Continuous majority-vote model

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We introduce a kinetic irreversible *XY* model and investigate its dynamic critical behavior through short-time Monte Carlo simulations on square lattices with periodic boundary conditions, starting from an ordered state. We find evidence that this system exhibits a Kosterlitz-Thouless-like phase for low values of the noise parameter. We present results for the correlation function exponent η for several noise values. We also find that the dynamic critical exponent z is in agreement with the value expected for local update Monte Carlo rules.

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I. INTRODUCTION

An important aspect in the study of statistical model systems is the role of symmetry. It is well known that equilibrium statistical models are in the same universality class when they share identical symmetries.

For far-from-equilibrium systems, results of a number of numerical simulations support the conjecture that models with the same symmetries and defined on the same lattice are also in the same universality class [1,2]. Most of these studies have been done on models of discrete symmetries—viz., on kinetic Ising [3–7] and Potts models [8–12] evolving under stochastic reversible dynamics. Models with an infinite number of states were analyzed as well [13].

An interesting class of nonequilibrium models consists of systems which evolve in time according to a dynamics such that in the stationary state the condition of detailed balance is not satisfied—that is, microscopic irreversible models [14].

The simplest example of an irreversible system that presents a nontrivial behavior is provided by the two-state majority-vote model [15]. In this model, each dynamical variable σ on a lattice site takes the state of the majority of their neighbors with probability $1-q$ and the opposite state with probability q. Therefore, the time evolution is governed by the flip rate

$$w_i = \frac{1}{2}\left\{ 1-(1-2q)\sigma_i \mathcal{S}\left(\sum_{\delta}^{z} \sigma_{i+\delta} \right) \right\}, \qquad (1)$$

where $\sigma_i = \pm 1$, $\mathcal{S}(x) = \mathrm{sgn}(x)$ for $x \neq 0$, $\mathcal{S}(0)=0$, and the summation runs over the z neighbors of the site i.

Numerical simulations of this model showed that it presents a dynamical phase transition from an ordered stationary state to a disordered one at a critical value of noise parameter q, which is in the same universality class as the equilibrium Ising model [15]. Therefore, microscopic reversible and irreversible models seem to be in the same universality class [2], at least when their dynamic variables have a finite number of states.

In this work, we introduce a generalization of the majority-vote model in which the dynamical variables can have continuous symmetry—i.e., an *infinite* number of states.

We investigate whether the system exhibits a phase transition at a finite value of the noise parameter and, if it is the case, whether the conjecture mentioned above could include models with continuous symmetries.

Here we consider the case where the dynamic variables are plane rotators $\vec{S}_i = (S_{x,i}, S_{y,i})$ residing on the sites of square lattices and $|\vec{S}_i| = 1$. Under a reversible dynamics this system corresponds to a bidimensional *XY* model in which an ordered phase is not possible. Nevertheless, this model undergoes a nonusual phase transition, due to unbinding of defects, known as the Kosterlitz-Thouless (KT) phase transition [16]. In the low-temperature phase the O(2) symmetry is preserved and the system remains critical in the sense that the spatial correlation length is infinity.

The equilibrium *XY* model has important physical applications, apart from the obvious one to *XY* magnets. For example, it describes the critical properties of the superfluid helium and it is also related to some models of the roughening transition of crystalline surfaces.

In the context of reversible dynamics this model has attracted much attention and its equilibrium [17–20] as well as out-of-equilibrium [13,21–24] properties have been investigated by a number of techniques.

In this paper we are going to investigate, through short-time Monte Carlo simulations, what features of the behavior of the *XY* model are preserved when it evolves according to a *microscopic irreversible* dynamics. In Sec. II we introduce the kinetic *XY* model and our simulational procedure. In Sec. III we present the results of the simulations along with a finite-size scaling analysis of the temporal behavior of the relevant observables. We conclude in Sec. IV with our final remarks.

