Reversible soliton motion

A. J. Scroggie,* J. Jeffers, G. McCartney, and G.-L. Oppo
Department of Physics, University of Strathclyde, 107 Rottenrow, Glasgow G4 ONG, Scotland, United Kingdom
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We show that spatial solitons on either phase- or amplitude-modulated backgrounds can change their direction of motion according to the modulation frequency. A soliton may, therefore, move up or down phase gradients or remain motionless regardless of where it is in relation to the background modulation. The general theory is in good agreement with numerical results in a variety of nonlinear systems.

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I. INTRODUCTION

Spatially localized structures are of wide scientific interest in disciplines as diverse as photonics [1], hydrodynamics [2], Bose-Einstein condensation [3], chemical oscillations [4], and biological morphogenesis [5]. The Swift-Hohenberg, Ginzburg-Landau, nonlinear Schrödinger, and other prototype equations provide common settings for their investigation and characterization [6]. Problems related to soliton behavior in the presence of perturbations have also been pursued for a number of years (see, for example, [7,8]). In optics, studies of the behavior of localized structures and spatial solitons on inhomogeneous backgrounds have shown that the breaking of translational invariance leads to their motion [9–13]. For example, cavity solitons in nonlinear absorbers [9] and frequency converters [12,13] move up phase gradients. Cavity solitons in detuned quadratic media can also move and come to rest away from extrema of a spatially varying parameter while seeking resonance with the cavity or move up phase gradients with oscillating velocities [13].

While the investigation of solitons on modulated backgrounds is not new, here we point out the existence of previously unrecognized phenomena associated with certain types of regular modulation. Specifically, we show that for a sinusoidal modulation of wave vector K, the soliton velocity is proportional to a simple, but nontrivial, function of K. This function turns out to be the spatial Fourier transform (FT) of a function related to the unperturbed soliton itself. When the FT changes sign upon variation of a control parameter, the direction of motion is therefore reversed. Furthermore, at frequencies where the FT is zero, solitons are stationary at any spatial location in spite of the background modulations. These results are general and are verified here in model equations such as the Swift-Hohenberg (SH) and parametrically driven Ginzburg-Landau (PDGL) equations, as well as models of photonic systems such as the degenerate optical parametric oscillator [14] and a nonlinear absorber in an optical cavity [9]. In optics, for example, such knowledge is of crucial importance when constructing robust arrays of solitons for information processing [1,15] and quantum imaging [16].

In Sec. II we make a precise statement of the class of problems we intend to study. We restate the well-known expression for the velocity of otherwise stationary solutions in the presence of spatial perturbations [10,11]. We then point out novel consequences of this analysis when the perturbation is sinusoidal. In Sec. III we investigate our predictions in several models of nonlinear optical systems, using both the perturbative results and numerical integration of the equations. In Sec. IV we demonstrate the generality of the phenomena by showing their occurrence in the PDGL and SH equations. In Sec. V we show briefly how the results can be generalized to two dimensions (2D). Section VI contains our conclusions.

II. MOTION INDUCED BY COSINUSOIDAL PERTURBATIONS

Consider a field governed by partial differential equations of the following general form:

$$\partial_t W = H(W,\nabla^2 W) + \mu g(W;\mathbf{x}). \tag{1}$$

The quantity W represents a column vector of field variables while H is a function containing both linear and nonlinear terms, but not depending explicitly on time t or the spatial coordinates. The real quantity μ parametrizes the magnitude of the spatial-symmetry-breaking term while $g(W;\mathbf{x})$ may be either homogeneous or inhomogeneous in W. Many pattern-forming systems can be described by models of this form, including optical cavities [1] and reaction-diffusion systems [5]. In the subsequent discussion we will restrict ourselves to one spatial dimension, so that $\nabla^2 \equiv \partial_{xx}$. Such a situation is not unphysical and in optics, for example, could correspond to a planar waveguide geometry, in which the fields are tightly confined in one spatial direction. Our results can, however, be generalized to 2D as we will show.

