Point process model of $1/f$ noise vs a sum of Lorentzians

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We present a simple point process model of $1/f^{\beta}$ noise, covering different values of the exponent β. The signal of the model consists of pulses or events. The interpulse, interevent, interarrival, recurrence, or waiting times of the signal are described by the general Langevin equation with the multiplicative noise and stochastically diffuse in some interval resulting in a power-law distribution. Our model is free from the requirement of a wide distribution of relaxation times and from the power-law forms of the pulses. It contains only one relaxation rate and yields $1/f^{\beta}$ spectra in a wide range of frequencies. We obtain explicit expressions for the power spectra and present numerical illustrations of the model. Further we analyze the relation of the point process model of $1/f$ noise with the Bernamont-Surdin-McWhorter model, representing the signals as a sum of the uncorrelated components. We show that the point process model is complementary to the model based on the sum of signals with a wide-range distribution of the relaxation times. In contrast to the Gaussian distribution of the signal intensity of the sum of the uncorrelated components, the point process exhibits asymptotically a power-law distribution of the signal intensity. The developed multiplicative point process model of $1/f^{\beta}$ noise may be used for modeling and analysis of stochastic processes in different systems with the power-law distribution of the intensity of pulsing signals.

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I. INTRODUCTION

$1/f$ fluctuations are widely found in nature; i.e., the power spectra $S(f)$ of a large variety of physical, biological, geophysical, traffic, financial, and other systems at low frequencies f have $1/f^{\beta}$ (with $0.5 \lesssim \beta \lesssim 1.5$) behavior [1–4]. Widespread occurrence of signals exhibiting such a behavior suggests that a generic mathematical explanation of $1/f$ noise might exist. The generic origins of two popular noises—white noise [no correlation in time, $S(f) \sim 1/f^0$] and Brownian noise [no correlation between increments, $S(f) \sim 1/f^2$]—are very well known and understood. It should be noted that the Brownian motion is the integral of white noise and that the operation of integration of the signal increases the exponent by 2 while the inverse operation of differentiation decreases it by 2. Therefore, $1/f$ noise cannot be obtained by the simple procedure of integration or differentiation of such convenient signals. Moreover, there are no simple, even linear stochastic differential equations generating signals with $1/f$ noise. Recently we derived a stochastic nonlinear differential equation for a signal exhibiting $1/f$ noise in any desirably wide range of frequency [5]. The physical interpretation of this highly nonlinear equation is not so clear and straightforward as that of the linear Langevin equation, generating the Brownian motion of the signal with $1/f^2$ spectrum. Therefore, $1/f$ noise is often represented as a sum of independent Lorentzian spectra with a wide range of relaxation times [6]. Summation or integration of the Lorentzians with the appropriate weights may yield $1/f$ noise.

Not long ago a simple analytically solvable model of $1/f$ noise was proposed [7], analyzed [8,9], and generalized [10]. The signal in the model consists of pulses or series of events (a point process). The interpulse times of the signal stochastically diffuse about some average value. The process may be described by an autoregressive iteration with a very small relaxation. The proposed model reveals one of the possible origins of $1/f$ noise—i.e., random increments of the time interval between the pulses (the Brownian motion in the time axis), sometimes resulting in a clustering of the signal pulses [7,8,10].

The power spectral density of such point process may be expressed as

$$S(f) \simeq 2\bar{I}^2\bar{\tau}P_k(0)/f. \tag{1}$$

Here $\bar{\tau}=\langle \tau_k \rangle$ is the expectation of the interpulse time $\tau_k = t_{k+1} - t_k$, with $\{t_k\}$ being the sequence of pulses occurrence times or arrival times t_k, whereas $P_k(\tau_k)$ is a steady-state distribution density of the interpulse time τ_k in k space and $\bar{I}$ is the average intensity of the signal:

$$I(t) = \sum_k A_k(t - t_k). \tag{2}$$

The function $A_k(t-t_k)$ represents the shape of the k pulse of the signal in the region of the pulse occurrence time t_k.

It is easy to show that the fluctuations and shapes of $A_k(t-t_k)$ for sharp pulses mainly influence the high-frequency power spectral density. Therefore, in a low-frequency region we can restrict our analysis to the noise originated from the correlations between the occurrence times t_k. Then we can simplify the signal to the point process

$$I(t) = \bar{a}\sum_k \delta(t - t_k), \tag{3}$$

with $\bar{a}$ being an average contribution to the signal of one pulse or one particle when it crosses the section of observation.

Point processes arise in different fields, such as physics, economics, cosmology, ecology, neurology, seismology, traffic flow, signaling and telecom networks, audio streams, and the Internet (see, e.g., [3,11–14] and references therein). The proposed point process model [7,8,10] can been modified and useful for the modeling and analysis of self-organized systems [15], atmospheric variability [16], large flares from γ-ray repeaters in astronomy [17], particles moving in viscous fluid [18], dynamical percolation [19], $1/f$ noise observed in cortical neurons and earthquake data [20], financial markets [21], cognitive experiments [4,22], Parkinsonian tremors [23], and time interval production in tapping and oscillatory motion of the hand [24].

The analytically solvable model and its generalizations [7–10] contain, however, some shortage of generality; i.e., it results only in exact $1/f$ (with $\beta=1$) noise and only if $P_k(\tau_k)\simeq$const when $\tau_k\to 0$. On the other hand, a numerical analysis of the generalized model with different restrictions for the diffusion of the interpulse time τ_k reveals $1/f^\beta$ spectra with $1\lesssim\beta\lesssim 1.5$ [10].

The aims of this paper are to generalize the analytical model, seeking to define the variety of time series exhibiting the power spectral density $S(f)\sim 1/f^\beta$ with $0.5\lesssim\beta\lesssim 2$, and to analyze the relation of the point process model with the Bernamont-Surdin-McWhorter model [6], representing the signal as a sum of the appropriate signals with different rates of the linear relaxation.

II. POWER SPECTRAL DENSITY OF THE POINT PROCESS

The point process is primarily and basically defined by the occurrence times t_k. The power spectral density of the point process (3) may be expressed as [7,8,10]

$$S(f)=\lim_{T\to\infty}\left\langle \frac{2}{T}\int_{t_i}^{t_f}\int_{t_i}^{t_f} I(t')I(t'')e^{i\omega(t''-t')}dt'dt''\right\rangle = \lim_{T\to\infty}\left\langle \frac{2\bar{a}^2}{T}\left|\sum_{k=k_{\min}}^{k_{\max}} e^{-i\omega t_k}\right|^2\right\rangle = \lim_{T\to\infty}\left\langle \frac{2\bar{a}^2}{T}\sum_{k=k_{\min}}^{k_{\max}}\sum_{q=k_{\min}-k}^{k_{\max}-k} e^{i\omega\Delta(k;q)}\right\rangle, \quad (4)$$

where $T=t_f-t_i\gg\omega^{-1}$ is the observation time, $\omega=2\pi f$, and

$$\Delta(k;q)\equiv t_{k+q}-t_k=\sum_{i=k}^{k+q-1}\tau_i \quad (5)$$

is the difference between the pulses occurrence times t_{k+q} and t_k. Here $k_{\min}$ and $k_{\max}$ are minimal and maximal values of the index k in the interval of observation T and the brackets $\langle\cdots\rangle$ denote averaging over realizations of the process.

It should be stressed that the spectrum is related to the underlying process and not to a realization of the process [25,26]. Therefore, averaging over realizations of the process is essential. Without the averaging over the realizations we obtain the squared modulus of the Fourier transform of the data—i.e., the periodogram which is fluctuating wildly and its variance is almost independent of T [25,26]. For calculation of the power spectrum of the actual signal without averaging over the realizations one should use the well-known procedures of the smoothing for spectral estimations [25–28].

Equation (4) may be rewritten as

$$S(f)=2\bar{a}^2\bar{\nu}+\lim_{T\to\infty}\left\langle \frac{4\bar{a}^2}{T}\sum_{q=1}^{N}\sum_{k=k_{\min}}^{k_{\max}-q}\cos[\omega\Delta(k;q)]\right\rangle, \quad (6)$$

where $N=k_{\max}-k_{\min}$ and

$$\bar{\nu}=\frac{1}{\bar{\tau}}=\left\langle \lim_{T\to\infty}\frac{N+1}{T}\right\rangle$$

is the mean number of pulses per unit time. The first term in the right-hand side of Eq. (6) represents the shot noise,

$$S_{\text{shot}}(f)=2\bar{a}^2\bar{\nu}=2\bar{a}\bar{I}, \quad (7)$$

with $\bar{I}=\bar{a}\bar{\nu}$ being the average signal.