II. KINETIC MODEL AND SIMULATION

In order to introduce a microscopic irreversible dynamics for continuous degrees of freedom closely related to that of Eq. (1) we first need a generalization of the spin-flip operation $\sigma_i \rightarrow -\sigma_i$ as in the Ising model. That has been done by Wolf in the context of a cluster Monte Carlo dynamics [25].

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It consists in the reflection of the spin with respect to the plane orthogonal to an arbitrary direction.

For an O(2) symmetry the system evolves according to the following rules. Each time step, a random direction $\hat{r}$ is selected and we scan the whole lattice. For each site we project $\vec{S}_i$ and its neighbor rotators onto $\hat{r}$. The component of $\vec{S}_i$ in the $\hat{r}$ direction is flipped with probability

$$w_i = \frac{1}{2}\left\{1 - (1-2q)\mathcal{S}(\vec{S}_i \cdot \hat{r})\mathcal{S}\left(\sum_{\delta}^{z} \vec{S}_{i+\delta} \cdot \hat{r}\right)\right\}. \qquad (2)$$

We might visit the lattice sites sequentially or pick them up at random. As in our test runs we did not observe any difference between both procedure, we opt to walk sequentially through the lattice. As usual, a Monte Carlo time step (MCS) corresponds to a sweep of the whole lattice. According to Eq. (2) one sees that for $q=1/2$ every spin rotates with probability $1/2$, whereas for $q=0$ the probability of the component of a spin ending up pointing contrary to its neighborhood vanish. So we might expect some kind of cooperative behavior for sufficiently low values of the noise parameter q.

It has been observed in a number of Monte Carlo simulations that the time evolution of several observables shows scaling behavior even for short time. This was analytically predicted only for dynamic evolution starting from a disordered state [26]. Nevertheless, short-time dynamical scaling can also be found starting from an ordered state. This has provided an efficient method to determine the conventional critical exponents. In the next section we are going to explore this to investigate the behavior of the kinetic model introduced here.

Therefore, let us introduce some useful quantities for our posterior analysis. In this paper we only consider the dynamic relaxation of the two-dimensional kinetic *XY* model from an ordered state. For this purpose we take the initial state to be $\vec{S}_i=(1,0)$ for all spins. This means that the x component of the magnetization,

$$m_x(t) = \frac{1}{L^2}\sum_i S_{x,i}, \qquad (3)$$

at $t=0$ is 1, where L denotes the lattice size. Then the system evolves according to the above dynamics for some value q of the noise. Due to the absence of conventional long-range order for two-dimensional systems with continuous symmetry, we expect $m_x(t)$ to go to zero for large enough t for any value of q. Nevertheless, if there exists a KT-like phase at low noises, the magnetization should show a short-time power-law behavior for noises near and below some critical q_c.

Other observables also provide useful information about the critical behavior of the system. In particular, we measure the second moment of the magnetization,

$$\chi_0(t) = \frac{1}{L^2}\left(\sum_i S_{x,i}\right)^2, \qquad (4)$$

and the Fourier transform of the equal-time two-point correlation function,

$$\chi_k(t) = \frac{1}{L^2}\sum_{i,r} S_{x,i}S_{x,i+r}\exp(ikr), \qquad (5)$$

with $k=2\pi/L$.

The last two equations allow us to introduce a time-dependent correlation length through

$$\xi(t) = \frac{1}{k}\sqrt{\left(\frac{\chi_0}{\chi_k} - 1\right)}. \qquad (6)$$

The equilibrium *XY* model has an exponential singularity; that is, the correlation length diverges exponentially. This behavior contrasts with that of a second-order transition, where the correlation length diverges with a power law. Also the spatial correlation function decays algebraically to zero. Therefore, assuming that the correlation function decays as

$$\Gamma(r) \sim \frac{1}{r^{-\eta}}\exp(-r/\xi_s), \qquad (7)$$

where ξ_s is the spatial correlation length, we obtain

$$\chi_k \sim \xi_s^{2-\eta}. \qquad (8)$$

And the short-time dependence of the quantities of interest comes from the relation $t\sim\xi_s^z$, where z is the so-called dynamical critical exponent. For instance, Eq. (8) implies

$$\chi_k(t) \sim t^{(2-\eta)/z}. \qquad (9)$$

In addition,

$$m_x(t) \sim t^{-\eta/2z} \qquad (10)$$

and

$$\xi(t) \sim t^{-1/z}. \qquad (11)$$

In the next section we are going to show these scaling laws are satisfied by the kinetic *XY* model introduced here.