We assume that Eq. (1) possesses a time-independent, localized solution $W_0(x)$ when μ is set to zero. Since the only explicit spatial dependence is now via the operator ∂_{xx}, Eq. (1) is invariant under spatial translations, and so $W_0(x-x_0)$ is also a solution for any x_0. (We are also assuming that the boundary conditions do not break the translational invariance.) This implies

$$\mathcal{L}(\partial_x W_0) = 0, \tag{2}$$

where $\mathcal{L}$ is the Jacobian of $H(W,\nabla^2 W)$ evaluated at W_0. In other words, $\mathcal{L}$ has at least one zero eigenvalue and corresponding eigenvector $\phi_0(x) \propto \partial_x W_0$.

*Electronic address: andrew@phys.strath.ac.uk

Now we consider $0<|\mu|\ll 1$, so that the localized solution persists, and assume that $g(W;x)$ is not a constant. All spatial positions are no longer equivalent and, in general, the localized solution will move. A singular perturbation analysis of Eq. (1) [11] yields, to $O(\mu)$, the velocity of the localized solution at a point x:

$$v(x) = -\langle \psi_0(a)|\mu g(W(a);a+x)\rangle, \tag{3}$$

where $\mathcal{L}^\dagger \psi_0=0$ and $\langle \psi_0|\phi_0\rangle=1$.

The appealingly straightforward interpretation of Eq. (3) is that the component of $\mu g(W;x)$ along the direction of the generator of translations for $W_0(x)$ [Eq. (2)] causes movement of the localized solution by a proportionate amount [11].

The velocity may vanish at isolated points in space. For example, with an underlying solution $W_0(x)$ of even parity, $v(x)$ will vanish at points about which $g(W;x)$ is symmetric, where the two possible spatial directions are locally equivalent. This result is in agreement with an independent argument showing that, in certain cases, $v(x)$ is proportional to the gradient of an applied (phase) modulation [9]. In the optics community in particular, both of these approaches have served to support the view that localized solutions climb the gradient of any externally applied phase modulation, a phenomenon seen in several theoretical (see [9,12] and, for more general parameter perturbations, [13]) and experimental [17] contexts.

The result expressed by Eq. (3) is well known [10,11,13]. For a physically important class of modulations, however, it implies a type of novel behavior which has not been reported before. Consider, therefore, the case of a cosinusoidal (or sinusoidal) perturbation $g(W;x)=\xi(W)\cos(Kx)$. In the common situation where the localized solution is an even function of space, $\xi(W)$ is also even while ψ_0 is odd. Therefore

$$v(x) = \langle \psi_0(a)|\mu\xi(W(a))\sin(Ka)\rangle \sin(Kx), \tag{4}$$

so that the velocity field factorizes into a spatially independent amplitude and a sinusoid with the same frequency as, but different phase from, the perturbation $g(W;x)$. The amplitude term

$$\mathcal{A}(K) \equiv \langle \psi_0(a)|\mu\xi(W(a))\sin(Ka)\rangle \tag{5}$$

is simply (proportional to) the Fourier sine transform of $\psi_0^*\cdot\mu\xi$ or, since $\psi_0^*\cdot\mu\xi$ is odd,

$$\mathcal{A}(K) = -i\sqrt{2\pi}\mathcal{F}(\psi_0^*\cdot\mu\xi(W)), \tag{6}$$

where $\mathcal{F}(\cdots)$ denotes a FT. From this it can be seen that $g(W;x)$ couples to the translational mode of the system only through the Fourier component of $\psi_0^*\cdot\xi$ at the same frequency. If $\mathcal{A}(K)$ changes sign for some value of K, a solution which previously moved up (down) the gradient of the applied modulation will move in the opposite direction. Moreover, if a particular frequency K_0 is absent from the Fourier spectrum, then $\mathcal{A}(K_0)$ is zero and the velocity vanishes simultaneously *at all points in space*. This corresponds to a function $g(W;x)$ in Eq. (1) which breaks the spatial symmetry but which is invisible to the underlying localized solution, at least with respect to its motion.