Equations (4)–(7) may be modified as

$$S(f)=2\bar{a}^2\sum_{q=-N}^{N}\left(\bar{\nu}-\frac{|q|}{T}\right)\chi_{\Delta(q)}(\omega) \quad (8)$$

and used for evaluation of the power spectral density of the nonstationary process or for the process of finite duration, as well. Here

$$\chi_{\Delta(q)}(\omega)=\overline{\langle e^{i\omega\Delta(q)}\rangle}=\int_{-\infty}^{+\infty}e^{i\omega\Delta(q)}\Psi_q(\Delta(q))d\Delta(q) \quad (9)$$

is the characteristic function of the distribution density $\Psi_q(\Delta(q))$ of $\Delta(q)$, a definition $\overline{\Delta(q)=-\Delta(-q)=\Delta(k;q)}$ is introduced, and the brackets $\langle\ldots\rangle$ denote averaging over realizations of the process and over the time (index k) [8,10]. For the nonstationary process or process of finite duration one should use the real distribution $\Psi_q(\Delta(q))$ with a finite interval of the variation of $\Delta(q)$ or calculate the power spectra directly according to Eq. (4).

When the second sum of Eq. (8) in the limit $T\to\infty$, due to the decrease of the characteristic function $\chi_{\Delta(q)}(\omega)$ for finite ω and large q, approaches zero,

$$\lim_{T\to\infty}\frac{1}{T}\sum_{q=-N}^{N}|q|\chi_{\Delta(q)}(\omega)\to 0,$$

we have from Eq. (8) a power spectrum in the form

$$S(f)=\lim_{T\to\infty}\left\langle \frac{2\bar{a}^2}{T}\sum_{k,q}e^{i\omega\Delta(k;q)}\right\rangle=2\bar{I}^2\bar{\tau}\sum_{q=-N}^{N}\chi_{\Delta(q)}(\omega). \quad (10)$$

III. STOCHASTIC MULTIPLICATIVE POINT PROCESS

According to the above analysis, the power spectrum of the point process signal is completely described by the set of the interpulse intervals $\tau_k=t_{k+1}-t_k$. Moreover, the low-frequency noise is defined by the statistical properties of the signal at large time scale—i.e., by the fluctuations of the time difference $\Delta(k;q)$ at large q, determined by the slow dynamics of the average interpulse interval $\tau_k(q)=\Delta(k;q)/q$ between the occurrence of pulses k and $k+q$. In such a case quite generally the dependence of the interpulse time τ_k on the occurrence number k may be described by the general Langevin equation with the drift coefficient $d(\tau_k)$ and a multiplicative noise $b(\tau_k)\xi(k)$,

$$\frac{d\tau_k}{dk} = d(\tau_k) + b(\tau_k)\xi(k). \tag{11}$$

Here we interpret k as a continuous variable while the white Gaussian noise $\xi(k)$ satisfies the standard condition

$$\langle \xi(k)\xi(k')\rangle = \delta(k-k'),$$

with the brackets $\langle\cdots\rangle$ denoting averaging over the realizations of the process. Equation (11) we understand in Ito interpretation.

A perturbative solution of Eq. (11) in the vicinity of τ_k yields

$$\tau_{k+j} \simeq \tau_k + d(\tau_k)j + b(\tau_k)\int_k^{k+j}\xi(l)dl, \tag{12}$$

$$\Delta(k;q) = \sum_{i=k}^{k+q-1}\tau_i \simeq \int_0^q \tau_{k+j}dj \simeq \tau_k q + d(\tau_k)\frac{q^2}{2} + b(\tau_k)\int_0^q dj\int_k^{k+j}\xi(l)dl. \tag{13}$$

After integration by parts we have

$$\Delta(k;q) = \tau_k q + d(\tau_k)\frac{q^2}{2} + b(\tau_k)\int_k^{k+q}(k+q-l)\xi(l)dl, \tag{14}$$

$$\langle\Delta(k;q)\rangle = \tau_k q + d(\tau_k)\frac{q^2}{2}. \tag{15}$$

Analogously, in the same approximation we can obtain the variance $\sigma_\Delta^2(k;q)=\langle\Delta(k;q)^2\rangle-\langle\Delta(k;q)\rangle^2$ of the time difference $\Delta(k;q)$,

$$\sigma_\Delta^2(k;q) = b^2(\tau_k)\frac{q^3}{3}. \tag{16}$$

A. Power spectral density

Substituting Eqs. (14) and (15) into Eq. (10) and replacing the averaging over k by the averaging over the distribution of the interpulse times τ_k we have the power spectrum

$$S(f) = 4\bar{I}^2\bar{\tau}\int_0^\infty d\tau_k P_k(\tau_k)\mathrm{Re}\int_0^\infty dq\,\exp\left\{i\omega\left[\tau_k q + d(\tau_k)\frac{q^2}{2}\right]\right\}$$
$$= 2\bar{I}^2\frac{\bar{\tau}}{\sqrt{\pi f}}\int_0^\infty P_k(\tau_k)\mathrm{Re}[e^{-i(x-\pi/4)}\,\mathrm{erfc}\,\sqrt{-ix}]\frac{\sqrt{x}}{\tau_k}d\tau_k, \tag{17}$$

where $x=\pi f\tau_k^2/d(\tau_k)$.

The replacement of the averaging over k and over realizations of the process by the averaging over the distribution of the interpulse times $\tau_k, P_k(\tau_k)$, is possible when the process is ergodic. Ergodicity is usually a common feature of the stationary process described by the general Langevin equation [29]. Therefore, we will consider the stationary processes of diffusion of the interpulse time τ_k described by Eq. (11) and restricted in a finite interval of the motion. Such restrictions may be introduced as some additional conditions to the stochastic equation. Similar restrictions, however, may be fulfilled by introducing some additional terms into Eq. (11), corresponding to the diffusion in some "potential well," as in Ref. [5].

The approach (17) is an improvement of the simplest model of pure $1/f$ noise [7,8] taking into account the second, drift, term $d(\tau_k)q^2/2$ in the expression for $\Delta(k;q)$. Note that for $d(\tau_k)\to 0$ from Eq. (17) we recover the known result (1).

According to Eqs. (1), (4), and (17) the small interpulse times and clustering of the pulses make the greatest contribution to $1/f^\beta$ noise. The power-law spectral density is very often related to the power-law behavior of other characteristics of the signal, such as the autocorrelation function, probability densities, and other statistics, and with the fractality of the signals, in general [3,30–35]. Therefore, we investigate the power-law dependences of the drift coefficient and of the distribution density on the time τ_k in some interval of the small interpulse times—i.e.,

$$d(\tau_k) = \gamma\tau_k^\delta, \quad P_k(\tau_k) = C\tau_k^\alpha, \quad \tau_{\min} \leqslant \tau_k \leqslant \tau_{\max}, \tag{18}$$

where the coefficient γ represents the rate of the signal's nonlinear relaxation and C has to be defined from the normalization.

The power-law distribution of the interpulse, interevent, interarrival, recurrence, or waiting time is observable in different systems from physics, astronomy, and seismology to the Internet, financial markets, and neural spikes (see, e.g., [3,14,15,36] and references therein).

One of the most direct applications of the model described by Eq. (18), perhaps, is for the modeling of computer network traffic [14] with the spreading of the packets of the requested files in the Internet traffic and exhibiting the power-law distribution of the interpacket time. Modeling of these processes is under way.

Because of the divergence of the power-law distribution and requirement of the stationarity of the process, stochastic diffusion may be realized over a certain range of the variable τ_k only. Therefore, we restrict the diffusion of τ_k to the interval $[\tau_{\min},\tau_{\max}]$ with the appropriate boundary conditions. Then the steady-state solution of the stationary Fokker-Planck equation with a zero flow corresponding to Eq. (11) is [29]

$$P_k(\tau_k) = \frac{C}{b^2(\tau_k)} \exp\left\{ 2\int_{\tau_{\min}}^{\tau_k} \frac{d(\tau)}{b^2(\tau)} d\tau \right\}. \quad (19)$$

For the particular power-law coefficients $d(\tau_k)$ and $b(\tau_k)$ [see, e.g., Eq. (26)] we can obtain the power-law stationary distribution density (18).

