-\pi$ to π. It means that the phase shift between spectral components f is different for each experiment (i.e., for different initial conditions) and, therefore, there is no synchronism whereas the considered frequency f is the same for both spectra $S_{1,2}(f)$. Similar distributions are observed for all spectral components f in the case of the asynchronous regime [see also Figs. 1(a) and 3(b)], though one can distinguish the maximum in the distribution on the frequency f close to the main frequency of the Fourier spectrum $S(f)$ as a prerequisite the beginning of synchronization.

When the systems demonstrate synchronous behavior the distributions $N(\Delta\phi_f)$ of the phase shift $\Delta\phi_f$ are quite different. In this case one can distinguish both synchronized and nonsynchronized spectral components characterized by distributions of the phase shift of different types. In Fig. 3(c) the distribution of $\Delta\phi_f$ for the synchronous spectral component

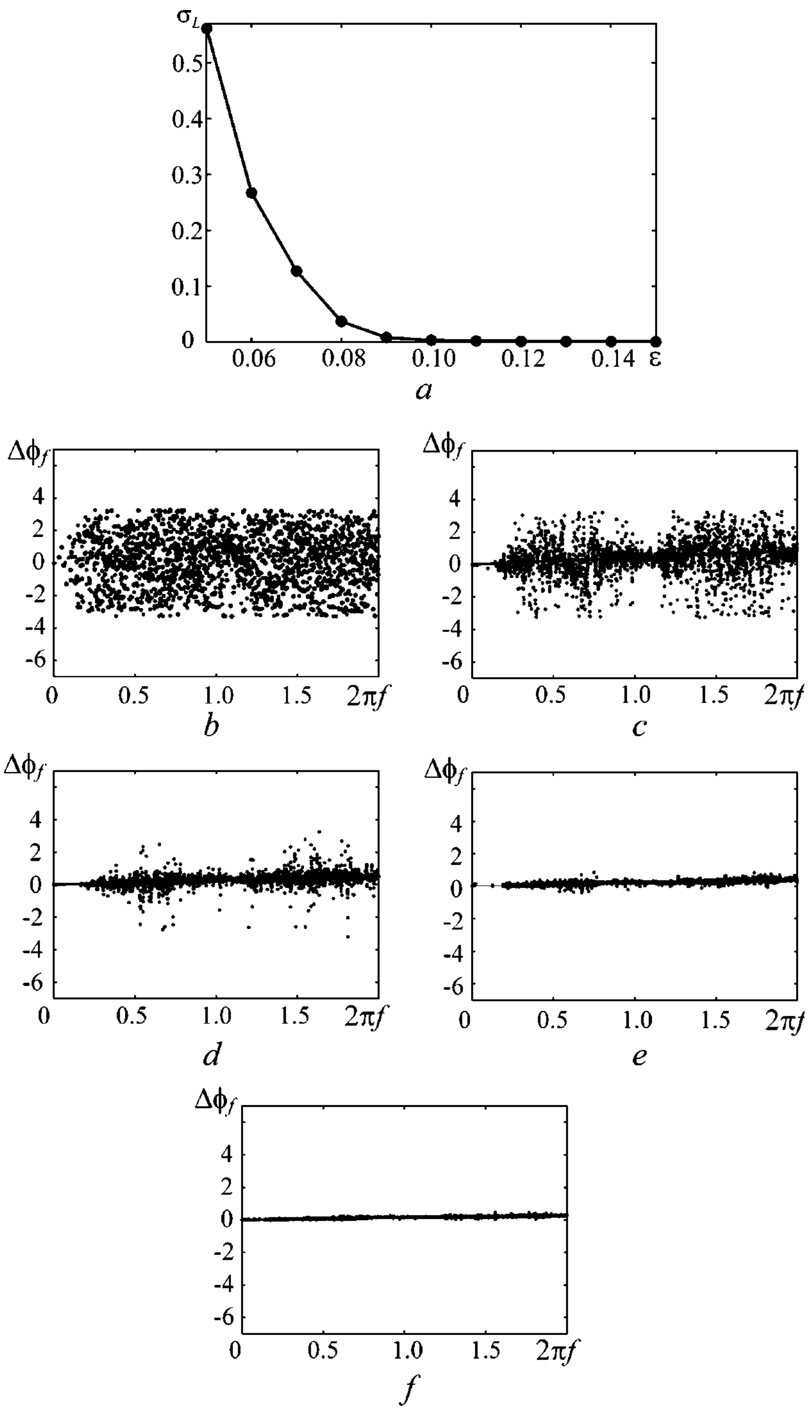

FIG. 1. (a) The value σ_L versus coupling parameter ε and (b)–(f) the phase difference $\Delta\varphi_f$ of various spectral components f of the Fourier spectra $S_{1,2}(f)$ of two coupled Rössler systems for different values of coupling strength ε. (b) The asynchronous dynamics for the coupling parameter $\varepsilon = 0.02$, (c) the chaotic synchronization regime $\varepsilon = 0.05$, (d) $\varepsilon = 0.08$, (e) $\varepsilon = 0.1$, and (f) $\varepsilon = 0.15$. The plots are constructed for the time series $x_{1,2}(t)$ with a length of 2000 dimensionless time units at a discretization step of $h = 0.2$ at a power level of $L = -40$ dB of Fourier spectra $S_{1,2}(f)$.