Although Eq. (3) only approximates the velocity to lowest order in the parameter μ, we will demonstrate in the following two sections that changes of sign (and hence "vanishings") of the velocity field can occur in several specific examples and are well described by Eqs. (4) and (5).

III. REVERSAL OF SOLITON MOTION IN OPTICAL SYSTEMS

We first consider the mean-field model for a degenerate optical parametric oscillator (DOPO) at resonance [14]:

$$\partial_t A_0 = \gamma[-A_0 + E_I - A_1^2] + \frac{i}{2}\partial_{xx}A_0,$$

$$\partial_t A_1 = -A_1 + A_0 A_1^* + i\partial_{xx}A_1, \tag{7}$$

where A_0 and A_1 denote the (complex) amplitudes of the pump and signal fields, respectively. Time has been normalized by the photon lifetime in the signal cavity and space by the square root of the diffraction coefficient. The parameter $\gamma=\gamma_0/\gamma_1$ is the ratio between the pump and signal cavity decay rates and E_I is the amplitude of the external pump field.

In the case of plane-wave pumping, E_I is a (real) constant function of x. Families of stable, localized solutions to Eqs. (7) (cavity solitons), distinguished by their widths, exist for $E_I>1$ and can be viewed as locked pairs of fronts, each of which asymptotically connects the two solutions $A_1^\pm \equiv \pm\sqrt{E_I-1}$ [12]. We consider the narrowest of such locked solutions, which is also the most stable [12]. A small phase modulation of the pump,

$$E_I = E_0 e^{i\mu\cos(Kx)}, \tag{8}$$

induces motion of the soliton. In practice, such a modulation can be achieved by placing a mask in front of the pump beam. Equation (8) is consistent with the assumptions leading to Eqs. (7) as long as the wave vector K is not too large: specifically, as long as $E_0\mu K^2=O(1)$. In what follows, E_0 is of order 1, and we will take $\mu=0.01$ with K always less than 7.

In Fig. 1 we plot the velocity of the soliton, measured at the point of the largest phase gradient, as a function of K, the wave vector of the modulation, for a fixed value of the pump amplitude, E_0. These results were obtained by integrating Eqs. (7) on a 1024-, 4096-, or 8192-point grid (depending on resolution constraints) using a split-step method and with periodic boundary conditions. The size of the integration domain was varied to allow sufficient resolution of K. Also shown are the values of $\mathcal{A}(K)$ calculated using Eq. (5). From Eq. (8), ξ is a constant vector, and the component of ψ_0 corresponding to $\mathrm{Im}(A_0)$ is the only one which couples to the field to produce soliton motion (inset to Fig. 1). Both the simulations and Eq. (3) indicate local extrema of $\mathcal{A}$ separated by zeros at $K\simeq 0.351, 0.497, 1.02, 1.66$, where $\mathcal{F}(\psi_0^*\cdot\xi)$ vanishes. At such points the coupling between the modulation and translational mode of the field vanishes, and there is no induced motion. Whether a zero of $\mathcal{A}$ actually occurs or not depends on the details of the underlying solution, which