III. RESULTS AND DISCUSSION

We perform Monte Carlo simulation in a kinetic two-dimensional *XY* model introduced in the previous section, restricting ourselves only to the relaxation from an ordered initial state. We consider full periodic square lattices of linear sizes $L=16$, 32, 64, and 256. Most of our results will be presented for a 64×64 lattice. The system is prepared in an initial state and then released to the dynamic evolution for some value of the noise q. All calculated quantities are averaged over several realizations—that is, over different time trajectories.

First we consider the relaxation of the x component of the magnetization for rather high noises. Figure 1 shows the magnetization as a function of time for two values of the noise q and $L=16$, 32, and 64. We performed 1000 samples; i.e., we averaged over 1000 time trajectories, for $q=0.08$ (open symbols) and $q=0.14$ (solid symbols) and each lattice size. From those data shown in Fig. 1 we see that the magnetization measured for both values of q decays exponentially with a characteristic time τ depending strongly on q. Notice also the weak dependence with the lattice size for

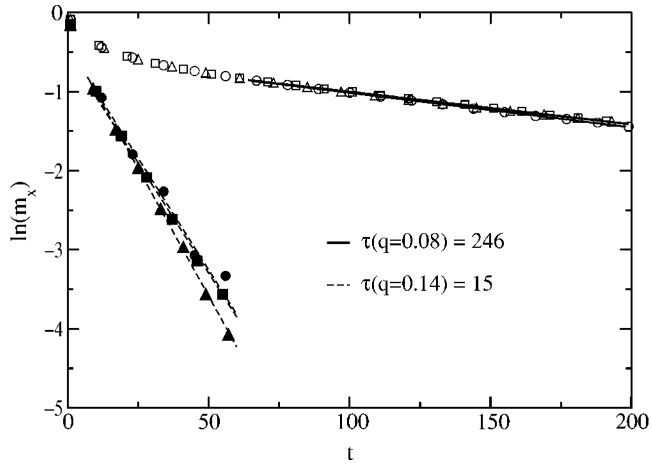

FIG. 1. The x component of the magnetization for high noises starting from the ordered state in a lin-log scale. The lines are linear fitting to the data from which we estimate the relaxation time, τ. Different symbols correspond to distinct lattice sizes. τ increases from about 16 times when the noise is lowered from $q=0.14$ to $q=0.08$, which might indicate the system is approaching criticality.

these high values of the noise parameter. Assuming that $m_x(t) \sim \exp(-t/\tau)$, we estimate $\tau(q=0.14)=15(1)$ MCS and $\tau(q=0.08)=246(1)$ MCS from a linear fitting to the data of Fig. 1. Whereas the exponential decaying is a signal that the system is not critical at all, the extraordinary increasing of the relaxation time when the noise is lowered from $q=0.14$ to $q=0.08$ might indicate we are approaching criticality.

