Anisotropic diffusion across an external magnetic field and large-scale fluctuations in magnetized plasmas

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The problem of random motion of charged particles in an external magnetic field is studied under the assumption that the Langevin sources produce anisotropic diffusion in velocity space and the friction force is dependent on the direction of particle motion. It is shown that in the case under consideration, the kinetic equation describing particle transitions in phase space is reduced to the equation with a Fokker-Planck collision term in the general form (nonisotropic friction coefficient and nonzero off-diagonal elements of the diffusion tensor in the velocity space). The solution of such an equation has been obtained and the explicit form of the transition probability is found. Using the obtained transition probability, the mean-square particle displacements in configuration and velocity space were calculated and compared with the results of numerical simulations, showing good agreement. The obtained results are used to generalize the theory of large-scale fluctuations in plasmas to the case of anisotropic diffusion across an external magnetic field. Such diffusion is expected to be observed in the case of an anisotropic *k* spectrum of fluctuations generating random particle motion (for example, in the case of drift-wave turbulence).

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I. INTRODUCTION

Diffusion of particles and energy in plasmas exposed to external magnetic field still remains one of the important problems of plasma physics and controlled fusion. To a large extent, this problem could be considered to be exhausted if the probability of the test particle transition in the phase space is known. Such a transition probability makes it possible to calculate the mean and mean-square particle displacements and thus to describe the efficiency of particle diffusion. Beside that, as is known, the test particle transition probability can be treated as a Green's function of the linearized equation for large-scale perturbations in plasmas [1–4], which gives the possibility to study electromagnetic and kinetic processes in the systems under consideration with regard to self-consistent electromagnetic interaction between particles. One more important point is that the transition probability in phase space determines the correlation function of the Langevin sources for fluctuation fields in plasmas, and thus the theory of large-scale fluctuations can be worked out [1–4].

The transition probability of a test particle under the action of some random fields can be calculated on the basis of various approaches. One of the well-known treatments is based on the use of the Langevin equations for particle motion in random fields with known statistical properties [5,6]. Another possibility is to solve the appropriate Fokker-Planck equation with the initial distribution described by a δ function. As was shown by Chandrasekhar [5], the two approaches are equivalent and lead to the same results. In the present paper, we use the formalism of the Fokker-Planck equation.

The first solution for the test particle transition probability in phase space was obtained by Chandrasekhar [5] for the system with no external fields and for the case of a parabolic external potential. Different particular limits of these solutions and their generalizations have been widely used in the theory of turbulence (quasilinear theory included) [7,8]. As regards the transition probability in phase space under the presence of an external magnetic field, discussion of the problem started only a few years ago [9–11]. The particular case of isotropic diffusion across the external magnetic field was studied and the application of the Fokker-Planck formalism [9,10] and the Langevin approach [11] led to the same results. These results give answers to a number of questions. In particular, they explain the transition from the ballistic motion to the diffusive regime and describe details of classical diffusion across an external magnetic field. At the same time, the above-mentioned results cannot be applied to the description of turbulent diffusion generated by drift-wave instabilities, which are characterized by an asymmetric random field spectrum [12,13].

The purpose of the present paper is to find an explicit form of the transition probability for a test particle exposed to an external magnetic field and random force field producing anisotropic cross-B diffusion. Using the expression for the transition probability, we can estimate the transport level by calculating mean-square particle displacements. This is done is Sec. III. The obtained analytical results have been compared with results of numerical simulations.

In the second part of this work, we present the theory of large-scale fluctuations worked out for the system under consideration. Finally, we performed a detailed analysis of electron density fluctuation spectra and found simple analytical expressions of it in asymptotic limits.

II. BASIC EQUATIONS

We consider a charged test particle exposed to the external magnetic field $\mathbf{B}_0=(0,0,B_0)$ and random force field

$\delta\mathbf{F}(t)$. Treating these random forces as Langevin sources, we can describe the particle dynamics by the Langevin equations

$$\frac{d\mathbf{r}}{dt} = \mathbf{v},$$

$$\frac{dv_i}{dt} = -\beta_i v_i + \frac{1}{m}(e[\mathbf{v}\times\mathbf{B}_0]_i + \delta F_i), \quad (1)$$

where β_i is the friction coefficient for a particle moving in the ith direction, i.e., we suggest that the friction coefficient could be different for various directions of motion. Statistical properties of the random fields are assumed to be known,

$$\langle \delta F_i \rangle = 0,$$

$$\frac{1}{m^2}\langle \delta F_i(t)\delta F_j(t')\rangle = 2D_{ij}\delta(t-t'). \quad (2)$$

As was shown by Chandrasekhar in [5], the Langevin equations (1) generate the generalized Liouville equation. It is easy to show that in this case the evolution of the one-particle distribution function can be described by the equation

$$\hat{L}_0 f(X,t) = 0, \quad (3)$$

where

$$\hat{L}_0 \equiv \frac{\partial}{\partial t} + v_i\frac{\partial}{\partial r_i} + [\mathbf{v}\times\mathbf{\Omega}]_i\frac{\partial}{\partial v_i} - \frac{\partial}{\partial v_i}\left(\beta_i v_i + D_{ij}\frac{\partial}{\partial v_j}\right), \quad (4)$$

$$\mathbf{\Omega} = (0,0,\Omega), \quad \Omega = eB_0/m, \quad X \equiv (\mathbf{r},\mathbf{v}).$$

We can see that $D_{ij}=D_{ji}$ plays the role of a diffusion coefficient in velocity space.