Then Eqs. (17) and (18) yield the power spectra with different slopes β—i.e.,

$$S(f) = \frac{2\bar{I}^2}{\sqrt{\pi}(2-\delta)f}\left(\frac{f_0}{f}\right)^{\alpha/(2-\delta)} I_\kappa(x_{\min},x_{\max}), \quad (20)$$

$$I_\kappa(x_{\min},x_{\max}) = \mathrm{Re}\int_{x_{\min}}^{x_{\max}} e^{-i(x-\pi/4)}\,\mathrm{erfc}(\sqrt{-ix})x^\kappa dx. \quad (21)$$

Here $\kappa=\alpha/(2-\delta)-\frac{1}{2}$, $x_{\min}=f/f_2$, $x_{\max}=f/f_1$,

$$f_0 = \frac{\gamma}{\pi}(C\bar{\tau})^{(2-\delta)/\alpha}, \quad f_1 = \frac{\gamma}{\pi\tau_{\max}^{2-\delta}}, \quad f_2 = \frac{\gamma}{\pi\tau_{\min}^{2-\delta}}. \quad (22)$$

Note that f_0 is indefinite when $\alpha\to 0$; however, $f_0^{\alpha/(2-\delta)}$ is definite and converges to $C\bar{\tau}$ in this limit.

We note the special cases of the power spectral density (20).

(i) $f_1 \ll f \ll f_2$, $-1<\kappa<1/2$,

$$S(f) = \frac{\Gamma(1+\kappa)\bar{I}^2}{\sqrt{\pi}(2-\delta)\cos[(\kappa/2+1/4)\pi]f}\left(\frac{f_0}{f}\right)^{\kappa+1/2}, \quad (23)$$

i.e., $S(f)\sim 1/f^{1+\alpha/(2-\delta)}$ and $S(f)\sim 1/f$ for $\alpha=0$, in accordance with Eq. (1).

(ii) $f\ll f_1$, $\kappa>-1$,

$$S(f) = \frac{\bar{I}^2}{(1+\alpha-\delta/2)}\left(\frac{f_0}{f_1}\right)^{\alpha/(2-\delta)}\sqrt{\frac{2}{\pi f_1 f}}, \quad (24)$$

i.e., for very low frequencies $S(f)\sim 1/\sqrt{f}$.

(iii) $f\gg f_2$, $\kappa<1/2$,

$$S(f) = \frac{\bar{I}^2}{\sqrt{\pi}(2-\alpha-\delta)}\left(\frac{f_0}{f_2}\right)^{\alpha/(2-\delta)}\frac{f_2}{f^2}, \quad (25)$$

i.e., for high frequencies $S(f)\sim 1/f^2$.

For very high frequencies $f\gg\tau_{\max}^{-1}$, however, we cannot replace the summation in Eq. (10) by the integration. Then from Eq. (6) or (10) one gets the shot noise $S(f)=2\bar{a}\bar{I}$, Eq. (7).

Equations (20) and (23)–(25) reveal that the proposed model of the stochastic multiplicative point process may result in power-law spectra over several decades of low frequencies with the slope β between 0.5 and 2.

The simplest and well-known process generating the power-law probability distribution function for τ_k is a multiplicative stochastic process with $b(\tau_k)=\sigma\tau_k^\mu$ and $\delta=2\mu-1$, written as [37]

$$\tau_{k+1} = \tau_k + \gamma\tau_k^{2\mu-1} + \sigma\tau_k^\mu\varepsilon_k. \quad (26)$$

Here γ represents the relaxation of the signal, while τ_k fluctuates due to the perturbation by normally distributed uncorrelated random variables ε_k with a zero expectation and unit variance and σ is a standard deviation of the white noise. According to Eq. (19) the steady-state solution of the stationary Fokker-Planck equation with a zero flow, corresponding to Eq. (26), gives the power-law probability density function for τ_k in k space,

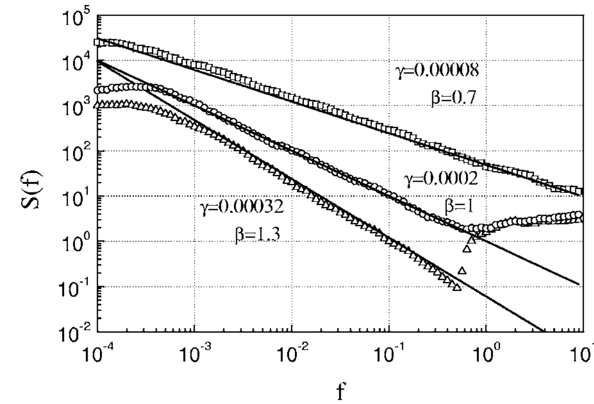

FIG. 1. Power spectral density (4) vs frequency of the signal generated by Eqs. (3) and (26). Numerical simulations are averaged over ten realizations of $N=10^6$ pulse sequences with the parameters $\bar{a}=1$, $\mu=1/2$, and $\sigma=0.02$ and different relaxations of the signal γ. We restrict the diffusion of the interpulse time to the interval $\tau_{\min}=10^{-6}$, $\tau_{\max}=1$ with the reflective boundary condition at $\tau_{\min}$ and transition to the white noise, $\tau_{k+1}=\tau_{\max}+\sigma\varepsilon_k$, for $\tau_k>\tau_{\max}$. The straight lines represent the analytical results according to Eq. (28).

$$P_k(\tau_k) = \frac{1+\alpha}{\tau_{\max}^{1+\alpha}-\tau_{\min}^{1+\alpha}}\tau_k^\alpha, \quad \alpha = \frac{2\gamma}{\sigma^2}-2\mu. \quad (27)$$

The power spectrum for the intermediate f, $f_1\ll f\ll f_2$, according to Eq. (23) is

$$S(f) = \frac{(2+\alpha)(\beta-1)\bar{a}^2\Gamma(\beta-1/2)}{\sqrt{\pi}\alpha(\tau_{\max}^{2+\alpha}-\tau_{\min}^{2+\alpha})\sin(\pi\beta/2)}\left(\frac{\gamma}{\pi}\right)^{\beta-1}\frac{1}{f^\beta}, \quad (28)$$

where

$$\beta = 1+\frac{\alpha}{3-2\mu}, \quad \frac{1}{2}<\beta<2. \quad (29)$$

For $\mu=1$ we have a completely multiplicative point process when the stochastic change of the interpulse time is proportional to itself. Multiplicativity is an essential feature of the financial time series, economics, some natural, and physical processes [38].

Another case of interest is with $\mu=1/2$, when the Langevin equation in the actual time takes the form

$$\frac{d\tau}{dt} = \gamma\frac{1}{\tau}+\sigma\xi(t), \quad (30)$$

i.e., the Brownian motion of the interpulse time with the linear relaxation of the signal $I\simeq\bar{a}/\tau$.

Figures 1 and 2 represent the spectral densities (4) with different slopes β of the signals generated numerically according to Eqs. (3) and (26) for the different parameters of the model. We see that the simple iterative equation (26) with the multiplicative noise produces signals with the power

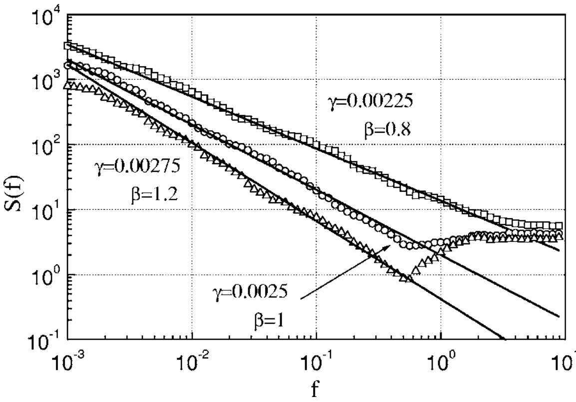

FIG. 2. The same as in Fig. 1 but for $\mu=1$, $\sigma=0.05$, and different parameters γ.

spectral density of different slopes, depending on the parameters of the model. The agreement of the numerical results with the approximate theory is quite good.