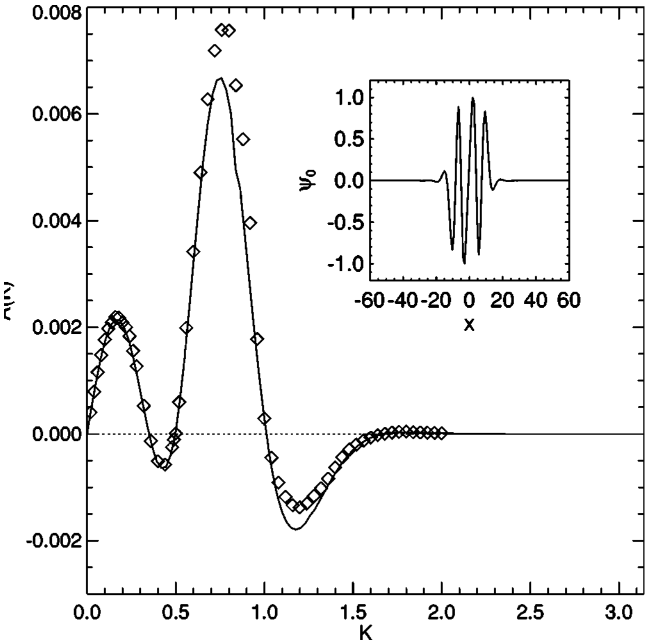

FIG. 1. Plot of $\mathcal{A}(K)$ for the DOPO model (7). The solid line is obtained from Eq. (5); the diamonds indicate values taken from simulations. $\mathcal{A}(K)=0$ is indicated by the horizontal dotted line. Parameters are $E_0=1.2$ and $\mu=0.01$. The inset shows the component of ψ_0 corresponding to $\mathrm{Im}(A_0)$.

change with the parameters of the system. Figure 2 shows the collision and mutual annihilation of the first two zeros as the pump amplitude is increased. Note that these phenomena persist away from cavity resonance.

The fact that the velocity $v\to 0$ as $K\to\infty$ is to be expected on physical grounds: below a certain length scale the soliton

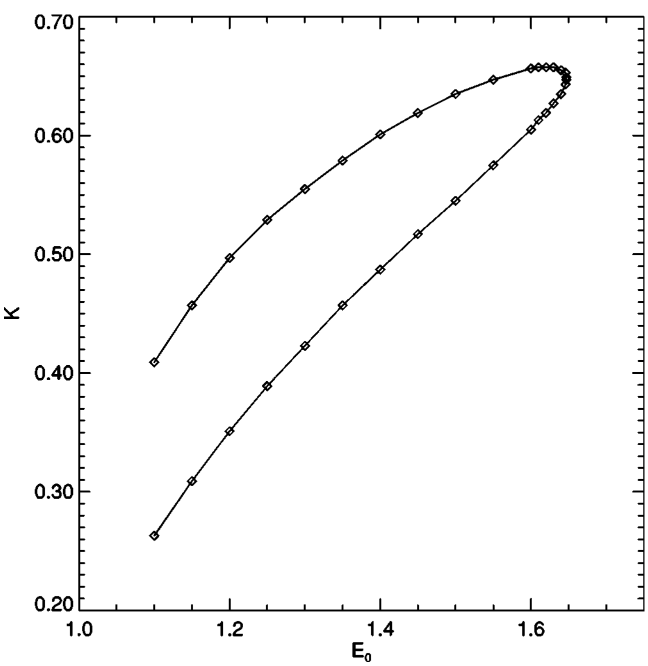

FIG. 2. Locations of the first two zeros of $\mathcal{A}(K)$ for the DOPO as a function of the pump amplitude, E_0. The data are taken from simulations of Eqs. (7). All other parameters as in Fig. 1.