The solution of the initial-value problem (3) with the initial condition $f(X,t)|_{t=t'}=f(X,t')$ is given by

$$f(X,t) = \int dX' W(X,X';t,t') f(X',t'), \quad (5)$$

where $W(X,X';t,t')$ satisfies the equation

$$\hat{L}_0 W(X,X';t,t') = 0 \quad (6)$$

with the initial condition

$$W(X,X';t',t') = \delta(X-X'). \quad (7)$$

According to Eqs. (6) and (7), the quantity $W(X,X';t,t')$ can be treated as the transition probability for a particle whose random motion is generated by the Langevin forces with statistical properties described by Eq. (2).

We notice that our model can be used to describe the diffusion of test particles due to the influence of electrostatic turbulence, thus we assume the diffusion and friction coefficients to be constant in zero-order approximation. In the general case, however, they should be treated self-consistently and their velocity and time dependence should be taken into account [8].

III. SOLUTION FOR TRANSITION PROBABILITY

Taking into account the independency of motion along and across the magnetic field (we assume $D_{xz}=D_{yz}=0$), the solution of Eq. (6) with the initial condition (7) can be represented in the form

$$W(X,X',t,t') = W_\perp(X_\perp,X'_\perp,t,t')W_\parallel(X_\parallel,X'_\parallel,t,t'). \quad (8)$$

The solution for the transition probability in the parallel direction was found by Chandrasekhar in [5],

$$W_\parallel(X_\parallel,X'_\parallel,\tau) = \frac{e^{\beta_z\tau}}{2\pi\sqrt{\Delta}}\exp\left[-\frac{1}{2\Delta}(a\rho^2 + bP^2 + 2h\rho P)\right], \quad (9)$$

where

$$\rho = e^{\beta_z\tau}v_z + v'_z, \quad P = z - z' + \frac{v_z - v'_z}{\beta_z},$$

$$a = \frac{2D_{zz}\tau}{\beta_z^2}, \quad b = \frac{D_{zz}}{\beta_z}(e^{2\beta_z\tau}-1),$$

$$h = -\frac{2D_{zz}}{\beta_z^2}(e^{\beta_z\tau}-1),$$

$$\Delta = ab - h^2, \quad \tau = t - t'.$$

Equation (6) for the perpendicular part of the transition probability (8) reads

$$\left\{\frac{\partial}{\partial t} + v_i\frac{\partial}{\partial r_i} + [\mathbf{v}_\perp\times\mathbf{\Omega}]_i\frac{\partial}{\partial v_i} - \frac{\partial}{\partial v_i}\left(\beta_i v_i + D_{ij}\frac{\partial}{\partial v_j}\right)\right\}W_\perp(X_\perp,X'_\perp,t,t') = 0, \quad (10)$$

$$i,j = \{x,y\}, \quad X_\perp \equiv \{\mathbf{r}_\perp,\mathbf{v}_\perp\}.$$

Such a Fokker-Planck-type kinetic equation would correspond to the following equations of motion for charged particle in viscous media across the external magnetic field:

$$\dot{x} = v_x,$$

$$\dot{y} = v_y,$$

$$\dot{v}_x = -\beta_x v_x + \Omega v_y,$$

$$\dot{v}_y = -\beta_y v_y - \Omega v_x. \quad (11)$$

There are four integrals of motion for the system (11),

$$\rho_x = \frac{e^{[(\beta_x+\beta_y)/2]\tau}}{\tilde{\Omega}}\left[v_x\left(\tilde{\Omega}\cos\tilde{\Omega}\tau + \frac{\beta_x-\beta_y}{2}\sin\tilde{\Omega}\tau\right) - v_y\Omega\sin\tilde{\Omega}\tau\right] - v'_x,$$

$$\rho_y = \frac{e^{[(\beta_x+\beta_y)/2]\tau}}{\tilde{\Omega}}\left[v_y\left(\tilde{\Omega}\cos\tilde{\Omega}\tau - \frac{\beta_x-\beta_y}{2}\sin\tilde{\Omega}\tau\right) + v_x\Omega\sin\tilde{\Omega}\tau\right] - v_y',$$

$$P_x = x - x' + \frac{\beta_y(v_x - v_x') + \Omega(v_y - v_y')}{\beta_x\beta_y + \Omega^2},$$

$$P_y = y - y' + \frac{\beta_x(v_y - v_y') - \Omega(v_x - v_x')}{\beta_x\beta_y + \Omega^2}, \tag{12}$$

where $\tilde{\Omega} \equiv [\Omega^2 - (\beta_x - \beta_y)^2/4]^{1/2}$.

In terms of variables ρ and P, the equation for the transition probability transforms into

$$\frac{\partial w_\perp(x,\tau)}{\partial\tau} - \sum_{i,j=1}^{4} a_{ij}(\tau)\frac{\partial^2 w_\perp(x,\tau)}{\partial x_i \partial x_j} = 0, \tag{13}$$

$$w_\perp(x,\tau) = e^{-(\beta_x+\beta_y)\tau}W_\perp(x,\tau),$$

$$\{x_1,x_2,x_3,x_4\} \equiv \{\rho_x,\rho_y,P_x,P_y\}.$$

The explicit form of the coefficients $a_{ij}(\tau)$ is presented in the Appendix.