In Fig. 2 we plot the magnetization as a function of time for a 64×64 lattice in a double-logarithmic scale and for noises 0.025, 0.03, 0.035, 0.04, 0.045, and 0.048. For each noise we performed 8 blocks of 5×10^3 samples 400 MCS long. Each block yields an estimate for $m_x(t)$ at a given time t, and from this we obtain our final estimate and estimate of its statistical error following standard procedures. We proceed in the same way to calculate all other quantities of interest and their respective statistical errors. The largest statistical errors turned out to be less than 0.5%. Therefore, the error bars are too small and they are left out in the figures for clarity. From Fig. 2(a) we see clearly that at early times the curves do not display power-law behavior. However, for t greater than a *microscopic* time scale, which depends weakly on q, all curves exhibit power-law behavior as shown in Fig. 2(b). In fact, visual inspection of these data shows that the magnetization decays as a power law for values of the noise $q\leqslant 0.048$ at least for intermediary times. This behavior strongly suggests that the system remains critical all the way down to $q=0$ and it is similar to that of the equilibrium *XY* model with the noise being analogous to the temperature. From Eq. (10) one sees that the slope of each curve displayed in Fig. 2(b) yields the critical exponent $\eta/2z$ at that particular noise. However, finite-size effects take place at long times and the behavior cross over to an exponential relaxation towards a steady state. Therefore, strictly speaking the power law is only obeyed in some subinterval of the time evolution. In order to estimate the exponent $\eta/2z$ we fit the data according to a straight line with t in the interval $[t_{\mathrm{inf}}, t_{\mathrm{sup}}]$, where t_{inf} and t_{sup} are inferior and superior cutoffs,

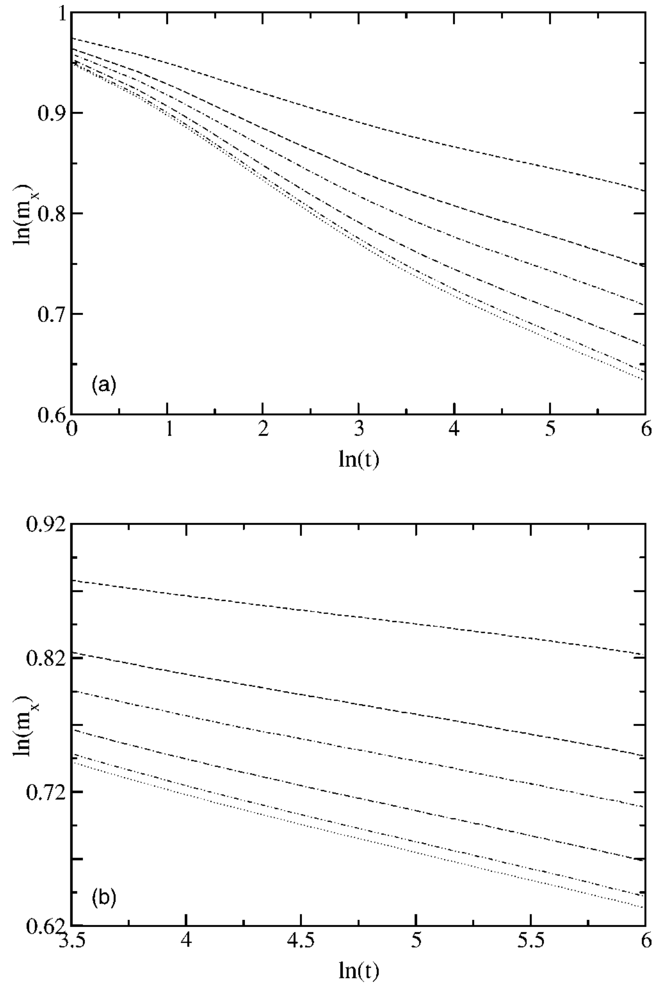

FIG. 2. The x component of the magnetization for a 64×64 lattice in a double-logarithmic scale. The starting state is the ordered one. In (a) t belongs to $[1,400]$ and in (b) only the long-time regime is shown. From above the noises are 0.025, 0.03, 0.035, 0.04, 0.045, and 0.048. The slope of each curve yields $\eta/2z$ at each q.

respectively. We selected the cutoffs guided by the χ square per degree of freedom as the best fitting criterion. The use of different time intervals yields an estimate for the systematic errors involved in our analysis. Moreover, we perform the fitting of the data directly to a power law according to Eq. (10)—i.e., giving equal weight to the data in both the short- and long-time regimes. Both procedures yield consistent results, and we take the mean as our final estimates for the exponents. At $q=0.048$ the best fitting was achieved in the time interval $[100,300]$ and we estimate $\eta/2z=0.059\,39(2)$. The quoted error is the sum of statistical and systematic error. We proceed in the same way to calculate $\eta/2z$ for the other noises, and we summarize the results in Table I along with the value of others relevant exponents as calculated below. Results from simulations carried out on a 256×256 lattice agree with those reported in Table I. Therefore, finite-size effects can be ignored within the statistical errors.