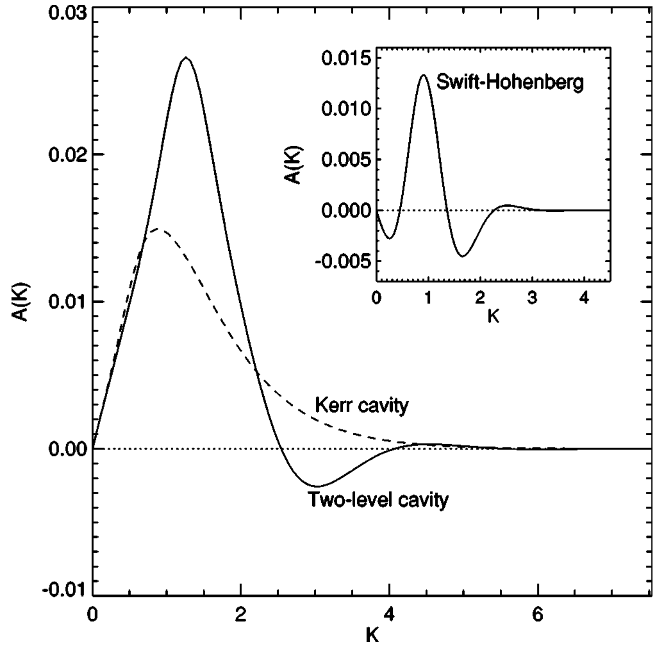

FIG. 3. Plot of $\mathcal{A}(K)$ for the two-level (solid line) and Kerr (dashed line) cavity models. For the two-level cavity, $\theta=-1.2$, $C=5.4$, and $E_0=6.65$. For the Kerr cavity, $\theta=1.6$ and $E_0=1.15$. In both cases $\mu=0.01$. The inset shows $\mathcal{A}(K)$ for the SH equation (10). Parameters are $\zeta=1$, $a=0.5$, $\Delta=0$, $E_0=1.4$, and $\mu=0.01$.

cannot distinguish the fine structure of the perturbation and only responds to its average value. At the other extreme, as $K\to 0$, the phase modulation is almost constant on the scale of ψ_0, and the phase gradient on which the soliton moves becomes shallower and shallower, causing the velocity to go to zero once again.

We can perform a similar analysis for a two-level atomic medium placed in an optical cavity, described by the equation [9,18]

$$\partial_t E = -(1+i\theta)E + E_I + \mathcal{G}(|E|^2)E + i\partial_{xx}E,$$

$$\mathcal{G}(|E|^2) = -\frac{2C(1-i\Delta)}{1+\Delta^2+|E|^2}, \tag{9}$$

where E is the intracavity optical field, θ is the cavity detuning, Δ is the detuning of the light field from atomic resonance, E_I is, again, an external driving field, and time and space are scaled to the photon lifetime and the diffraction length, respectively.

Equation (9) is known to possess cavity soliton solutions (optical bullet holes) [9,18]. If we select one such solution, modulate the pump phase [Eq. (8)], and perform the same analysis as for the DOPO, using Eq. (3), we find similar behavior: a zero of the velocity and concomitant difference in the direction of motion on either side of the zero (Fig. 3). Numerical integration of Eq. (9) confirms that for $K<2.5$ ($K>2.5$) solitons ascend (descend) the applied phase gradient. If we redefine $\mathcal{G}$ in Eq. (9) as $\mathcal{G}(|E|^2)=i|E|^2$, the resulting equation describes an optical cavity containing a (self-focusing) Kerr medium. Again, this system exhibits soliton

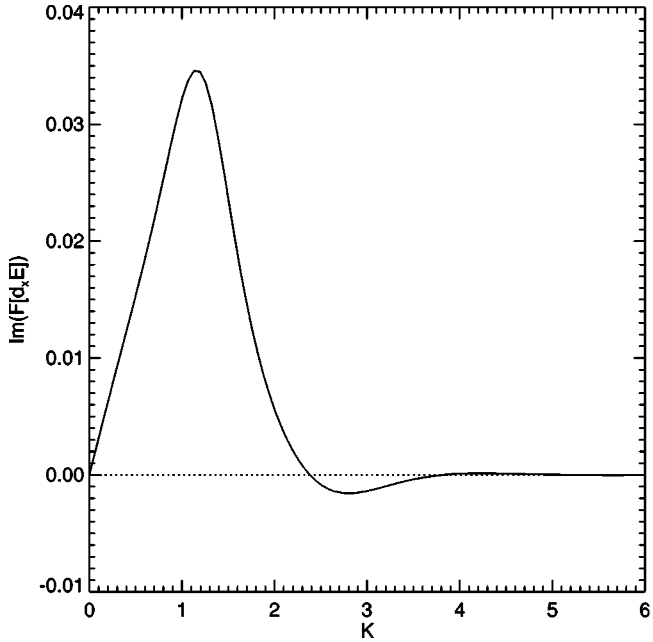

FIG. 4. The imaginary part of the Fourier transform of $d_x E$ for the two-level cavity soliton [Eq. (9)]. $\theta=-1.2$, $C=5.4$, and $E_0=6.65$, as in Fig. 3.