It should be noted that the low-frequency noise is insensitive to the small additional fluctuations of the particular occurrence times t_k. Therefore, we can interpret that Eqs. (11), (26), and (30) describe the slow diffusive motion of the average interpulse time, superimposed by some additional randomness.

On the other hand, numerical investigations have shown that the proposed model is stable with respect to variation of the dynamics of the interpulse time τ_k. The substitution of the reflecting boundaries for τ_k by an appropriate confining potential as in Ref. [5] does not change the result.

B. Distribution density of the signal intensity

The origin for appearance of $1/f$ fluctuations in the point process model described by Eqs. (2)–(30) is related to the slow, Brownian fluctuations of the interpulse time τ_k as a function of the pulse number k, when the average interpulse time $\tau_k(q)$ is a slowly fluctuating function of the arguments k and q. In such a case the transition from the occurrence number k to the actual time t according to the relation $dt=\tau_k dk$ yields the probability distribution density $P_t(\tau_k)$ of τ_k in the actual time t,

$$P_t(\tau_k) = P_k(\tau_k)\tau_k/\bar{\tau}. \tag{31}$$

The signal averaged over the time interval τ_k according to Eq. (3) is $I=\bar{a}/\tau_k$. Therefore, the distribution density of the intensity of the point process (3) averaged over the time interval τ_k is

$$P(I) = \frac{\bar{a}\bar{I}}{I^3} P_k\left(\frac{\bar{a}}{I}\right). \tag{32}$$

If $P_k(\tau_k) \simeq \text{const}$ when $\tau_k \to 0$ (the condition for the exhibition for the pure $1/f$ noise in the point process model), the distribution density of the signal is

$$P(I) \sim I^{-3}. \tag{33}$$

For the generalized multiplicative processes (3), (11), and (18) we have from Eqs. (27) and (32) the distribution density of the signal intensity,

$$P(I) = \frac{\lambda - 1}{\tau_{\max}^{\lambda-1} - \tau_{\min}^{\lambda-1}} \frac{\bar{a}^{\lambda-1}}{I^{\lambda}}, \quad \lambda = 3 + \alpha. \tag{34}$$

The power-law distribution of the signals is observable in a large variety of systems ranging from earthquakes to the financial time series [3,12,21,30–35,37,39].

One of the simplest models generating the Brownian fluctuations of the interpulse time τ_k is an autoregressive model [7,8,10] with random increments and linear relaxation of the interpulse time—i.e., the model described by the iterative equation

$$\tau_{k+1} = \tau_k - \gamma(\tau_k - \bar{\tau}) + \sigma\varepsilon_k. \tag{35}$$

Here $\bar{\tau}$ is the average interpulse time, γ is the rate of the linear relaxation, $\{\varepsilon_k\}$ denotes the sequence of uncorrelated normally distributed random variables with zero expectation and a unit variance, and σ is the standard deviation of this white noise. The model (3), (10), and (35) then results in the power spectral density [8]

$$S(f) = \bar{I}^2 \frac{\alpha_H}{f}, \quad \alpha_H = \frac{2}{\sqrt{\pi}} K e^{-K^2}, \quad K = \frac{\bar{\tau}\sqrt{\gamma}}{\sigma}. \tag{36}$$

The distribution density of the intensity of the signal according to Eqs. (19) and (32) then is

$$P(I) = \frac{K\bar{I}^2}{\sqrt{\pi}I^3} \exp\left\{-\frac{\gamma\bar{a}^2}{\sigma^2}\left(\frac{1}{\bar{I}} - \frac{1}{I}\right)^2\right\}. \tag{37}$$

Restricting the diffusion of the interpulse time τ_k by the reflective boundary condition at $\tau_{min}>0$ and for $\tau_{min}\to 0$ we have the truncated distribution density of the signal intensity,

$$P_r(I) = \frac{2K\bar{I}^2}{\sqrt{\pi}[1 + \text{erf}(K)]} \exp\left\{-K^2\left(1 - \frac{\bar{I}}{I}\right)^2\right\} \frac{1}{I^3}, \quad I > 0. \tag{38}$$

In the asymptotic $I \gg \bar{I}$ and $I \gg 2K^2\bar{I}$ from Eq. (38) we have

$$P_r(I) \simeq \alpha_H^r \frac{\bar{I}^2}{I^3} \sim \frac{1}{I^3}, \tag{39}$$

i.e., the power-law distribution density of the signal. Here

$$\alpha_H^r = \frac{\alpha_H}{1 + \text{erf}(K)}. \tag{40}$$

The restriction of motion of τ_k by the reflective boundary condition at $\tau_k=0$ reduces the effective (average) value of $P_k(0)=\frac{1}{2}[P_k(\tau_k\to +0)+P_k(\tau_k\to -0)]$ in Eq. (1) and, consequently, the power spectral density approximately 2 times in comparison with the theoretical result (36) obtained without the restriction, because $P_k(\tau_k\to -0)=0$ for the restricted motion. More exactly, in such a case the power spectral density may be expressed by Eq. (36) with α_H^r instead of α_H—i.e.,

$$S_r(f) = \bar{I}^2 \frac{\alpha_H^r}{f}. \tag{41}$$

C. Correlation function of the point process

The correlation function $C(s)$ of the point process (3) may be expressed as

$$C(s) = \left\langle \frac{\bar{a}^2}{T} \sum_{k,q} \delta(t_{k+q} - t_k - s) \right\rangle = \bar{I}\bar{a} \sum_q \int_{-\infty}^{+\infty} \Psi_q(\Delta(q)) \delta(\Delta(q) - s) d\Delta(q) = \bar{I}\bar{a} \sum_q \Psi_q(s), \tag{42}$$

where the brackets $\langle \cdots \rangle$ denote averaging over the realizations of the process and over time (index k) as well. Such averaging coincides with averaging over the distribution of the time difference $\Delta(q)$, $\Psi_q(\Delta(q))$.

From Eq. (42) for the approximation

$$\Delta(k;q) \equiv t_{k+q} - t_k = \sum_{l=k+1}^{k+q} \tau_l \simeq \tau(q) q, \quad q \geqslant 0, \tag{43}$$

we have an expression for the correlation function in the simplest approximation [10],

$$C(s) \simeq \bar{I}\bar{a} \sum_q \int_{\tau_{\min}}^{\tau_{\max}} P_k(\tau_k) \delta(\tau_k q - s) d\tau_k = \bar{I}\bar{a}\delta(s) + \bar{I}\bar{a} \sum_{q \neq 0} P_k\left(\frac{s}{q}\right) \frac{1}{|q|}. \tag{44}$$

Replacing the summation in Eq. (44) by the integration we have an approximate expression for the correlation function of the point processes (3) and (11) or (35),

$$C(s) \simeq \bar{I}\bar{a} \int_0^\infty P_k\left(\frac{s}{q}\right) \frac{dq}{q}, \quad s \geqslant 0, \quad C(-s) = C(s). \tag{45}$$

IV. SIGNAL AS A SUM OF UNCORRELATED COMPONENTS

As was already mentioned above, $1/f$ noise is often modeled as the sum of the Lorentzian spectra with the appropriate weights of a wide-range distribution of the relaxation times τ^{rel}. It should be noted that summation of the spectra is allowed only if the processes with different relaxation times are isolated one from another [6,40]. For the construction of the signal $I(t)$ with $1/f$ noise spectrum from the stochastic equations with a wide-range distribution of the relaxation times (and rates $\gamma_l = 1/\tau_l^{rel}$) one should express the signal as a sum of N uncorrelated components [9],

$$I(t) = \sum_{l=1}^{N} I_l(t), \tag{46}$$

where every component I_l satisfies the stochastic differential equation

$$\dot{I}_l = -\gamma_l (I_l - \bar{I}_l) + \sigma_l \xi_l(t). \tag{47}$$

Here $\bar{I}_l$ is the average value of the signal component I_l, $\xi_l(t)$ is the δ-correlated white noise, $\langle \xi_l(t) \xi_{l'}(t') \rangle = \delta_{l,l'} \delta(t-t')$, and σ_l is the intensity (standard deviation) of the white noise.