The solution of Eq. (13) with the initial condition

$$w_\perp(x_1,x_2,x_3,x_4,0) = \delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4)$$

has the form of a multidimensional Gaussian distribution

$$w_\perp(\rho_x,\rho_y,P_x,P_y,\tau) = \frac{1}{4\pi^2}\left(\frac{1}{\det C}\right)^{1/2} \times \exp\left[-\frac{1}{2}\sum_{i,j=1}^{4} c_{ij}^{-1}(\tau)x_i x_j\right]. \tag{14}$$

Here c_{ij} are the elements of a matrix C,

$$c_{ij}(\tau) = 2\int_0^\tau a_{ij}(\tau')d\tau',$$

and c_{ij}^{-1} are the elements of the inverse matrix C^{-1}.

Using the obtained solution (14), it is possible to calculate the mean-square displacements,

$$\langle \Delta r_i \Delta r_j\rangle_{\mathbf{v},\tau} = \int d\Delta\mathbf{r}\int d\Delta\mathbf{v}\,\Delta r_i \Delta r_j \times W(\mathbf{r}+\Delta\mathbf{r},\mathbf{v}+\Delta\mathbf{v},\mathbf{r},\mathbf{v};\tau),$$

$$\langle \Delta v_i \Delta v_j\rangle_{\mathbf{v},\tau} = \int d\Delta\mathbf{r}\int d\Delta\mathbf{v}\,\Delta v_i \Delta v_j \times W(\mathbf{r}+\Delta\mathbf{r},\mathbf{v}+\Delta\mathbf{v},\mathbf{r},\mathbf{v};\tau), \tag{15}$$

and their values averaged over velocity distribution,

$$\langle \Delta r_i \Delta r_j\rangle_{\tau} = \int \langle \Delta r_i \Delta r_j\rangle_{\mathbf{v},\tau} f(\mathbf{v})d\mathbf{v},$$

$$\langle \Delta v_i \Delta v_j\rangle_{\tau} = \int \langle \Delta v_i \Delta v_j\rangle_{\mathbf{v},\tau} f(\mathbf{v})d\mathbf{v}. \tag{16}$$

In the particular case of a Maxwellian velocity distribution and $\beta_x = \beta_y = \beta$, the mean-square displacement in the x direction has the form

$$\begin{aligned}\langle \Delta x^2\rangle_\tau ={}& 2\frac{\beta^2 D_{xx} + 2\beta\Omega D_{xy} + \Omega^2 D_{yy}}{(\beta^2+\Omega^2)^2}\tau + \frac{1-e^{-2\beta\tau}}{(\beta^2+\Omega^2)\beta}\frac{D_{xx}+D_{yy}}{2} + \frac{1}{(\beta^2+\Omega^2)^3}\Bigg\{[\beta - e^{-2\beta\tau}(\beta\cos 2\Omega\tau - \Omega\sin 2\Omega\tau)]\\ &\times\left((\beta^2-\Omega^2)\frac{D_{xx}-D_{yy}}{2} + 2\beta\Omega D_{xy}\right) + [\Omega - e^{-2\beta\tau}(\Omega\cos 2\Omega\tau + \beta\sin 2\Omega\tau)]\left((\beta^2-\Omega^2)D_{xy} - 2\beta\Omega\frac{D_{xx}-D_{yy}}{2}\right)\Bigg\}\\ &- 4\frac{\beta D_{xx} + \Omega D_{xy}}{(\beta^2+\Omega^2)^3}\{(\beta^2-\Omega^2)(1-e^{-\beta\tau}\cos\Omega\tau) + 2\beta\Omega e^{-\beta\tau}\sin\Omega\tau\}\\ &+ 4\frac{\beta D_{xy} + \Omega D_{yy}}{(\beta^2+\Omega^2)^3}\{-2\beta\Omega(1-e^{-\beta\tau}\cos\Omega\tau) + (\beta^2-\Omega^2)e^{-\beta\tau}\sin\Omega\tau\} + \frac{1+e^{-2\beta\tau}-2e^{-\beta\tau}\cos\Omega\tau}{\beta^2+\Omega^2}v_{th}^2,\end{aligned} \tag{17}$$

$$\begin{aligned}\langle \Delta v_x^2\rangle_\tau ={}& \frac{1-e^{-2\beta\tau}}{\beta}\frac{D_{xx}+D_{yy}}{2} + (e^{-2\beta\tau} + 1 - 2e^{-\beta\tau}\cos\Omega\tau)v_{th}^2 + \frac{\beta}{\beta^2+\Omega^2}\left(\cos 2\Omega\tau + \frac{\Omega}{\beta}\sin 2\Omega\tau - e^{-2\beta\tau}\right)\left(\frac{D_{xx}-D_{yy}}{2}\cos 2\Omega\tau\right.\\ &\left.+ D_{xy}\sin 2\Omega\tau\right) + \frac{\Omega}{\beta^2+\Omega^2}\left(\cos 2\Omega\tau - \frac{\beta}{\Omega}\sin 2\Omega\tau - e^{-2\beta\tau}\right)\left(D_{xy}\cos 2\Omega\tau - \frac{D_{xx}-D_{yy}}{2}\sin 2\Omega\tau\right).\end{aligned} \tag{18}$$

The expressions for $\langle \Delta y^2 \rangle_\tau$ and $\langle \Delta v_y^2 \rangle_\tau$ can be obtained by switching D_{xx} and D_{yy} and setting the opposite sign of D_{xy}.