$\Delta\varphi_f$ is shown. One can see that it looks like a δ function, which means the phase shift $\Delta\phi_f$ is always the same after the transient finished. Obviously, this phase shift $\Delta\phi_f$ does not depend on initial conditions.

For the nonsynchronized spectral components the distributions $N(\Delta\phi_f)$ are different [see Fig. 3(d)]. Evidently, in this case the phase shift $\Delta\phi_f$ is varied from experiment to experiment. At the same time, the tendency to synchronization of these spectral components can be observed. The distribution $N(\Delta\phi_f)$ looks Gaussian. With the coupling parameter increasing the dispersion of it decreases and the spectral components f of the considered Fourier spectra $S_{1,2}(f)$ tend to be synchronized. The same effect can be observed in Figs. 1(b)–1(f). With an increase of the coupling parameter ε, the points on the $(f, \Delta\phi_f)$ plane tend to fit a straight line with slope $k = 2\pi\tau$ and their scattering decreases.

So the general criterion of synchronism of identical spectral components f of coupled systems is the establishment of the phase shift (9) after the transient finished. It is important to note that the case of classical synchronization of periodical oscillations also obeys the considered criterion (9) (see, e.g., [23]).

Let us consider now the quantitative characteristic of synchronization. In [12] the measure of synchronization based

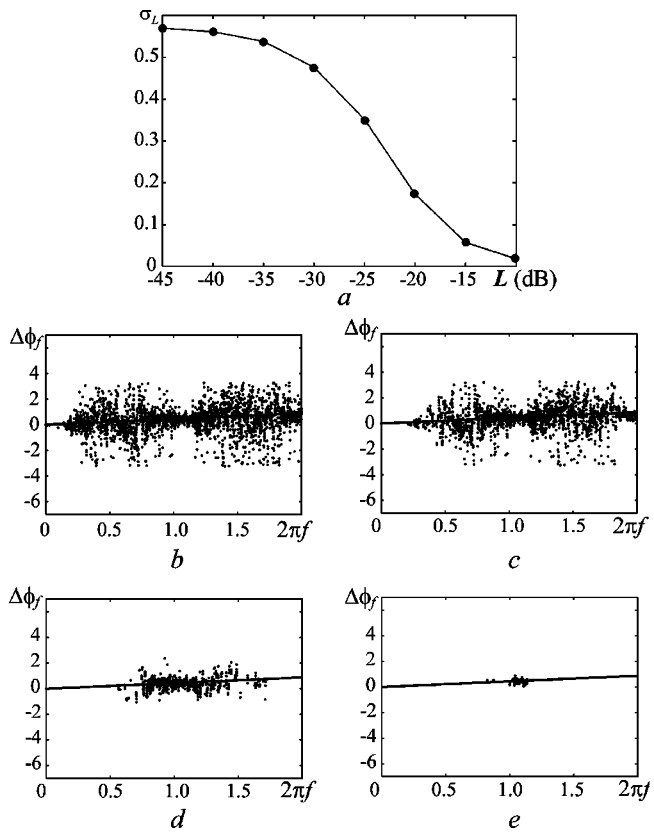

FIG. 2. (a) The value σ_L versus power L at which the spectral components f_j of the Fourier spectra $S_{1,2}(f)$ are taken into account in Eq. (7). (b)–(e) The phase difference $\Delta\varphi_f$ of various spectral components f of the Fourier spectra $S_{1,2}(f)$ of two coupled Rössler systems for various power levels $L=-40$ dB (b), $L=-30$ dB (c), $L=-20$ dB (d), and $L=-10$ dB (e) for coupling strength $\varepsilon=0.05$.