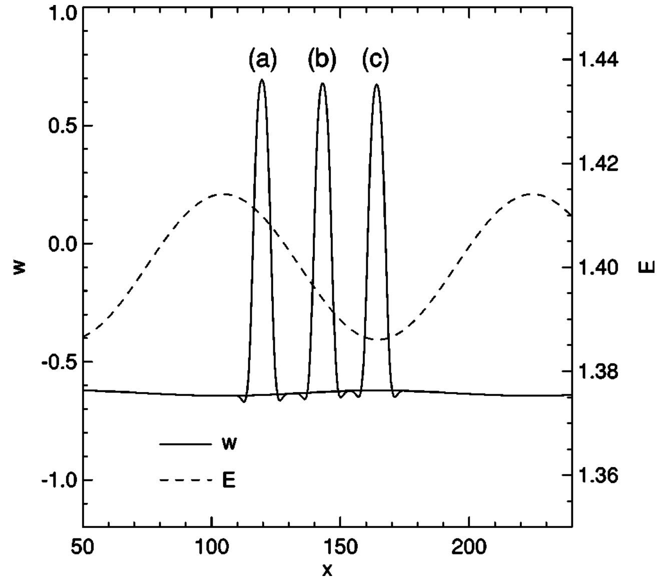

FIG. 5. Time sequence of an SH soliton [Eq. (10)] descending the gradient of an amplitude modulated pump (dashed line). Parameters are $\zeta=1$, $a=0.5$, $\Delta=0$, $E_0=1.4$, and $\mu=0.01$, as in the inset to Fig. 3. The soliton is shown at (a) $t=0$, (b) $t=30\,000$, and (c) $t=120\,000$ cavity lifetimes.

solutions [19], but for a typical soliton, no zeros in the velocity are observed (Fig. 3). This illustrates the fact that, although the phenomenon is general, it is not universal, even among systems which are structurally similar.

Because the operator $\mathcal{L}$ [Eq. (2)] is not in general self-adjoint, we usually cannot calculate the adjoint null eigenvector ψ_0 which appears in Eq. (6). Empirically, however, we often notice a strong resemblance between the components of ψ_0 and those of ϕ_0, the null eigenvector of $\mathcal{L}$. Since ϕ_0 is just the spatial derivative of the soliton [Eq. (2)], a zero in the FT of a component of ϕ_0 (except at $K=0$) implies a zero in the FT of the corresponding component of the soliton itself. A zero in the FT of the unperturbed soliton is, therefore, a good indicator of the existence and location of a value of K at which the soliton motion vanishes (compare Figs. 4 and 3). This means that the soliton itself can be used as a good diagnostic for the occurrence of reversible motion, without recourse to $\mathcal{L}^\dagger$ and its eigenvectors. From an experimental point of view, only the soliton data are accessible or even meaningful. Such an indicator, approximate though it may be, is therefore potentially extremely useful.

IV. SH AND PDGL EQUATIONS

The true generality of the phenomena reported in the previous two sections is demonstrated by looking at models not particular to optics. We first consider the real SH equation, one of the standard model equations in the study of pattern formation [2,6], which we write in the form

$$\partial_t w = (E-1)w - \zeta w^3 - a(\Delta + \nabla^2)^2 w \qquad (10)$$

and in which all variables and parameters are real. As with Eq. (7) for the DOPO, Eq. (10) possesses equivalent homogeneous solutions $w_\pm = \pm\sqrt{(E-1)/\zeta}$ for $E>1$, domain walls connecting them, and locked pairs of domain walls in the form of solitons. In this case, since the field is real, we consider an amplitude modulated driving term $E=E_0\exp[\mu\cos(Kx)]$.