We turn now to the calculation of the other exponents. In Fig. 3 we plot the Fourier transform of the equal-time two-point correlation function χ_k as a function of t in a double-

TABLE I. The slope of the curves of $m_x(t)$, $\chi_k(t)$, the dynamic critical exponent z, and the correlation function exponent η. The exponent z was obtained through the reciprocal of the slope of the curve of $\xi(t)$ and agrees with that calculated using the measured values of $\eta/2z$ and $(2-\eta)/z$. The exponent η comes from $\eta/2z$ and z.

q	$\eta/2z$	$(2-\eta)/z$	z	η
0.025	0.0252(6)	0.92(3)	2.1(1)	0.106(5)
0.030	0.0310(3)	0.87(2)	2.14(18)	0.133(3)
0.035	0.0382(3)	0.85(4)	2.16(20)	0.168(2)
0.040	0.0446(1)	0.83(2)	2.17(9)	0.194(2)
0.045	0.0537(6)	0.830(3)	2.133(7)	0.229(3)
0.048	0.05939(2)	0.814(4)	2.140(9)	0.254(2)

logarithmic scale for the same noises and lattice size as in Fig. 2. Again, visualization of the curves in this figure shows that Eq. (9) is satisfied. Therefore, by fitting the data to straight lines we estimate the exponent $(2-\eta)/z$ for each q. Also we double checked the results by fitting the data directly to a power-law according to Eq. (9). It is worth noticing that different quantities display power-law behavior in distinct time intervals, so we carried out a careful analysis to estimate the systematic error which comes from using different time intervals to fit the data. The calculated values for the $(2-\eta)/z$ exponent are displayed in the third column of Table I.

Having independent estimates for $\eta/2z$ and $(2-\eta)/z$ we can calculate the dynamical critical exponent z and the associated critical exponent of the correlation function η. Another way of getting estimates for z is through Eq. (11). For this purpose, we plot the time-dependent correlation length as a function of time in Fig. 4 for the same simulational parameters as in the previous two figures. Again, we see that after a certain period of time ξ displays power-law behavior and the slope of each curve yields an estimate for $1/z$. Both procedures yield compatible results. In the fourth column of Table I we present our final estimates for the dynamical critical exponent z. We observe that within our quoted error bars z does not depend on q.

Finally, we complete Table I listing in the last column the correlation function exponent η calculated from z and $\eta/2z$.

In Table I, we summarize all the values of the critical exponents discussed above. First, we notice that the correlation function exponent η shows a strong dependence with the noise q, with a rather linear trend towards 0.25 when $q \to q_c$ from below. This behavior is qualitatively similar to that observed in the equilibrium *XY* model. A quantitative comparison is not possible for we did not calculate the exact value of q_c. The critical dynamic exponent z is estimated through a weighted average of the data shown in the fourth column of Table I and turns out to be $z=2.136(6)$, in agreement with $z=2.16(2)$ obtained for the *XY* model evolving under reversible dynamics [27]. Our estimate for z is also compatible with those of two-dimensional systems with a second-order transition such as the Ising or the three-state Potts model and with a KT-like transition such as the six-state clock model [13]. On the other hand, this estimate is slightly higher than $z\approx 2$ recently obtained in the context of reversible dynamics [24].