The distribution density $P(I_l)$ of the component I_l is Gaussian,

$$P(I_l) = \sqrt{\frac{\gamma_l}{\pi}} \frac{1}{\sigma_l} \exp\left\{ -\frac{\gamma_l}{\sigma_l^2} (I_l - \bar{I}_l)^2 \right\}. \tag{48}$$

The distribution density $P(I)$ of the signal $I(t)$, Eq. (46), expressed as a sum of uncorrelated Gaussian components, is Gaussian as well,

$$P(I) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(I - \bar{I})^2}{2\sigma^2} \right\}, \tag{49}$$

with the average value $\bar{I}$ and the variance σ^2 expressed as

$$\bar{I} = \sum_l \bar{I}_l, \quad \sigma^2 = \sum_l \frac{\sigma_l^2}{2\gamma_l}. \tag{50}$$

Therefore, the Bernamont-Surdin-McWhorter model based on the sum of signals with a wide range distribution of the relaxation times always results in a Gaussian distribution of the signal intensity. However, not all signals exhibiting $1/f$ noise are Gaussian [2]. Some of them are non-Gaussian, exhibiting a power-law distribution or even fractal [3,30–35].

Equations (46) and (47) result in an expression for the correlation function of the signal (46),

$$C(s) = \sum_l \frac{\sigma_l^2}{2\gamma_l} e^{-\gamma_l s}, \quad s \geqslant 0. \tag{51}$$

The correlation function (51) yields the power spectrum

$$S(f) = \sum_l \frac{2\sigma_l^2}{\gamma_l^2 + \omega^2}, \quad \omega = 2\pi f. \tag{52}$$

Introducing the distribution of the relaxation rates, $g(\gamma)$, we can replace the summation in Eqs. (46) and (50)–(52) by the integration and express the power spectrum of the signal (46) as

$$S(f) = \int_{\gamma_{\min}}^{\gamma_{\max}} \frac{2\sigma^2(\gamma) g(\gamma)}{\gamma^2 + \omega^2} d\gamma = \frac{1}{\pi f} \int_{y_{\min}}^{y_{\max}} \frac{\sigma^2(\omega y) g(\omega y)}{1 + y^2} dy. \tag{53}$$

Here $\gamma_{\min}$ and $\gamma_{\max}$ are minimal and maximal values of the relaxation rate, respectively.

A. Signals with the pure 1/f power spectrum

Equation (53) yields the pure $1/f$ power spectrum only in the case when $\sigma^2(\omega y) g(\omega y) = A = \text{const}$. In such a case the correlation function (51) may be expressed as

$$C(s) = \frac{A}{2} \int_{\gamma_{\min}}^{\gamma_{\max}} e^{-\gamma s} \frac{d\gamma}{\gamma} = \frac{A}{2} \int_{\tau_{\min}^{rel}}^{\tau_{\max}^{rel}} e^{-s/\tau^{rel}} \frac{d\tau^{rel}}{\tau^{rel}}, \tag{54}$$

while the power spectrum (53) yields

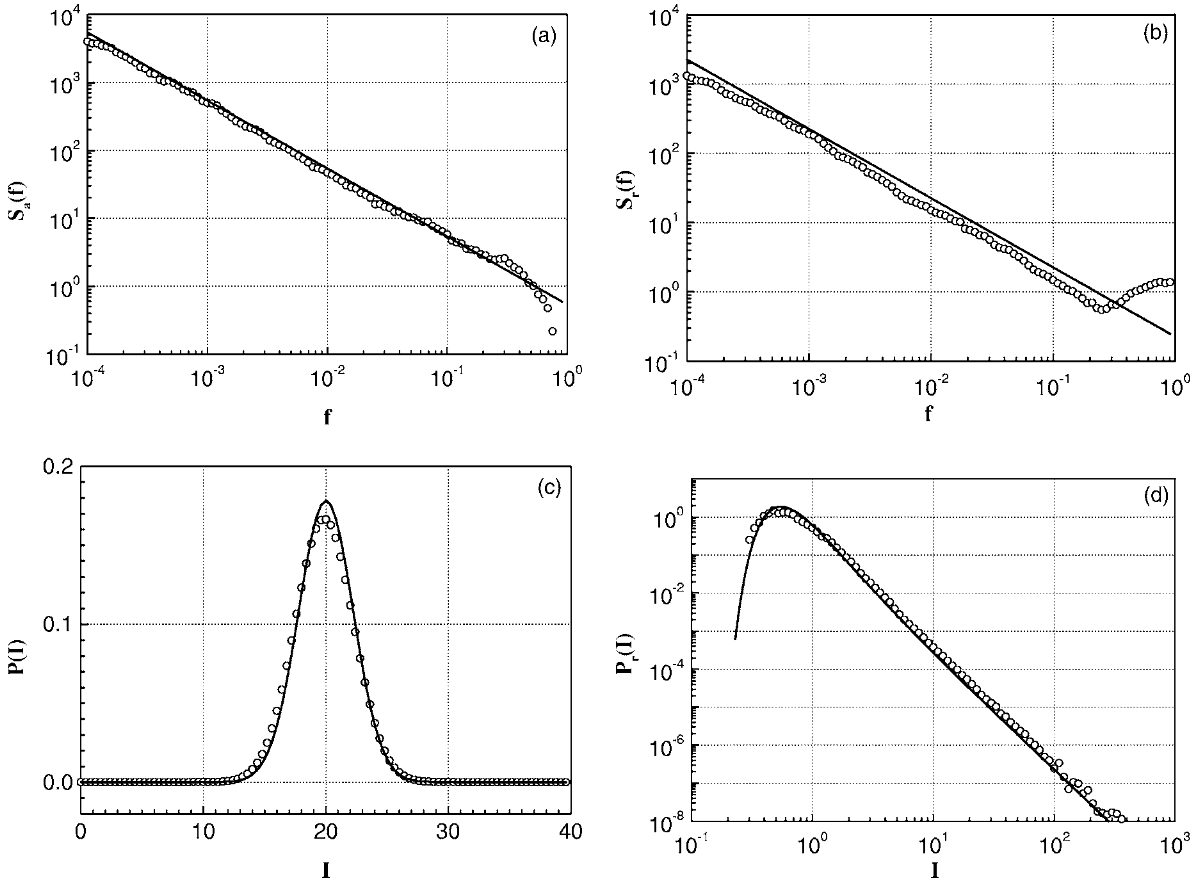

FIG. 3. Power spectra: (a) numerically calculated for the average signal (56) from $N=10$ components (47) with $\bar{I}=20$, $\sigma_l^2(\gamma_l)g(\gamma_l)=\mathrm{const}$, and uniform distribution of lg γ_l with γ_l values in the interval 10^{-4}–10^0—i.e., with $g(\gamma_l)\sim\gamma_l^{-1}$, $\sigma_l^2(\gamma_l)\sim\gamma_l$, and $\sigma_1(\gamma_1)=0.1$ (open circles) in comparison with theoretical results (58) (straight line); (b) for the point process (3), (4), and (35), with $\bar{a}=1$, $\bar{\tau}=1$, $\sigma=0.01$, and $\gamma=0.0001$ averaged over 10 realizations of 10^5 pulse sequences (open circles) in comparison with the theoretical results according to Eq. (41) (straight line). (c) and (d) Numerically calculated distribution densities of the corresponding signals (open circles) in comparison with the theoretical results (49), (57), and (38) (solid lines), respectively.

$$S(f)=\frac{A}{\pi f}\left[\arctan\left(\frac{\gamma_{\max}}{\omega}\right)-\arctan\left(\frac{\gamma_{\min}}{\omega}\right)\right]\simeq\frac{A}{2f},$$
$$\gamma_{\min}\ll\omega\ll\gamma_{\max}. \tag{55}$$

