Reciprocity relations between ordinary temperature and the Frieden-Soffer Fisher temperature

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Frieden and Soffer conjectured some years ago [Phys. Rev. E **52**, 2274 (1995)] the existence of a "Fisher temperature" T_F that would play, with regards to Fisher's information measure I, the same role that the ordinary temperature T plays in relation to Shannon's logarithmic measure. Here we exhibit the existence of reciprocity relations between T_F and T and provide an interpretation with reference to the meaning of T_F for the canonical ensemble.

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I. WHY A FISHER TEMPERATURE?

Frieden and Soffer conjectured some years ago [1,2] the existence of a "Fisher temperature" T_F that would play, with regards to Fisher's information measure I, the same role that the ordinary temperature T plays in relation to Shannon's logarithmic measure S [3,4]. In a series of more recent publications, this conjecture was amply validated by showing that the Legendre transform structure of thermodynamics can be replicated without changes if ones substitutes I for the Shannon entropy S [5–8], which yields then a "Fisher thermodynamics."

This Fisher thermodynamics is exactly equivalent to the conventional one, except that instead of the Shannon-Boltzmann-Gibbs (SBG) entropy S one uses Fisher's I [5–8]. A question still lingers, though: we have a SBG pair (S,T) and a Fisher pair (I,T_F). What is the relation between T and T_F?

In other words, in parallel to $(1/T)=dS/dU$ (U is the mean energy) [9], we have $(1/T_F)=dI/dU$ [5–8]. We need a thermometer to measure T_F, and this is best achieved by finding a relationship between the two temperatures. In this Brief Report we purport to provide a first answer with respects to the relation between T and T_F.

II. BRIEF FISHER CONSIDERATIONS

Estimation theory [2] provides one with a powerful result with reference to a system that is specified by a physical parameter θ. Let $\mathbf{x}$ be a stochastic variable and $p_\theta(\mathbf{x})$ the probability density for this variable, which depends on the parameter θ. If an observer (i) makes a measurement of $\mathbf{x}$ and wishes to best infer θ from this measurement, calling the resulting estimate $\tilde{\theta}=\tilde{\theta}(\mathbf{x})$ and (ii) wonders how well θ can be determined, then estimation theory asserts [2] that the best possible estimator $\tilde{\theta}(\mathbf{x})$, after a very large number of $\mathbf{x}$ samples is examined, suffers a mean-square error e^2 from θ, which obeys a relationship involving Fisher's I—namely, $Ie^2=1$—where the Fisher information measure (FIM) I is of the form

$$I=\left\langle\left(\frac{\partial \ln p_\theta}{\partial\theta}\right)^2\right\rangle. \tag{1}$$

The FIM is additive [2]. If we have n independent parameters θ_i, Eq. (1) becomes a sum of n terms of the form given above [2]. The "best" estimator is called the *efficient* estimator. Any other estimator must have a larger mean-square error. The only proviso to the above result is that all estimators be unbiased—i.e., satisfy $\langle\tilde{\theta}(\mathbf{x})\rangle=\theta$. Thus, Fisher's information measure has a lower bound, in the sense that, no matter what parameter of the system we choose to measure, I has to be larger or equal than the inverse of the mean-square error associated with the concomitant experiment. This result, i.e.,

$$Ie^2 \geqslant 1, \tag{2}$$

is referred to as the Cramer-Rao bound and constitutes a very powerful statistical result [2].

III. FORMALISM

We start by defining the well-known density operator that describes a system at equilibrium [3,4]:

$$\hat{\rho}=\frac{1}{Z}\exp\left(-\sum_{i=1}^{M}\chi_i\hat{A}_i\right). \tag{3}$$

The χ_i are Lagrangian multipliers associated to the M observables $\hat{A}_i$, whose expectation values are given by

$$\langle\hat{A}_i\rangle=\mathrm{Tr}\,\hat{\rho}\hat{A}_i \quad (i=1,\ldots,M), \tag{4}$$

where the partition function Z has the form $Z(\chi_i)=\mathrm{Tr}[\exp(-\sum_{i=1}^{M}\chi_i\hat{A}_i)]$ [10]. In our present considerations we assume that these multipliers *have already been determined*.