on the normalized energy of synchronous time scales has been introduced. The analogous quantity ρ may be defined for Fourier spectra $S(f)$ as

$$\rho_{1,2} = \frac{1}{P}\int_{F_s} |S_{1,2}(f)|^2 df, \tag{10}$$

where F_s is the set of synchronized spectral components and

$$P = \int_0^{+\infty} |S_{1,2}(f)|^2 df \tag{11}$$

is the full energy of chaotic oscillations. In fact, the value of ρ is the part of the full system energy corresponding to synchronized Fourier components. This measure ρ is 0 for the nonsynchronized oscillations and 1 for the case of complete and lag synchronization regimes as well as the quantity introduced in [12]. When the systems undergo a transition from asynchronous behavior to the lag synchronization regime the measure of synchronism takes a value between 0 and 1, which corresponds to the case when there are both synchronized and nonsynchronized spectral components in the Fourier spectra $S_{1,2}(f)$.

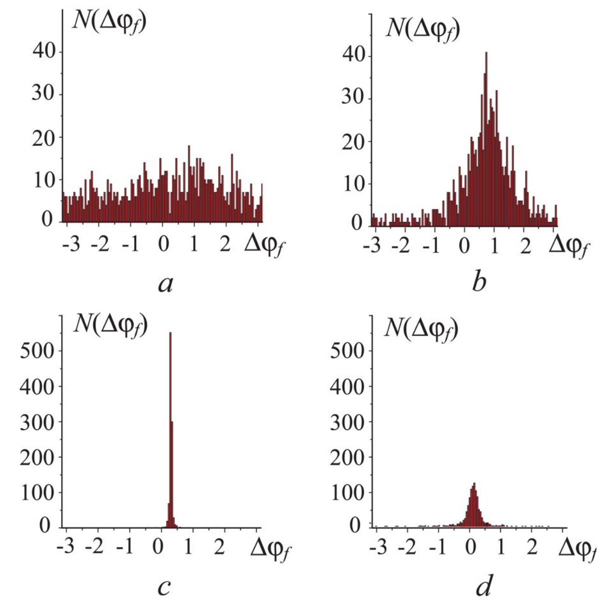

FIG. 3. Distribution $N(\Delta\phi_f)$ of the phase difference $\Delta\varphi_f$ obtained from a series of 10^3 experiments for Rössler systems (8). (a) The asynchronous dynamics takes place ($\varepsilon=0.02$), and the distribution of the phase shift $\Delta\varphi_f$ has been obtained for the spectral components $f\simeq 0.0711$, (b) $\varepsilon=0.02$, $f\simeq 0.1764$, (c) the distribution of phase shift for synchronous spectral component $f\simeq 0.1764$ ($\varepsilon=0.08$), and (d) the analogous distribution for asynchronous spectral component $f\simeq 0.0711$ for the same coupling parameter $\varepsilon=0.08$. Compare with Fig. 1.

For the real data represented by a discrete time series of finite length one has to use the discrete analog of the Fourier transform while the integrals in relation (10) should be replaced by the sums

$$\rho_{1,2} = \frac{1}{P}\sum_{j_s} |S_{1,2}(f_{j_s})|^2 \Delta f, \tag{12}$$

where

$$P = \sum_j |S_{1,2}(f_j)|^2 \Delta f. \tag{13}$$

While the sum in Eq. (12) is being calculated only the synchronized spectral components f_{j_s} should be taken into account.

Figure 4 presents the dependence of the synchronization measure ρ for the first Rössler oscillator of system (8) on the coupling parameter ε. It is clear that the part of the energy corresponding to the synchronized spectral components grows with an increase in the coupling strength.