For the signal expressed not as a sum (46) but as an average of N uncorrelated components,

$$I_a(t)=\frac{1}{N}\sum_{l=1}^{N}I_l(t), \tag{56}$$

all characteristics (48)–(55) are similar, except that the average value $\bar{I}_a$ of the averaged signal (56) is N times smaller than that according to Eq. (50), while the expressions for the correlation function $C(s)$, Eqs. (51) and (54), for the power spectrum $S(f)$, Eqs. (52), (53), and (55), and for the variance σ_a^2, Eq. (50), should be divided by N^2—i.e.,

$$\bar{I}_a=\frac{1}{N}\sum_l\bar{I}_l,\quad \sigma_a^2=\frac{1}{N^2}\sum_l\frac{\sigma_l^2}{2\gamma_l}, \tag{57}$$

$$S_a(f)\simeq\frac{A}{2N^2f}, \tag{58}$$

$$C_a(s)=\frac{1}{2N^2}\int_{\gamma_{\min}}^{\gamma_{\max}}\frac{e^{-\gamma s}}{\gamma}\sigma^2(\gamma)g(\gamma)d\gamma. \tag{59}$$

When replacing the summation in Eqs. (46), (50)–(53), and (56)–(59) by the integration, the distribution density of the relaxation rates, $g(\gamma)$, should be normalized to the number of uncorrelated components N,

$$\int_{\gamma_{\min}}^{\gamma_{\max}}g(\gamma)d\gamma=N. \tag{60}$$

We see the similarity of expressions (45) and (59) for the correlation function of the point process model and that of the sum of signals with different relaxation rates, respectively. In general, however, different distributions $P_k(\tau_k)$ of the interpulse time τ_k when $P_k(0)\neq 0$—e.g., exponential, Gaussian, and continuous distributions—with the slowly fluctuating interpulse time τ_k may result in $1/f$ noise. There-

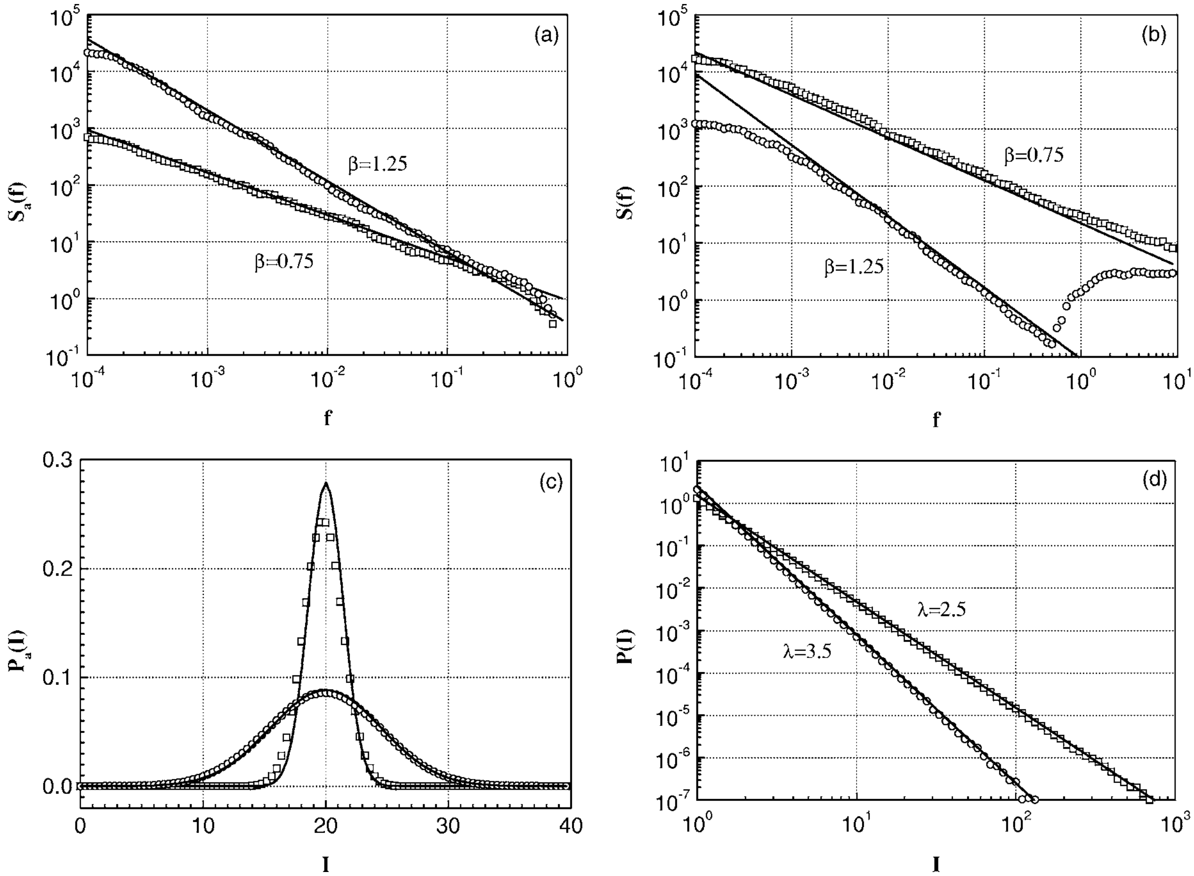

FIG. 4. Power spectra: (a) numerically calculated for signal (47), (56), and (61) from ten components with $\bar{I}=20$, $A=100$, and $\eta=-0.25$ (open circles) and $\eta=0.25$ (open squares), in comparison with theoretical results (64) (straight line); (b) for the point process (3), (4), and (26) with the parameters $\bar{a}=1$, $\mu=0.5$, $\sigma=0.02$, and $\gamma=0.0001$ (open squares) and $\gamma=0.0003$ (open circles) averaged over 10 realizations of 10^6 pulse sequences in comparison with the theoretical results (28) (straight lines). (c) and (d) Numerically calculated distribution densities of the corresponding signals in comparison with the theoretical results (65), (66), and (34), respectively (solid lines).

fore, the point process model is, in some sense, more general than the model based on the sum of the Lorentzian spectra.

In Fig. 3 examples of the pure $1/f$ power spectra for the average (56) of signals (47) generated for different relaxation rates γ_l and with the corresponding intensities of the white noise σ_l^2 and those of the autoregressive point process (3), (4), and (35), are presented together with the distribution densities of the corresponding signals. We see the similarity of the spectra but very different distributions of the intensity of the signals: the signal of the sum of the Lorentzians is Gaussian while that of the point process is approximately of power law type, asymptotically $P(I)\sim I^{-3}$.

B. Signals with the power spectral density of different slopes β

Using the sum of different Lorentzian signals we can generate not only a signal with the pure $1/f$ spectrum but the signal with any predefined slope β of $1/f^{\beta}$ power spectral density, as well. Indeed, let us investigate the case when

$$\sigma^2(\gamma)g(\gamma) = A\gamma^{\eta}, \tag{61}$$

where A and η are some parameters. Substitution of Eq. (61) into Eq. (53) yields the power spectral density

$$S(f) = \frac{A}{\pi f}\int_{\gamma_{\min}/\omega}^{\gamma_{\max}/\omega} \frac{(\omega y)^{\eta}}{1+y^2}dy = \frac{A}{\omega^{1-\eta}}\left\{\left[\frac{\gamma_{\max}}{\omega}\right]^{\eta+1}\Phi\left(-\left[\frac{\gamma_{\max}}{\omega}\right]^2,1,\frac{\eta+1}{2}\right) - \left[\frac{\gamma_{\min}}{\omega}\right]^{\eta+1}\Phi\left(-\left[\frac{\gamma_{\min}}{\omega}\right]^2,1,\frac{\eta+1}{2}\right)\right\}, \tag{62}$$

where $\Phi(z,s,a)$ is a Lerch's phi transcendent. In the limit when $\gamma_{\min}\to 0$ and $\gamma_{\max}\to\infty$ we can approximate the power spectral density (62) as

$$S(f) \simeq \frac{(2\pi)^{\eta}A}{2\cos(\pi\eta/2)}\frac{1}{f^{1-\eta}}; \tag{63}$$

i.e., we have the generalization of the result (55).

For the average signal (56) we have

$$S_a(f) \simeq \frac{(2\pi)^{\eta}A}{2N^2\cos(\pi\eta/2)}\frac{1}{f^{1-\eta}}. \tag{64}$$

In order to obtain an arbitrary β of the $1/f^{\beta}$ power spectral density we should choose in Eq. (61) $\eta=1-\beta$.