Following Mandelbrot [11–13] we (i) *associate* the above Lagrange multipliers to parameters to be estimated via *Fisher considerations* involving a FIM that depends upon $\hat{\rho}$ and (ii) write down this FIM as a sum of M terms, each one associated to the estimation of the parameter χ_i, i.e.,

$$I=\sum_{i=1}^{M}\Gamma_i\left\langle\left(\frac{\partial\ln\hat{\rho}}{\partial\chi_i}\right)^2\right\rangle, \tag{5}$$

where the Γ_i are suitable constants related to the (conventional) wish of having a dimensionless I, as discussed in [14,15]. After replacing Eq. (3) into Eq. (5) we then find that I is intimately connected to our observables' fluctuations, as pointed out long ago by Mandelbrot [11,16]:

$$I = \sum_{i=1}^{M} \Gamma_i \langle (\hat{A}_i - \langle \hat{A}_i \rangle)^2 \rangle. \quad (6)$$

If we wish to have a dimensionless I, Γ_i has the dimension of $[1/\dim(\hat{A}_i)^2]$. Now, it is well known (and straightforwardly verified) that the statistical fluctuations of an observable obey the relation [11,16]

$$\langle (\hat{A}_i - \langle \hat{A}_i \rangle)^2 \rangle = - \frac{\partial \langle \hat{A}_i \rangle}{\partial \chi_i} \quad (7)$$

(the χ_i have dimension of $[1/\dim(\hat{A}_i)]$, which allows us to recast the Fisher measure in the fashion

$$I = - \sum_{i=1}^{M} \Gamma_i \frac{\partial \langle \hat{A}_i \rangle}{\partial \chi_i}. \quad (8)$$

IV. EXTREMIZATION OF *I* SUBJECT TO CONSTRAINTS

As stated above, the thermodynamics Legendre structure can be neatly re-obtained if one extremizes FIM subject to constraints instead of doing this using the Boltzmann entropy [5–8]. We deal then with the same mean values $\langle \hat{A}_i \rangle$ used above, but, of course, different Lagrange multipliers will ensue. Let us call these new Fisher multipliers γ_i and borrow from the well-known thermodynamic relation that links information measure, Lagrange multipliers (here the Fisher ones), and expectation values [3,5]:

$$\gamma_i = \frac{\partial I}{\partial \langle \hat{A}_i \rangle}. \quad (9)$$

It is now clear that, introducing the above result into Eq. (8), we get an expression for the Fisher multipliers γ_i in terms of the Shannon ones (χ_j):

$$\gamma_i = - \sum_{j=1}^{M} \Gamma_j \frac{\partial}{\partial \langle \hat{A}_i \rangle} \frac{\partial \langle \hat{A}_j \rangle}{\partial \chi_j}; \quad \dim(\gamma_i) = \dim(1/\hat{A}_i), \quad (10)$$

a relation which could be used to determine them. It might seem at this point natural to ask what happens if we consider a canonical distribution in which the Lagrange multipliers are the γ_i instead of the χ_i. We discuss this question below for classical systems within the strictures of the canonical ensemble.

V. EQUIPARTITION THEOREM

In classical statistical mechanics there exists a useful general result concerning the energy E of a system expressed as a function of $2N$ generalized coordinates ξ_i (for instance, N coordinates r_i and N momenta p_i). Thus, $E = E(\xi_1, \ldots, \xi_{2N})$. The result holds in the case of the following (frequent) occurrence.

(i) The energy splits additively into the form $E = \epsilon_i(\xi_i) + E'(\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_{2N})$, where $\epsilon_i(\xi_i)$ involves only the variable ξ_i and the remaining part E' does not depend on ξ_i.

(ii) The function $\epsilon_i(\xi_i)$ is quadratic in ξ_i.