V. SPECTRAL COMPONENT BEHAVIOR IN THE PRESENCE OF SYNCHRONIZATION

Let us now consider how the closed frequency components of two coupled oscillators behave with an increase of the coupling strength ε. As a model of such a situation let us select two mutually coupled Van der Pol oscillators

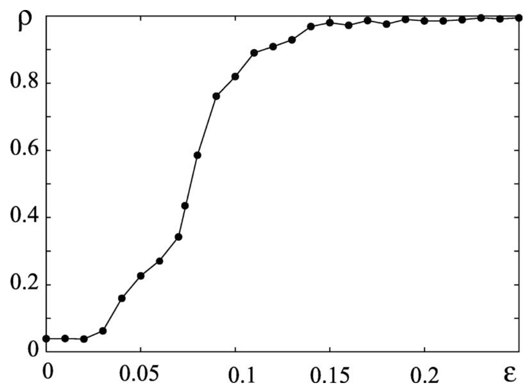

FIG. 4. The dependence of the synchronization measure ρ for the first Rössler system (8) on the coupling parameter ε.

$$\ddot{x}_{1,2} - (\lambda - x_{1,2}^2)\dot{x}_{1,2} + \Omega_{1,2}^2 x_{1,2} = \pm\varepsilon(x_{2,1} - x_{1,2}), \quad (14)$$

where $\Omega_{1,2}=\Omega\pm\Delta$ are slightly mismatched natural cyclic frequencies and $x_{1,2}$ are the variables describing the behavior of the first and second self-sustained oscillators, respectively. The parameter ε characterizes the coupling strength between oscillators. The nonlinearity parameter $\lambda=0.1$ has been chosen small enough in order to make the oscillations of self-sustained generators close to the single-frequency ones.

An asymmetrical type of coupling in system (14) ensures the appearance of the synchronous regime which is similar to the lag synchronization in chaotic systems. For such a type of coupling the oscillations in the synchronous regime are characterized by one frequency $\omega=2\pi f$ while the small phase shift $\Delta\varphi_f$ between time series $x_{1,2}(t)$, decreasing when the coupling strength increases, takes place.

Using the method of complex amplitudes, the solution of Eq. (14) can be found in the form

$$x_{1,2} = A_{1,2}e^{i\omega t} + A_{1,2}^{*}e^{-i\omega t},$$

$$\dot{A}_{1,2}e^{i\omega t} + \dot{A}_{1,2}^{*}e^{-i\omega t} = 0, \quad (15)$$

where an asterisk means complex conjugation and ω is the cyclic frequency at which oscillations in system (14) are realized. One can reduce Eqs. (14) and (15) to the form

$$\dot{A}_{1,2} = \frac{1}{2}(\lambda - |A|^2)A + i\frac{1}{2\omega}[(\Omega_{1,2}^2 - \omega^2)A_{1,2} \mp \varepsilon(A_{2,1} - A_{1,2})] \quad (16)$$

by means of averaging over the fast-changing variables.

Choosing complex amplitudes in the form of

$$A_{1,2} = r_{1,2}e^{\varphi_{1,2}}, \quad (17)$$

one can obtain the equations for the amplitudes $r_{1,2}$ and phases $\varphi_{1,2}$ of the coupled oscillators as follows:

$$\dot{r}_{1,2} = \frac{1}{2}(\lambda - |r_{1,2}|^2)r_{1,2} \pm \frac{\varepsilon r_{2,1}}{2\omega}\sin(\varphi_{1,2} - \varphi_{2,1}),$$

$$\dot{\varphi}_{1,2} = \frac{\Omega_{1,2}^2 - \omega^2 \pm \varepsilon}{2\omega} \mp \frac{\varepsilon r_{2,1}}{2\omega r_{1,2}}\cos(\varphi_{1,2} - \varphi_{2,1}). \quad (18)$$

The oscillations of two Van der Pol generators (14) are synchronized when conditions

$$\dot{r}_{1,2} = 0, \quad \dot{\varphi}_{1,2} = 0 \quad (19)$$

are satisfied. Assuming that the phase difference of oscillations $\Delta\varphi=\varphi_2-\varphi_1$ is small enough and taking into account only components of first infinitesimal order over $\Delta\varphi$, one can obtain the relation for the phase shift,

$$\Delta\varphi_{1,2} = \frac{\lambda\sqrt{\Omega^2 + \Delta^2 \pm 2\sqrt{\Omega\Delta(\varepsilon + \Omega\Delta)}}}{2\varepsilon + 4\Omega\Delta}, \quad (20)$$

and frequency,

$$\omega_{1,2} = \sqrt{\Omega^2 + \Delta^2 \pm 2\sqrt{\Omega\Delta(\varepsilon + \Omega\Delta)}}, \quad (21)$$

which correspond to the stable and nonstable fixed points of the system (18). From relations (20) and (21) one can see that the phase difference $\Delta\varphi$ of coupled generators is directly proportional to the frequency of oscillations ω and inversely proportional to the coupling parameter ε for the small values of detuning parameter Δ:

$$\Delta\varphi \simeq \frac{\lambda\omega}{2\varepsilon}. \quad (22)$$

So in the synchronous regime the phase shift $\Delta\varphi$ for synchronized frequencies obeys the relation

$$\Delta\varphi \sim \omega\varepsilon^{-1}. \quad (23)$$

It is important to note that the time delay between synchronized spectral components,

$$\tau = \frac{\Delta\varphi}{\omega} \sim \varepsilon^{-1}, \quad (24)$$

does not depend upon the frequency, and therefore, the time delays for all frequencies f are equal to each other. Accordingly, the phase shift $\Delta\varphi_f$ for the frequency f obeys relation (5) which is the necessary condition for the appearance of lag synchronization. Evidently, if the type of coupling between oscillators is selected in such a manner that the phase shift $\Delta\varphi_f$ of synchronized spectral components satisfies the condition (23), the appearance of the lag synchronization regime is possible for large enough values of the coupling strength. Otherwise, if the established phase shift does not satisfy the condition (23), realization of the lag synchronization regime in the system is not possible for such a kind of coupling. So relation (23) can be considered as the criterion of the possibility of the existence (or, otherwise, impossibility of the existence) of the lag synchronization regime in coupled dynamical systems.

The regularity (24) takes place for a large number of dynamical systems and, probably, is universal. Let us consider manifestations of this regularity for several examples of coupled chaotic dynamical systems.

As the first example we consider the coupled Rössler systems (8) described above. Obviously, one has to consider the phase shift $\Delta\varphi_f$ (or time shift τ) of synchronized spectral components to verify relation (24). As has been shown above, spectral components characterized by a large value of

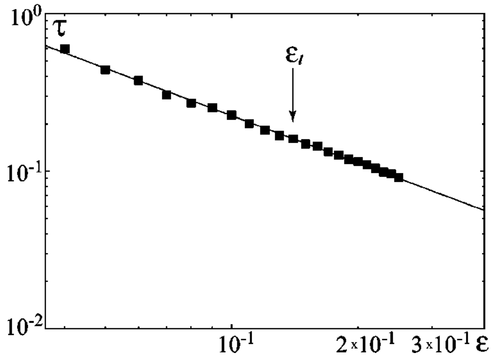

FIG. 5. The dependence of time shift τ between base spectral components (solid squares) on the coupling parameter ε for two coupled Rössler systems (8). The straight line corresponds to the power law $\tau \sim \varepsilon^{-1}$. The value of the coupling parameter $\varepsilon_l \simeq 0.14$ corresponding to the appearance of the lag synchronization regime is shown by the arrow.

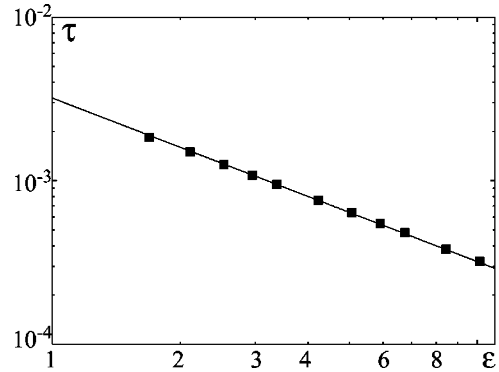

FIG. 6. The dependence of time shift τ between time series $x_1(t)$ and $x_2(t)$ (solid squares) on the coupling parameter ε for two unidirectionally coupled chaotic oscillators (25) and (26). The straight line corresponds to the power law $\tau \sim \varepsilon^{-1}$.

energy become synchronized first when the coupling strength increases. So the main spectral components f_m of the Fourier spectra of coupled systems are synchronized in the most lengthy range of the coupled parameter value. Therefore, it is appropriate to consider the time shift τ of the main spectral components for coupling strength values $\varepsilon > 0.05$.

In Fig. 5 the dependence of the time lag τ between Fourier-spectra-based frequency components of interacting chaotic oscillators on the coupling parameter ε is shown. The base frequency $\omega_m = 2\pi f_m$ of the spectrum is close to $\omega = 1$ and slightly changes with an increase of the coupling parameter. In Fig. 5 one can see that after entrainment of Fourier-spectrum-based spectral components of interacting oscillators (which corresponds to the establishment of the time-scale synchronization regime; see also [13]) the time lag τ, which is between them, obeys the universal power law (24).