In the case of isotropic diffusion ($D_{xx}=D_{yy}=D, D_{xy}=0$), Eqs. (17) and (18) reduce to

$$\langle \Delta x^2 \rangle_\tau = \frac{2D}{\beta^2+\Omega^2}\left\{\tau + \frac{1-e^{-2\beta\tau}}{2\beta} - \frac{2}{\beta^2+\Omega^2}[\beta - e^{-\beta\tau}(\beta\cos\Omega\tau - \Omega\sin\Omega\tau)]\right\} + \frac{1+e^{-2\beta\tau}-2e^{-\beta\tau}\cos\Omega\tau}{\beta^2+\Omega^2}v_{th}^2, \tag{19}$$

$$\langle \Delta v_x^2 \rangle_\tau = \frac{1-e^{-2\beta\tau}}{\beta}D + (1+e^{-2\beta\tau}-2e^{-\beta\tau}\cos\Omega\tau)v_{th}^2, \tag{20}$$

which is in agreement with the results presented in [10,11]. By putting $\Omega=0$ in Eq. (19) and looking at the asymptotic behavior in time, we can easily recognize the Einstein law for conventional Brownian particle diffusion $\langle \Delta x^2 \rangle_\tau = 2D^r\tau$ with the diffusion coefficient in configuration space defined as $D^r = D/\beta^2$.

For small time Eq. (19) gives

$$\langle \Delta x^2 \rangle_\tau = \frac{2D}{3}\tau^3 + v_{th}(\tau^2 - \beta\tau^3),$$

which corresponds to the ballistic regime of motion.

IV. NUMERICAL SIMULATIONS

In order to justify our analytical expressions, we performed numerical studies of particle diffusion in an external constant magnetic field and a prescribed stochastic electric field which represents a turbulent background. The last one is taken as a superposition of N^2 modes with random phases. We have solved the following test particle equations of motion numerically:

$$\frac{dx}{dt} = v_x, \quad \frac{dy}{dt} = v_y,$$

$$\frac{dv_x}{dt} = \frac{e}{m}[Bv_y + \delta E_x(\mathbf{r},t)], \tag{21}$$

$$\frac{dv_y}{dt} = \frac{e}{m}[-Bv_x + \delta E_y(\mathbf{r},t)],$$

where $\delta\mathbf{E}(\mathbf{r},t) = -\boldsymbol{\nabla}\,\delta\Phi(\mathbf{r},t)$ and

$$\delta\,\Phi(\mathbf{r},t) = \sum_{i=1}^{N}\sum_{j=1}^{N}\delta\,\Phi_{ij}\cos(\omega_{\mathbf{k}ij}t - k_{xi}x - k_{yj}y + \alpha_i + \beta_j).$$

In the last expression, α_i and β_j are random numbers equally distributed between 0 and 2π; $\delta\Phi_{ij}$ is the spectrum of a potential which was chosen to be Gaussian,

$$\delta\Phi_{ij}^2 = \delta\Phi_0^2\frac{50}{\pi N^2}\exp\left[-\left(\frac{k_{xi}}{\Delta k}\right)^2 - \left(\frac{k_{yi}}{\Delta k}\right)^2\right],$$

$$k_{xi} = 2.5i\Delta k/N, \quad i=1,\ldots,N,$$

$$k_{yj} = 2.5j\Delta k/N, \quad j=1,\ldots,N.$$

We use the dispersion relation $\omega_{\mathbf{k}ij} = c_s k/(1+k^2\rho_s^2)$, where ρ_s is the Larmor radius, $\rho_s = c_s/\Omega$. The fluctuation intensity is defined by the dimensionless parameter $\sigma \equiv (e/mc_s^2)\delta\Phi_0$.

Particle trajectories were calculated for different realizations of random phases and then the mean and mean-square displacements were found as averages over realizations.

For both analytical and numerical calculations, we normalize time by $2\pi\Omega$ and length by $2\pi\rho_s$. The diffusion coefficient in velocity space was normalized by $2\pi\rho_s^2\Omega^3$ and the friction coefficient by Ω. The dimensionless parameters chosen for simulations are $\sigma=1$, $\Delta k\rho_s=0.5$. The number of modes for each direction is $N=15$; the number of realizations is between 2000 and 10 000.

The obtained results are shown in Figs. 1–5. The mean-square displacements in configuration (Fig. 1) and velocity space (Fig. 3) were compared with those calculated using analytical expressions (17) and (18) (Figs. 2 and 4). As we can see, the derived analytical formulas describe the time behavior of the mean-square displacements in both configuration and velocity space rather well. In particular, they show

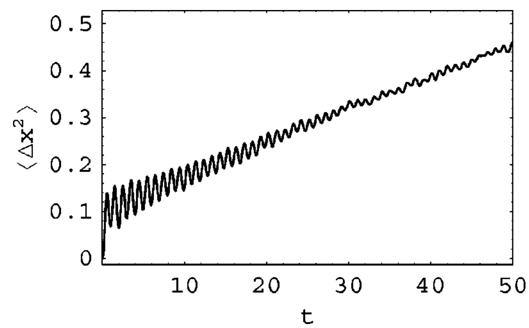

FIG. 1. Mean-square displacement in configuration space: simulation results.