IV. CONCLUSIONS

In conclusion, we introduced a kinetic irreversible *XY* model and investigated its behavior through short-time

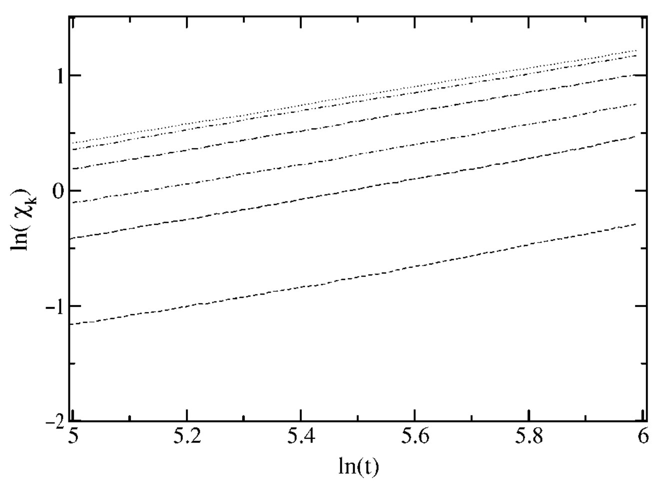

FIG. 3. The Fourier transform of the equal-time two-point correlation function χ_k as a function of time in a double-logarithmic scale for a 64×64 lattice. Noises and line types are the same as in Fig. 2. The slope of each curve yields $(2-\eta)/z$ at each q.

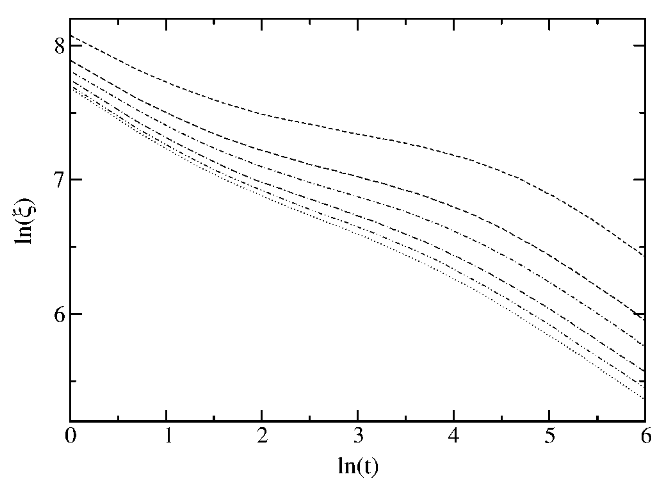

FIG. 4. The time-dependent correlation length ξ versus t in a double-logarithmic scale. Noises and line types are the same as in Fig. 2. After a certain period of time the curves are well fitted by straight lines whose slopes yield $1/z$.

Monte Carlo simulations in square lattices. We focused only on the relaxation from an ordered state. The nonequilibrium dynamic process is closely related to that of the majority-vote model with a noise in which the dynamic variables have a finite number of states. The results show that there exists a low-noise KT-like phase where the measured correlation function exponent η depends on the noise and the system is critical in the sense that the correlation length is infinity. Although in this paper we did not attempt to obtain the critical noise where the KT phase begins, we did find $\eta = 0.254(2)$ for a noise $q=0.048$ which is surprisingly close to the value 0.25 predicted by the Kosterlitz-Thouless theory at the onset of the KT phase in the context of the equilibrium *XY* model.

We obtained rather accurate value of the dynamic critical exponent z which turned out to be very close to 2, in agreement with what should be expected for local Monte Carlo update rules.

It is worth mentioning that we ignored any possible corrections to scaling throughout this paper. Nevertheless, in a recent work on dynamic *XY* models evolving under a reversible dynamics Zheng and co-workers have observed power-law corrections to scaling for the relaxation from an ordered state [24]. Therefore, one has to consider with some caution the systematic errors reported in the present work since they do not take into account such effects.

Finally, we notice that the irreversible dynamic system investigated in this work presents a critical behavior very close to the corresponding equilibrium one; i.e., they seem to be in the same universality class.

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