The distribution density $P_a(I_a)$ of the average signal $I_a(t)$ is Gaussian

$$P_a(I_a) = \frac{1}{\sqrt{2\pi}\sigma_a} e^{-(I-\bar{I}_a)^2/2\sigma_a^2}, \qquad (65)$$

with the variance σ_a^2 expressed as

$$\sigma_a^2 = \frac{1}{2N^2}\int_{\gamma_{\min}}^{\gamma_{\max}} \frac{\sigma^2(\gamma)g(\gamma)}{\gamma} d\gamma = \frac{A(\gamma_{\max}^{\eta} - \gamma_{\min}^{\eta})}{2N^2\eta}. \qquad (66)$$

The correlation function in such a case according to Eq. (59) is

$$C_a(s) = \frac{A}{2N^2}\int_{\gamma_{\min}}^{\gamma_{\max}} e^{-\gamma s}\gamma^{\eta-1} d\gamma$$

$$= \frac{A}{2N^2 s^{\eta}}[\Gamma(\eta,\gamma_{\min}s) - \Gamma(\eta,\gamma_{\max}s)], \qquad (67)$$

where $\Gamma(a,z)$ is the incomplete gamma function.

Figure 4 demonstrates the possibility to generate stochastic signals exhibiting similar $1/f^{\beta}$ power spectral densities with different slopes β by summation of signals with different relaxation rates and according to the multiplicative point process model. The distribution densities of the corresponding signals are, however, completely different.

V. CONCLUSIONS

The generalized multiplicative point processes (3), (11), (18), and (26), may generate time series exhibiting the power spectral density $S(f) \sim 1/f^{\beta}$ with $0.5 \lesssim \beta \lesssim 2$, Eqs. (17), (23), and (28)—i.e., with the slope observable in a large variety of systems. Such a spectral density is caused by the stochastic diffusion of the interpulse time, resulting in a power-law distribution. The power-law distribution of the interpulse, inter-event, interarrival, recurrence, or waiting times is observed in different systems from physics, astronomy, and seismology to the Internet, financial markets, neural spikes, and human cognition.

Furthermore, the power-law distribution of the interpulse time results in a power-law distribution of the stochastic signal, $P(I) \sim I^{-\lambda}$, with $2 \lesssim \lambda \lesssim 4$—i.e., the phenomenon observable in a large variety of processes, from earthquakes to the financial time series—as well. The proposed model relates and connects the power-law autocorrelation and spectral density with the power-law distribution of the signal intensity into a consistent theoretical approach. The generated time series of the model are fractal since they exhibit jointly the power-law probability distribution and the power-law autocorrelation of the signal.

In addition, we have analyzed the relation of the point process model with the Bernamont-Surdin-McWhorter model of $1/f$ noise, representing the signal as a sum of the appropriate signals with different rates of the linear relaxation. From the performed analysis we can conclude that the multiplicative point process model of $1/f$ noise when the signal consisting of pulses with a stochastic motion of the interpulse time is more general and complementary to the model based on the sum of signals with a wide-range distribution of the relaxation times. In contrast to the Gaussian distribution of the intensity of sum of the uncorrelated components, the point process model generating $1/f$ noise exhibits power-law distribution of the intensity of the signal. Moreover, it is free from the requirement of a wide-range distribution of the relaxation times. Obviously, the multiplicative point process model of $1/f^{\beta}$ noise may be used for modeling and analysis of stochastic processes in different systems exhibiting pulsing signals.

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[1] J. B. Johnson, Phys. Rev. **26**, 71 (1925); W. Schottky, *ibid.* **28**, 74 (1926); D. W. Allan, Proc. IEEE **54**, 221 (1966); C. W. J. Granger, Econometrica **34**, 150 (1966); R. F. Voss and J. Clarke, Nature (London) **258**, 317 (1975); W. H. Press, Comments. Astrophys. **7**, 103 (1978); M. Gardner, Sci. Am. **238**(4), 16 (1978); A. Lawrence, M. G. Watson, K. A. Pounds, and M. Elvis, Nature (London) **325**, 694 (1987).

[2] F. N. Hooge, T. G. M. Kleinpenning, and L. K. J. Vadamme, Rep. Prog. Phys. **44**, 479 (1981); P. Dutta and P. M. Horn, Rev. Mod. Phys. **53**, 497 (1981); S. M. Kogan, Usp. Fiz. Nauk **145**, 285 (1985) [Sov. Phys. Usp. **28**, 170 (1985)]; M. B. Weissman, Rev. Mod. Phys. **60**, 537 (1988); C. M. Van Vliet, Solid-State Electron. **34**, 1 (1991); G. P. Zhigal'skii, Usp. Fiz. Nauk **167**, 623 (1997) [Phys. Usp. **40**, 599 (1997)]; E. Milotti, e-print physics/0204033; H. Wong, Microelectron. Reliab. **43**, 585 (2003).

[3] S. Thurner, S. B. Lowen, M. C. Feurstein, C. Heneghan, H. G. Feichtinger, and M. C. Teich, Fractals **5**, 565 (1997).

[4] D. L. Gilden, T. Thornton, and M. W. Mallon, Science **267**, 1837 (1995).

[5] B. Kaulakys and J. Ruseckas, Phys. Rev. E **70**, 020101(R) (2004).

[6] J. Bernamont, Ann. Phys. (Leipzig) **7**, 71 (1937); M. Surdin, J. Phys. Radium **10**, 188 (1939); F. K. du Pré, Phys. Rev. **78**, 615 (1950); A. Van der Ziel, Physica (Amsterdam) **16**, 359 (1950); A. L. McWhorter, in *Semiconductor Surface Physics*, edited by R. H. Kingston (University of Pennsylvania, Philadelphia, 1957), p. 207; V. Palenskis, Litov. Fiz. Sb. **30**, 107 (1990) [Lith. Phys. J. **30**(2), 1 (1990)]; F. N. Hooge, in *Proceedings of the 14th International Conference on Noise in Physical Systems and* 1/*f Fluctuations*, Leuven, Belgium, 1997, edited by C. Claeys and E. Simoen (World Scientific, Singapore, 1997), p. 3.

[7] B. Kaulakys, e-print adap-org/9806004.