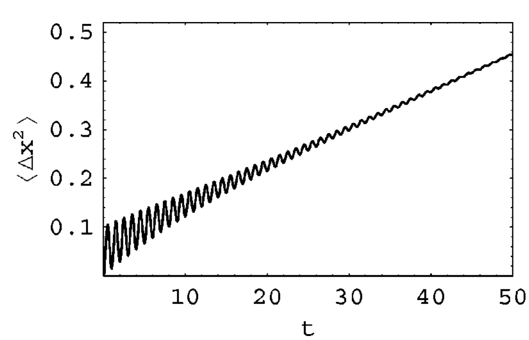

FIG. 2. Mean-square displacement in configuration space: analytical expression with $D=0.0038$ and $\beta=0.012$.

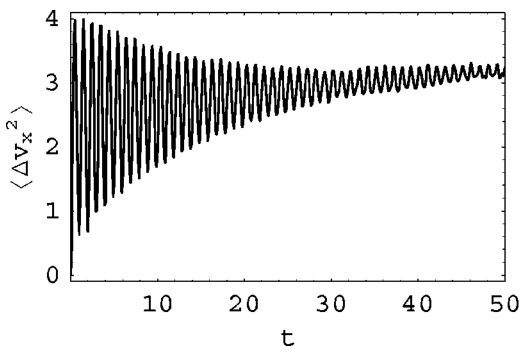

FIG. 3. Mean-square displacement in the velocity space: simulation results.

that the mean-square displacements are accomplished with oscillations which are damped by a friction. The mean-square displacement in configuration space has a linear time asymptotic (Figs. 1, 2, and 5) while the mean-square displacement in velocity space is saturated (Figs. 3 and 4).

In order to study the effects of anisotropy, we have multiplied the intensity of different components of random force by a factor 2 and compared the obtained mean-square displacements (see Fig. 5). As we can see, the increase of the intensity of random force in the y direction, i.e., the increase of D_{yy}, leads to an increase of the mean-square displacement in the x direction, which is in agreement with the asymptotic expression, following from Eq. (17),

$$\langle \Delta x^2 \rangle_\tau = 2\frac{\beta^2 D_{xx} + 2\beta\Omega D_{xy} + \Omega^2 D_{yy}}{(\beta^2+\Omega^2)^2}\tau. \tag{22}$$

V. LARGE-SCALE FLUCTUATIONS

In order to describe large-scale fluctuations in plasmas, it is necessary to take into account particle interactions through a self-consistent electric field. This means that the right-hand part of Eq. (1) should be supplemented with the force term responsible for such interaction. In turn, this force generates an additional self-consistent term in the kinetic equation for $f(X,t)$, which transforms to

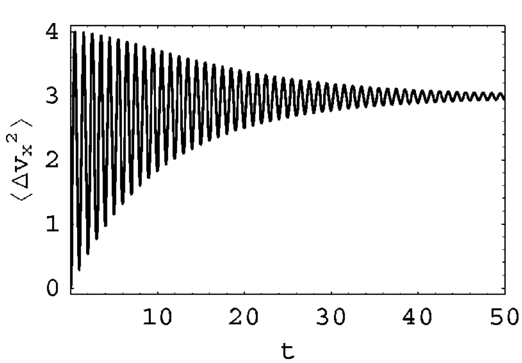

FIG. 4. Mean-square displacement in the velocity space: analytical expression with $D=0.0038$ and $\beta=0.012$.

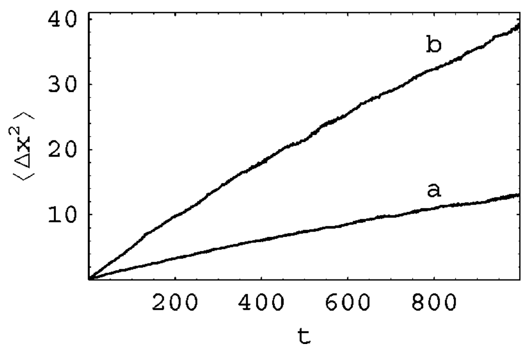

FIG. 5. The results of simulations showing the mean-square displacements in configuration space in anisotropic case: (a) $\sigma_x=2$, $\sigma_y=1$; (b) $\sigma_x=1$, $\sigma_y=2$.

$$\hat{L}_0 f(X,t) + \frac{e}{m}\mathbf{E}(\mathbf{r},t)\cdot\frac{\partial f(X,t)}{\partial \mathbf{v}} = 0, \tag{23}$$

where $\mathbf{E}(\mathbf{r},t)$ is the self-consistent field.

Assuming then that at the times $t > \tau_{ph}$ (where τ_{ph} is the physically infinitesimal time with respect to which the distribution function is introduced) $f(X,t)$ and $\mathbf{E}(\mathbf{r},t)$ are random functions and using the representation

$$f(X,t) = f_0(\mathbf{v}) + \delta f(X,t),$$

one obtains the following linearized equation for the distribution function fluctuations:

$$\hat{L}_0 \delta f(X,t) = -\frac{e}{m}\delta\,\mathbf{E}(\mathbf{r},t)\cdot\frac{\partial f_0(\mathbf{v})}{\partial \mathbf{v}}. \tag{24}$$

The formal solution of this equation given in terms of the transition probability is

$$\delta f(X,t) = \delta f^{(0)}(X,t) - \frac{e}{m}\int_{-\infty}^{t} dt' \int dX' W(X,X';t,t') \times \delta\mathbf{E}(\mathbf{r}',t')\cdot\frac{\partial f_0}{\partial \mathbf{v}'}. \tag{25}$$