As the second example we consider the chaotic synchronization of two unidirectionally coupled Van der Pol–Duffing oscillators [2,24,25]. The drive generator is described by a system of dimensionless differential equations

$$\dot{x}_1 = -\nu_1[x_1^3 - \alpha x_1 - y_1],$$

$$\dot{y}_1 = x_1 - y_1 - z_1,$$

$$\dot{z}_1 = \beta y_1, \tag{25}$$

while the behavior of the response generator is defined by the system

$$\dot{x}_2 = -\nu_2[x_2^3 - \alpha x_2 - y_2] + \nu_2 \varepsilon (x_1 - x_2),$$

$$\dot{y}_2 = x_2 - y_2 - z_2,$$

$$\dot{z}_2 = \beta y_2, \tag{26}$$

where $x_{1,2}$, $y_{1,2}$, and $z_{1,2}$ are dynamical variables, characterizing states of the drive and response generators, respectively. The values of the control parameters are chosen as $\alpha = 0.35$, $\beta = 300$, $\nu_1 = 100$, and $\nu_2 = 125$, and the difference of the parameters ν_1 and ν_2 provides the slight nonidentity of the considered generators.

In Fig. 6 the dependence of the time lag τ between time realizations of coupled oscillators on the coupling parameter value ε is shown. In this range of coupling parameter values the lag synchronization regime is realized. Obviously, the time lag τ also obeys the power law $\tau \sim \varepsilon^n$ with exponent $n = -1$, which corresponds to relation (24).

VI. UNSTABLE PERIODIC ORBITS

It is important to note another manifestation of the power law (24). It is well known that the unstable periodic orbits (UPO's) embedded in chaotic attractors play an important role in the system dynamics [26–28] including the cases of chaotic synchronization regimes [29–31]. The synchronization of two coupled chaotic systems in terms of unstable periodic orbits has been discussed in detail in [32]. It has been shown that UPO's are also synchronized with each other when chaotic synchronization in the coupled systems is realized [32]. Let us consider the synchronized saddle orbits $m:n$ $(m = n = 1, 2, \ldots)$, where m and n are the length of the unstable periodic orbits of the first and second Rössler systems (8), respectively. It was shown that such synchronized orbits may be both "in phase" and "out of phase," but only "in-phase" orbits exist in all range of coupling parameter values starting from the point of the beginning of the synchronization (see [32] for details). It is known that the time shift between synchronized "in-phase" orbits decreases with an increase in coupling strength. As the UPO's have an influence on the system dynamics (and on the Fourier spectra of the considered systems, too), it seems to be interesting to examine whether the time shift τ between UPO's obeys the power law (24).

To calculate the synchronized saddle orbits we have used the Schmelcher-Diakonos (SD) method [33,34] in the same way as it had been done in [32]. The UPO embedded in the chaotic attractor of the first Rössler system and the time series $x_{1,2}(t)$ corresponding to the "in-phase" synchronized UPO's realized in system (8) for coupling strength $\varepsilon = 0.07$ is shown in Fig. 7. One can see the presence of the time shift τ

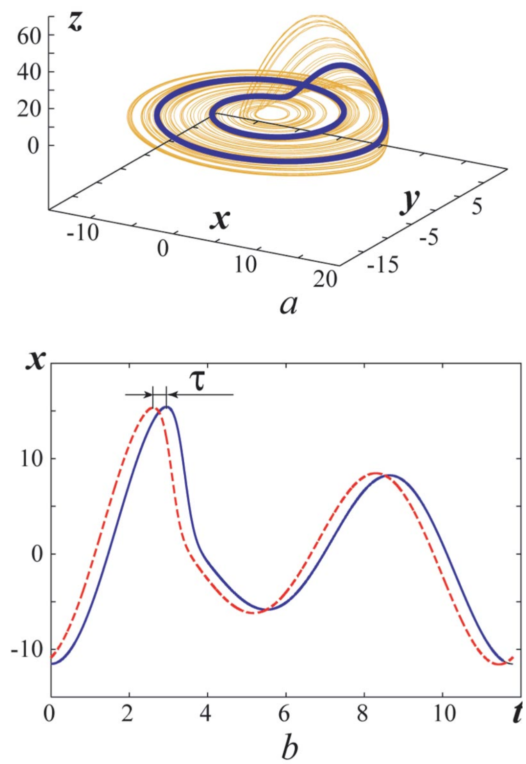

FIG. 7. (a) The unstable periodic orbit of length $m=2$ embedded in the chaotic attractor of the first system. (b) Time series $x_1(t)$ and $x_2(t)$ corresponding to the "in-phase" unstable saddle orbits of length $m=2$ in the first (solid line) and the second (dashed line) Rössler systems, respectively. The coupling parameter is chosen as $\varepsilon=0.07$. The time shift τ is denoted by means of the arrow.