[8] B. Kaulakys and T. Meškauskas, Phys. Rev. E **58**, 7013 (1998); B. Kaulakys, Phys. Lett. A **257**, 37 (1999).
[9] B. Kaulakys, Lith. J. Phys. **40**, 281 (2000).
[10] B. Kaulakys and T. Meškauskas, Nonlin. Anal.: Mod. Contr. (Vilnius) **4**, 87 (1999); B. Kaulakys and T. Meškauskas, Microelectron. Reliab. **40**, 1781 (2000); B. Kaulakys, *ibid.* **40**, 1787 (2000).
[11] K. L. Schick and A. A. Verveen, Nature (London) **251**, 599 (1974); F. Gruneis, Physica A **123**, 149 (1984); F. Gruneis and H.-J. Baiter, *ibid.* **136**, 432 (1986); F. Gruneis and T. Musha, Jpn. J. Appl. Phys., Part 1 **25**, 1504 (1986); T. Musha, *ibid.* **26**, 2022 (1987); F. Gruneis, M. Nakao, and M. Yamamoto, Biol. Cybern. **62**, 407 (1990); F. Gruneis, M. Nakao, Y. Mizutani, M. Yamamoto, M. Meesmann, and T. Musha, *ibid.* **68**, 193 (1993); F. Gruneis, Fluct. Noise Lett. **1**, R119 (2001); **4**, L413 (2004).
[12] H. R. Bittner, P. Tosi, C. Braun, M. Meesmann, and K. D. Kniffki, Geol. Rundsch. **85**, 110 (1996); R. Console and M. Murru, J. Geophys. Res. **106**, 8699 (2001); L. Telesca, V. Cuomo, and V. Lapenna, Fluct. Noise Lett. **2**, L357 (2002); J. M. Halley and P. Inchausti, *ibid.* **4**, R1 (2004); M. Lindman, K. Jonsdottir, R. Roberts, B. Lund, and R. Bodvarsson, Phys. Rev. Lett. **94**, 108501 (2005).
[13] T. Musha and H. Higuchi, Jpn. J. Appl. Phys. **15**, 1271 (1976); X. Zhang and G. Hu, Phys. Rev. E **52**, 4664 (1995); M. Y. Choi and H. Y. Lee, *ibid.* **52**, 5979 (1995).
[14] I. Csabai, J. Phys. A **27**, L417 (1994); W. Willinger, M. S. Taqqu, R. Sherman, and D. V. Wilson, IEEE/ACM Trans. Netw. **5**, 71 (1997); I. Kaj and I. Marsh, in *Quality of Service in Multiservice IP Networks*, edited by M. Ajmone Marsan, G. Corazza, M. Listanti, and A. Roveri, Lecture Notes in Computer Science Vol. 2601 (Springer-Verlag, Heidelberg, 2003), p. 35; A. J. Field, U. Harder, and P. G. Harrison, IEE Proc.-Commun. **151**, 355 (2004); S. Cohen, M. S. Taqqu, Methodol. Comput. Appl. Probab. **6**, 363 (2004); L. A. N. Amaral, A. Diaz-Guilera, A. A. Moreira, A. L. Goldberger, and L. A. Lipsitz, Proc. Natl. Acad. Sci. U.S.A. **101**, 15 551 (2004).
[15] J. Davidsen and H. G. Schuster, Phys. Rev. E **62**, 6111 (2000); J. Davidsen and M. Paczuski, *ibid.* **66**, 050101(R) (2002); Y. Kamitani, H. Suyari, and I. Matsuba, Electron. Commun. Jpn., Part 2: Electron. **87**, 72 (2004).
[16] J. I. Yano, K. Fraedrich, and R. Blender, J. Clim. **14**, 3608 (2001); W. Muller, R. Blender, and K. Fraedrich, Nonlinear Processes Geophys. **9**, 37 (2002); W. W. Tung, M. W. Moncrieff, and J. B. Gao, J. Clim. **17**, 2736 (2004); J. I. Yano, R. Blender, C. D. Zhang, and K. Fraedrich, Q. J. R. Meteorol. Soc. **130**, 1697 (2004); K. Fraedrich, U. Luksch, and R. Blender, Phys. Rev. E **70**, 037301 (2004).
[17] C. Guidorzi, F. Frontera, E. Montanari, M. Feroci, L. Amati, E. Costa, and M. Orlandini, Astron. Astrophys. **416**, 297 (2004).
[18] M. Agu, H. Akabane, and T. Saito, J. Phys. Soc. Jpn. **70**, 2798 (2001).
[19] M. Celasco and R. Eggenhoffner, Eur. Phys. J. B **23**, 415 (2001).
[20] J. Davidsen and H. G. Schuster, Phys. Rev. E **65**, 026120 (2002).
[21] V. Gontis, Lith. J. Phys. **41**, 551 (2001); e-print cond-mat/0201514; V. Gontis and B. Kaulakys, in *Proceeding of the 17th International Conference on Noise and Fluctuations Prague, 2003*, edited by J. Sikula (Czech Noise Research Laboratory, Brno University of Technology, Brno, Czech Republic, 2003), p. 622; V. Gontis, B. Kaulakys, M. Alaburda, and J. Ruseckas, Solid State Phenom. **97–98**, 65 (2004); V. Gontis and B. Kaulakys, Physica A **343**, 505 (2004); **344**, 128 (2004).
[22] A. Coza and V. V. Morariu, Physica A **320**, 449 (2003); E.-J. Wagenmakers, S. Farrell, and R. Ratcliff, Psychonom. Bull. Rev. **11**, 579 (2004).
[23] J. B. Gao, Med. Biol. Eng. Comput. **42**, 345 (2004).
[24] D. Delignieres, L. Lemoine, and K. Torre, Hum. Mov. Sci. **23**, 87 (2004).
[25] M. B. Priestley, *Spectral Analysis and Time Series* (Academic, San Diego, 1981).
[26] J. Timmer and M. Konig, Astron. Astrophys. **300**, 707 (1995).
[27] S. L. Marpl,Jr., *Digital Spectral Analysis with Applications* (Prentice-Hall, Englewood Cliffs, NJ, 1987).
[28] P. Stoica and R. L. Moses, *Introduction to Spectral Analysis* (Prentice-Hall, Englewood Cliff, NJ, 1997).
[29] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences* (Springer-Verlag, Berlin, 1985).
[30] B. B. Mandelbrot, J. Business **36**, 394 (1963); I. McHardy and B. Czerny, Nature (London) **325**, 696 (1987); G. W. Wornell, Proc. IEEE **81**, 1428 (1993); H. Scher, G. Margolin, R. Metzler, J. Klafter, and B. Berkowitz, Geophys. Res. Lett. **29**, 1061 (2002); P. Bak, K. Christensen, L. Danon, and T. Scanlon, Phys. Rev. Lett. **88**, 178501 (2002); L. Giraitis, P. Kokoszka, R. Leipus, and G. Teyssiere, Acta Appl. Math. **78**, 285 (2003).
[31] B. B. Mandelbrot, *Fractals and Scaling in Finance* (Springer, New York, 1997).
[32] B. B. Mandelbrot, *Multifractals and* $1/f$ *Noise: Wild Self-Affinity in Physics* (Springer, New York, 1999).
[33] R. N. Mantegna and H. E. Stanley, *An Introduction to Econophysics: Correlations and Complexity in Finance* (Cambridge University Press, Cambridge, U.K., 2000).
[34] *Handbook of Heavy Tailed Distributions in Finance*, edited by S. T. Rachev (Elsevier B. V., Amsterdam, 2003).
[35] X. Gabaix, P. Gopikrishnan, V. Plerou, and E. Stanley, Nature (London) **423**, 267 (2003).
[36] S. B. Lowen and M. C. Teich, IEEE Trans. Inf. Theory **39**, 1669 (1993); M. Usher, M. Stemmler, and Z. Olami, Phys. Rev. Lett. **74**, 326 (1995); M. Takayasu, H. Takayasu, and T. Sato, Physica A **233**, 824 (1996); M. S. Wheatland, Astrophys. J. Lett. **536**, L109 (2000); J. W. Kirchner, X. Feng, and C. Neal, Nature (London) **403**, 524 (2000); E. Scalas, Physica A **287**, 468 (2000); F. Lepreti, V. Carbone, and P. Veltri, Astrophys. J. **555**, L133 (2001); P. Richmond, Eur. Phys. J. B **20**, 523 (2001); L. Sabatelli, S. Keating, J. Dudley, and P. Richmond, *ibid.* **27**, 273 (2002); A. Erramilli, M. Roughan, D. Veitch, and W. Willinger, Proc. IEEE **90**, 800 (2002); S. Abe and N. Suzuki, Europhys. Lett. **61**, 852 (2003); P. L. Conti, A. Lijoi, and F. Ruggeri, Appl. Stochastic Models Bus. Ind. **20**, 305 (2004); F. Selcuk, Physica A **333**, 306 (2004); T. Kaizoji and M. Kaizoji, *ibid.* **336**, 563 (2004); A. Corral, Phys. Rev. Lett. **92**, 108501 (2004); A.-H. Sato, Phys. Rev. E **69**, 047101 (2004); M. E. J. Newman, e-print cond-mat/0412004, Contemp. Phys. (to be published); S. M. D. Queiros, C. Anteneodo, and C. Tsallis, e-print physics/0503024.
[37] V. Gontis, Nonlin. Anal.: Mod. Contr. (Vilnius) **7**, 43 (2002); e-print cond-mat/0211317.

[38] A.-H. Sato, H. Takayasu, and Y. Sawada, Phys. Rev. E **61**, 1081 (2000); Z. F. Huang and S. Solomon, Physica A **306**, 412 (2002).

[39] P. Gopikrishnan, M. Meyer, L. A. N. Amaral, and H. E. Stanley, Eur. Phys. J. B **3**, 139 (1998); W. J. Reed and B. D. Hughes, Phys. Rev. E **66**, 067103 (2002); N. Scafetta and B. J. West, Phys. Rev. Lett. **92**, 138501 (2004); C. B. Yang, J. Phys. A **37**, L523 (2004); L. A. N. Amaral, D. J. B. Soares, L. R. da Silva, L. S. Lucena, M. Saito, H. Kumano, N. Aoyagi, and Y. Yamamoto, Europhys. Lett. **66**, 448 (2004); A. Fujihara, T. Ohtsuki, and H. Yamamoto, Phys. Rev. E **70**, 031106 (2004).

[40] F. N. Hooge, IEEE Trans. Electron Devices **41**, 1926 (1994); F. N. Hooge, Physica B **239**, 223 (1997).