Here $\delta f^{(0)}(X,t)$ is the fluctuation in the appropriate system, but with no self-consistent interaction through the electric field. It satisfies the equation

$$\hat{L}_0 \delta f^{(0)}(X,t) = 0. \tag{26}$$

$\delta\mathbf{E}(\mathbf{r},t) = -\nabla\,\delta\Phi(\mathbf{r},t)$ is the electric field fluctuation satisfying the Poisson equation

$$\Delta\delta\Phi(\mathbf{r},t) = -en\int d\mathbf{v}\,\delta f(X,t). \tag{27}$$

Substituting Eq. (25) into Eq. (27), we obtain an inhomogeneous equation for the fluctuation potential

$$\Delta\delta\Phi(\mathbf{r},t) + \frac{e^2n}{m}\int_{-\infty}^{t} dt' \int d\mathbf{v}\int dX'\,W(X,X';t,t') \times \frac{\partial\delta\Phi(\mathbf{r}',t')}{\partial\mathbf{r}'}\cdot\frac{\partial f_0}{\partial\mathbf{v}'} = -en\int d\mathbf{v}\,\delta f^{(0)}(X,t). \tag{28}$$

We can see that the quantity $\delta f^{(0)}(X,t)$ plays the role of a Langevin source for the electric field fluctuation.

Taking into account Eqs. (3), (5), and (26), it is easy to find the correlation function for these sources,

$$\langle\delta f^{(0)}(X,t)\delta f^{(0)}(X',t')\rangle = \frac{1}{n}\{W(X,X';t,t')f(X',t')\Theta(t-t') + W(X',X;t',t)f(X,t)\Theta(t'-t)\}, \tag{29}$$

where $\Theta(t)$ is the Heaviside function.

Together with Eq. (28), we have a coupled set of equations for the description of large-scale fluctuations. In the potential case, the solution of these equations gives

$$\langle\delta n_\sigma^2\rangle_{\mathbf{k}\omega} = \left|\frac{1+\sum_{\sigma'\neq\sigma}\chi_{\sigma'}(\mathbf{k},\omega)}{\varepsilon(\mathbf{k},\omega)}\right|^2\langle\delta n_\sigma^{(0)2}\rangle_{\mathbf{k}\omega} + \left|\frac{\chi_\sigma(\mathbf{k},\omega)}{\varepsilon(\mathbf{k},\omega)}\right|^2\sum_{\sigma'\neq\sigma}\langle\delta n_{\sigma'}^{(0)2}\rangle_{\mathbf{k}\omega}. \tag{30}$$

In this expression, $\varepsilon(\mathbf{k},\omega)$ is the dielectric response function,

$$\varepsilon(\mathbf{k},\omega) = 1 + \sum_\sigma \chi_\sigma(\mathbf{k},\omega), \tag{31}$$

where the dielectric susceptibility for each species is defined as

$$\chi_\sigma(\mathbf{k},\omega) = -i\frac{\omega_{p\sigma}^2}{k^2}\int d\mathbf{v}\int d\mathbf{v}'\,W_{\sigma\mathbf{k}\omega}(\mathbf{v},\mathbf{v}')\mathbf{k}\cdot\frac{\partial f_{0\sigma}(\mathbf{v}')}{\partial\mathbf{v}'}. \tag{32}$$

Here $\omega_{p\sigma}$ is the corresponding plasma frequency and $f_{0\sigma}(\mathbf{v})$ is the equilibrium distribution function.

The spectral density of the sources is given by

$$\langle n_\sigma^{(0)2}\rangle_{\mathbf{k}\omega} = n_\sigma\int d\mathbf{v}'\int d\mathbf{v}\,W_{\sigma\mathbf{k}\omega}(\mathbf{v},\mathbf{v}')f_{0\sigma}(\mathbf{v}') + \text{c.c.}, \tag{33}$$

where n_σ is the mean density and c.c. means complex conjugated.

As we can see, both dielectric susceptibility (32) and sources spectral density (33) are defined by the space-time Fourier transformed transition probability

$$W_{\sigma\mathbf{k}\omega}(\mathbf{v},\mathbf{v}') = \int_0^\infty d\tau e^{i\omega\tau}\int_{-\infty}^{\infty} d\mathbf{R}e^{-i\mathbf{k}\cdot\mathbf{R}}W_\sigma(X,X';t,t'),$$

$$\mathbf{R} = \mathbf{r}-\mathbf{r}'; \quad \tau = t-t'.$$

It is possible to show that in the case of equilibrium Maxwellian distributions, Eqs. (32) and (33) reduce to

$$\chi_\sigma(\mathbf{k},\omega) = \frac{k_\sigma^2}{k^2}\left[1 + i\omega\int_0^\infty d\tau e^{i\omega\tau}e^{-(k_ik_j/2)\langle\Delta r_i\Delta r_j\rangle_{\sigma\tau}}\right], \tag{34}$$

$$\langle n_\sigma^{(0)2}\rangle_{\mathbf{k}\omega} = n_\sigma\int_0^\infty d\tau e^{i\omega\tau}e^{-(k_ik_j/2)\langle\Delta r_i\Delta r_j\rangle_{\sigma\tau}} + \text{c.c.} \tag{35}$$

in agreement with fluctuation-dissipation theorem

$$\langle n_\sigma^{(0)2}\rangle_{\mathbf{k}\omega} = \frac{T_\sigma k^2}{2\pi e_\sigma^2\omega}\,\text{Im}\,\chi_\sigma(\mathbf{k},\omega). \tag{36}$$

Here, $k_\sigma^2=4\pi e_\sigma^2 n_\sigma/T_\sigma$, $\langle\Delta r_i\Delta r_j\rangle_\tau$ is the mean value of the product of particle displacements in different directions, given by Eqs. (15) and (16).