In these circumstances $\langle \epsilon_i \rangle = kT/2$, with k Boltzmann's constant and T the temperature. This is the equipartition theorem [9]. The mean value of each independent quadratic term in the energy E equals $kT/2$, where $\beta = 1/kT$ is the (Shannon-Boltzmann-Gibbs) Lagrange multiplier associated with the the mean-energy constraint $\langle E \rangle = \int d\tau f E$. Its demonstration assumes that the thermal equilibrium Boltzmann-Gibbs equilibrium probability distribution

$$f = \frac{1}{Z} e^{-\beta E}, \quad (11)$$

with $d\tau$ the phase-space volume element. Setting $\Gamma = 1/k^2 T_0^2$, with T_0 an arbitrary but fixed reference temperature, yields a dimensionless Fisher information measure (8) for the canonical ensemble:

$$I = - \frac{1}{k^2 T_0^2} \frac{\partial \langle E \rangle}{\partial \beta}. \quad (12)$$

VI. RECIPROCITY

Since we assume equipartition, we immediately find [9]

$$\langle E \rangle = N \beta^{-1}, \quad (13)$$

implying

$$\frac{\partial \langle E \rangle}{\partial \beta} = - N \beta^{-2} = - \frac{\langle E \rangle^2}{N}, \quad (14)$$

entailing that, according to Eq. (10), the Fisher multiplier (defined as $\gamma = 1/kT_F$) is

$$\gamma = \frac{1}{kT_F} = - \frac{1}{k^2 T_0^2} \frac{\partial}{\partial \langle E \rangle} \frac{\partial \langle E \rangle}{\partial \beta} = \frac{2}{k^2 T_0^2 \beta}. \quad (15)$$

Since the multipliers are inverse temperatures, we obtain the interesting relationship

$$T_F = \frac{T_0^2}{2T}, \quad (16)$$

our main result here, which, on reflection, should not surprise anyone since it is a well-known fact that whenever I grows, Shannon's S decreases and vice versa [2]. Note that the Fisher information (12) adopts now the following appearance:

$$I = \frac{\langle E \rangle^2}{N k^2 T_0^2}, \quad \frac{\partial I}{\partial \langle E \rangle} = \frac{2}{k^2 T_0^2 \beta}, \quad (17)$$

where we have used equipartition ($\langle E \rangle = N/\beta$), leading to

$$\frac{1}{\beta} = \frac{k^2 T_0^2}{2} \frac{\partial I}{\partial \langle E \rangle}, \quad \langle E \rangle \equiv \langle E \rangle_\beta. \quad (18)$$

With reference to Eq. (15), let us introduce now the Fisher result $\gamma = 2/(k^2 T_0^2 \beta)$ as the multiplier entering the canonical probability distribution f in Eq. (11) and repeat the preceding discussion, starting with the relation that takes now the place of Eq. (13), here

$$\langle E\rangle_{\gamma} = N\gamma^{-1}. \tag{19}$$

One has

$$\frac{\partial\langle E\rangle_{\gamma}}{\partial\gamma} = -N\gamma^{-2} = -\frac{\langle E\rangle_{\gamma}^{2}}{N}. \tag{20}$$

We ask ourselves what is now the new Fisher multiplier γ_2. The answer is, using Eqs. (19) and (15),

$$\gamma_2 = -\frac{1}{k^2T_0^2}\frac{\partial}{\partial\langle E\rangle_{\gamma}}\frac{\partial\langle E\rangle_{\gamma}}{\partial\gamma} = \frac{2}{k^2T_0^2}\frac{\langle E\rangle_{\gamma}}{N} = \gamma\beta\frac{\langle E\rangle_{\gamma}}{N}, \tag{21}$$

i.e.,

$$\gamma_2 = \beta, \tag{22}$$

which is indeed consistent with Eq. (18). Here we encounter reciprocity. The "Fisher multiplier" γ_2 is the inverse Boltzmann-Gibbs-Shannon temperature that verifies [cf. Eq. (17)]

$$\frac{1}{kT} = \beta = \frac{\partial I(\gamma)}{\partial\langle E\rangle_{\gamma}},$$

$$\frac{1}{kT_F} = \gamma = \frac{\partial I(\beta)}{\partial\langle E\rangle_{\beta}}, \tag{23}$$

in self-explanatory notation. Equation (17) can be written in either the "β" language or in the "γ" one, indistinctly.

VII. CONCLUDING REMARKS

We have in this Brief Report provided two results that we deem important for the Fisher practitioners: namely, (a) $T_F = T_0^2/2T$, with T_0 an arbitrary but fixed reference Boltzmann temperature, and (b) the reciprocity relations given by Eq. (23).

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