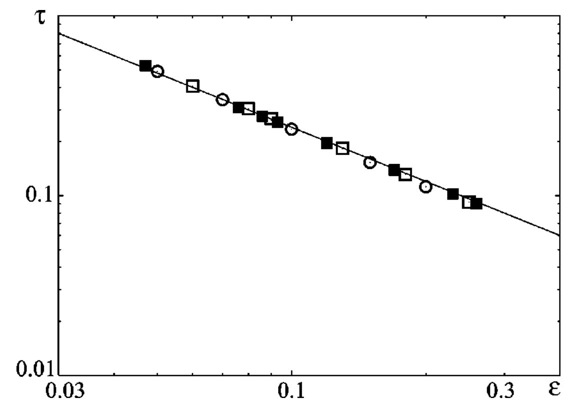

FIG. 8. The dependence of the time shift τ between time series $x_1(t)$ and $x_2(t)$ corresponding to the synchronized saddle orbits of the first and second Rössler systems on the coupling parameter ε. The open squares correspond to unstable orbits with length $m=1$, the open circles demonstrate the time shift τ for the orbits with length $m=2$, and the solid squares show this dependence for orbits with length $m=3$. The straight line corresponds to the power law $\tau \sim \varepsilon^{-1}$.

between these time series which can be easily calculated.

The calculated time shift τ between such "in-phase" synchronized saddle orbits appears to obey the power law $\tau \sim \varepsilon^{-1}$ as well as the spectral components of the Fourier spectra do (see Fig. 8). We have examined this relation for "in-phase" UPO's with length $m=1$–6 and found that the time shift dependence on the coupling strength agrees with power law (24) well, but the data are shown in Fig. 8 only for UPO's with length $m=1$–3 for clearness and simplicity. So the power law (24) seems to be universal and is manifested in different ways.

VII. CONCLUSION

In conclusion, we have considered the chaotic synchronization of coupled oscillators by means of Fourier spectra; several regularities have been observed.

The chaotic synchronization of coupled oscillators is manifested in the following way. Starting from a certain coupling parameter value synchronization of the main spectral components of the Fourier spectra of interacting chaotic oscillators takes place. Therefore, for these spectral components f, condition (9) is satisfied. In this case one can detect the presence of the time-scale synchronization regime (see [12,17]). If for the considered systems one can introduce correctly the instantaneous phase of chaotic signal [12,20], one will also detect easily the phase synchronization by means of a traditional approach (see, e.g., [1]).

With a further increase of the coupling parameter, more spectral components become synchronized. If the coupling between interacting systems is selected in such a way that the lag synchronization regime can be realized, then the time shift between synchronized spectral components obeys the power law (24). The spectral components characterized by the large value of the energy become synchronized first. Accordingly, the part of the energy falling on the synchronized spectral components increases from 0 (asynchronous dynamics) to 1 (the lag synchronization regime). Synchronization of all frequency components corresponds to the appearance of the lag synchronization regime. With a further increase of the coupling strength, the time lag τ obeying relation (24) tends to be zero, and related oscillations tend to demonstrate the complete synchronization regime. In this case the time shift τ between synchronized components does not depend on the frequency f of the considered components (it is the same for all synchronized components) and obeys the power law (24) with exponent $n=-1$. The time shift between synchronized "in-phase" UPO's embedded in chaotic attractors also obeys the same power law.

So in the present paper the mechanism of the appearance of the chaotic synchronization regime in coupled dynamical systems, based on the arising of the phase relation between frequency components of the Fourier spectra of interacting chaotic oscillators, has been discussed. The obtained results concerning the power law (24) may be also considered as a criterion of the possible existence (or, otherwise, impossibility of the existence) of the lag synchronization regime in coupled dynamical systems [i.e., the lag synchronization regime cannot be observed in the coupled chaotic oscillator system unless the time shift between synchronized components obeys the power law (24)].

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