Using Eqs. (30) and (36), one obtains for the equilibrium fluctuations

$$\langle\delta n_e^2\rangle_{\mathbf{k}\omega} = \frac{Tk^2}{2\pi e_e^2\omega}\,\text{Im}\,\frac{[1+\chi_i(\mathbf{k},\omega)]\chi_e(\mathbf{k},\omega)}{\varepsilon(\mathbf{k},\omega)}, \tag{37}$$

$$\langle\delta n_i^2\rangle_{\mathbf{k}\omega} = \frac{Tk^2}{2\pi e_i^2\omega}\,\text{Im}\,\frac{[1+\chi_e(\mathbf{k},\omega)]\chi_i(\mathbf{k},\omega)}{\varepsilon(\mathbf{k},\omega)}, \tag{38}$$

$$\langle\delta\Phi^2\rangle_{\mathbf{k}\omega} = \frac{8\pi T}{\omega}\,\text{Im}\left(-\frac{1}{k^2\varepsilon(\mathbf{k},\omega)}\right). \tag{39}$$

The static values of these correlation functions are given by

$$\langle\delta n_e^2\rangle_{\mathbf{k}} = n_e\frac{k^2+k_i^2}{k^2+k_i^2+k_e^2}, \tag{40}$$

$$\langle\delta n_i^2\rangle_{\mathbf{k}} = n_i\frac{k^2+k_e^2}{k^2+k_i^2+k_e^2}, \tag{41}$$

$$\langle\delta\Phi^2\rangle_{\mathbf{k}} = \frac{4\pi T}{k^2}\frac{k_e^2+k_i^2}{k^2+k_i^2+k_e^2}. \tag{42}$$

As is seen, they are independent of kinetic coefficients D and β.

If the friction and diffusion in the velocity space can be neglected, $\beta_i=0$, $D_{ij}=0$, the transition probability (14) reduces to the one describing unperturbed particle motion in the external magnetic field. Then Eqs. (30)–(39) recover the

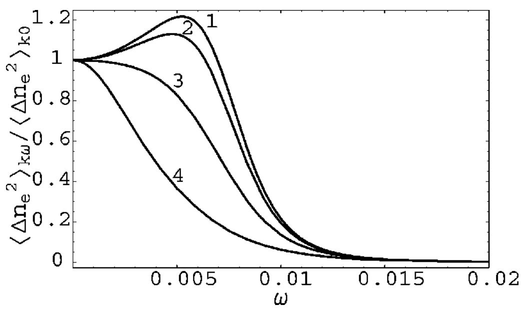

FIG. 6. Fluctuation spectra $\langle \Delta n_e^2 \rangle_{\mathbf{k},\omega} / \langle \Delta n_e^2 \rangle_{\mathbf{k},0}$ of isothermal plasma for different values of Δ: (1) $\Delta=0.1$, (2) $\Delta=0.3$, (3) $\Delta=1$, (4) $\Delta=3$; $k/k_e=0.1$, $T_e/T_i=1$, $\theta=10°$, $\phi=45°$.

appropriate results of the collisionless theory of electrostatic fluctuations (see, for example, [14]). Such an approximation is valid at $\beta\tau \ll 1$, $k_i k_j D_{ij}^{\sigma} \tau^3 \ll 1$. In the opposite case of low-frequency fluctuations ($\beta\tau \gg 1$, i.e., $\omega \ll \beta$) with large-scale spatial correlations ($R \gg v_{th}/\beta$, i.e., $k \ll \beta_\sigma / v_{th\sigma}$, where $v_{th\sigma}$ is thermal particle velocity), the above equations are considerably simplified. In particular,

$$W_{\sigma\mathbf{k}\omega} \equiv \int d\mathbf{v} \int d\mathbf{v}' W_{\sigma\mathbf{k}\omega}(\mathbf{v},\mathbf{v}') f_{0\sigma}(\mathbf{v}') = \frac{i}{\omega + k_i k_j D_{ij\sigma}^r} \tag{43}$$

and thus

$$\chi_\sigma(\mathbf{k},\omega) = \frac{i k_\sigma^2}{k^2} \frac{k_i k_j D_{ij\sigma}^r}{\omega + i k_i k_j D_{ij\sigma}^r}, \tag{44}$$

$$\langle \delta n_\sigma^{(0)2} \rangle_{\mathbf{k}\omega} = \frac{2 n_\sigma k_i k_j D_{ij\sigma}^r}{|\omega + i k_i k_j D_{ij\sigma}^r|^2}. \tag{45}$$

Here $D_{ij\sigma}^r$ is the diffusion coefficient in real space defined as follows:

$$D_{ij\sigma}^r = \frac{1}{2} \lim_{\beta\tau \gg 1} \frac{\langle \Delta r_i \Delta r_j \rangle_{\tau\sigma}}{\tau}. \tag{46}$$

Using Eq. (17) and the appropriate relations for $\langle \Delta x \Delta y \rangle_\tau$ and $\langle \Delta y^2 \rangle_\tau$, we obtain

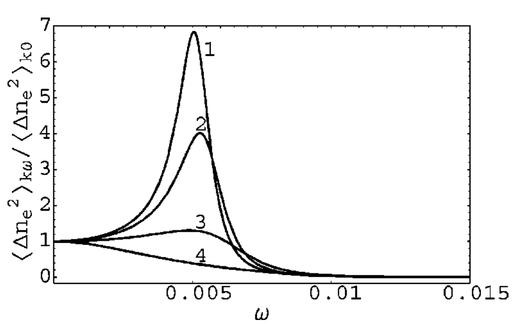

FIG. 7. Fluctuation spectra $\langle \Delta n_e^2 \rangle_{\mathbf{k},\omega} / \langle \Delta n_e^2 \rangle_{\mathbf{k},0}$ of nonisothermal plasma for different values of Δ (nonisothermal case): (1) $\Delta=0.1$, (2) $\Delta=0.3$, (3) $\Delta=1$, (4) $\Delta=3$; $k/k_e=0.1$, $T_e/T_i=2$, $\theta=10°$, $\phi=45°$.