A typical example of $\mathcal{A}(K)$ is given in the inset to Fig. 3. Again, we note the presence of zeros (at $K\simeq 0.45, 1.36$) and changes of sign, as well as the fact that solitons descend the gradient of the amplitude modulation for small K, in contrast with the previous examples. These predictions are again confirmed by direct simulation of Eq. (10). For example, Fig. 5 shows a time sequence of a soliton descending the gradient of the amplitude-modulated pump, in agreement with Fig. 3.

Figure 3 demonstrates the existence of the phenomenon, not only in yet another model, but also for amplitude (as opposed to phase) modulation. Indeed, adding an analogous amplitude modulation to the pump term in Eqs. (7) produces a curve similar in appearance to that in Fig. 3. This is not surprising, since Eq. (10) is the order parameter equation describing the DOPO close to threshold and for $|\Delta_1| \ll 1$, where Δ_1 is the signal field detuning [12,20]. We also note that, over certain ranges of wave vector K, it is sometimes possible for the modulation to destabilize the narrowest soliton in favor of the next highest-order soliton [12]. This destabilization can also occur in the PDGL equation below, but is outside the scope of this paper and will be discussed elsewhere.

As a second example of the generality of the phenomena, we mention the PDGL equation [21,22]

$$\partial_t A = \eta A - \zeta |A|^2 A + D\nabla^2 A + pA^*, \qquad (11)$$

where the parameters can be complex. Stationary solutions of Eq. (11) again include localized states in the form of locked pairs of domain walls. When, for example, a cosinusoidal amplitude modulation is added to the parameter p, application of Eq. (3) or direct simulation of Eq. (11) again indicate the reversal of motion and existence of zeros of the velocity of these solitons for certain values of K [we omit a plot of $\mathcal{A}(K)$ for the PDGL equation to avoid repetition].

V. TWO DIMENSIONS

It is straightforward to generalize our results to two spatial dimensions and "square" perturbations of the form [9]

$$g(W;x,y) = \xi(W)[A\cos(K_1 x) + B\cos(K_2 y)], \qquad (12)$$

where A and B are constants. In this case the Jacobian has two independent null eigenvectors, corresponding to motion in the x and y directions. Each translational mode couples only to the appropriate cosine term in Eq. (12) and the components of velocity in the x and y directions can each be expressed in the form of Eq. (4):

$$v_x = A\mu\langle\psi_{0x}(a_x,a_y)|\xi(W(a_x,a_y))\sin(K_1 a_x)\rangle\sin(K_1 x),$$

$$v_y = B\mu\langle\psi_{0y}(a_x,a_y)|\xi(W(a_x,a_y))\sin(K_2 a_y)\rangle\sin(K_2 y). \qquad (13)$$

In particular, it is possible to choose values of K_1 and K_2 such that a localized structure climbs the modulation gradient in one spatial direction and descends it (or remains stationary) in the orthogonal direction. This has been confirmed in simulations of the 2D version of Eq. (9).

VI. CONCLUSIONS

In conclusion, we have shown that localized solutions of a large class of equations may move up or down the gradient of an imposed cosinusoidal perturbation, depending on the wave vector of the modulation. More surprisingly, there may exist modulation frequencies at which the motion of the localized solutions vanishes identically, at every point in space. This behavior owes its existence to changes of sign in the Fourier spectrum of a function associated with the localized solution itself. We have demonstrated these phenomena in models specific to optics, where such phase and amplitude modulations have been proposed [9,13] and employed [17] to control the positions of spatial solitons. The behavior is more general, however, as evidenced by its existence in the SH and PDGL models. It could also occur in BEC solitons on modulated backgrounds, such as optical lattices.

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