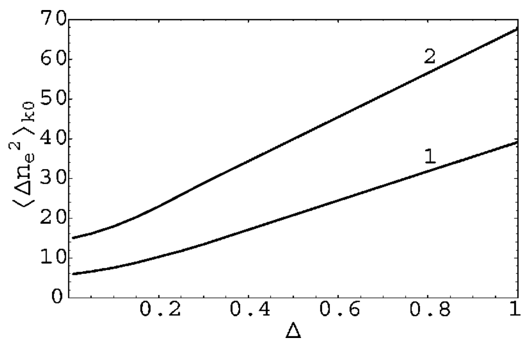

FIG. 8. Dependence of $\langle \Delta n_e^2 \rangle_{\mathbf{k},0}$ on diffusion for (1) isothermal, $T_e/T_i=1$ and (2) nonisothermal $T_e/T_i=2$ cases; $k/k_e=0.1$; $\theta=10°$; $\phi=45°$.

$$D_{xx}^r = \frac{\beta^2 D_{xx} + 2\beta\Omega D_{xy} + \Omega^2 D_{yy}}{(\beta^2 + \Omega^2)^2},$$

$$D_{yy}^r = \frac{\Omega^2 D_{xx} - 2\beta\Omega D_{xy} + \beta^2 D_{yy}}{(\beta^2 + \Omega^2)^2}, \tag{47}$$

$$D_{xy}^r = \frac{(\beta^2 - \Omega^2) D_{xy} - \beta\Omega (D_{xx} - D_{yy})}{(\beta^2 + \Omega^2)^2}.$$

Here and in what follows, we omit the subscript σ in all the cases when it does not lead to misunderstandings.

As we can see even with $D_{xy}=0$, anisotropy ($D_{xx} \neq D_{yy}$) generates an off-diagonal term for the diffusion in the configuration space.

It is convenient to present the results for anisotropic diffusion in the traditional form. For this purpose, we introduce the notation

$$D_L^r = \frac{k_i k_j}{k^2} D_{ij}^r. \tag{48}$$

In terms of this notation,

$$\chi_\sigma(\mathbf{k},\omega) = i \frac{k_\sigma^2}{k^2} \frac{k^2 D_{L\sigma}^r}{\omega + i k^2 D_{L\sigma}^r}, \tag{49}$$

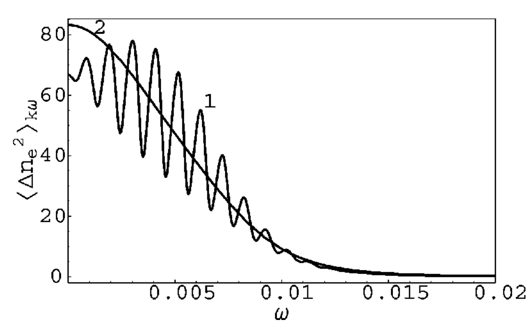

FIG. 9. Diffusion influence on the low-frequency fluctuation spectra fluctuation spectra at large θ: (1) $\Delta=0.001$, (2) $\Delta=1$; $k/k_e=0.1$, $T_e/T_i=1$, $\theta=85°$, $\phi=45°$.

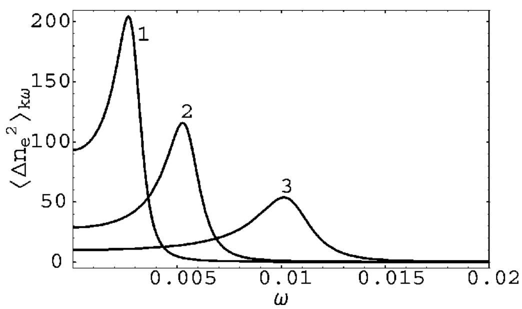

FIG. 10. Fluctuation spectra in nonisothermal plasma for different values of k/k_e: (1) $k/k_e=0.05$, (2) $k/k_e=0.1$, (3) $k/k_e=0.2$; $\Delta=0.3$; $T_e/T_i=2$; $\theta=10°$, $\phi=45°$.

$$\langle \delta n_\sigma^{(0)2} \rangle_{\mathbf{k}\omega} = \frac{2n_\sigma k^2 D^r_{L\sigma}}{|\omega + ik^2 D^r_{L\sigma}|^2}. \tag{50}$$

Equations (49) and (50) are of the same form as for the isotropic case. It is necessary to remember that the quantity D^r_L is dependent on the direction of $\mathbf{k} = (k\sin\theta\cos\phi; k\sin\theta\sin\phi; k\cos\theta)$,

$$D^r_L \equiv D^2_L(\theta,\phi) = [D^r_{xx}\cos^2\phi + D^r_{yy}\sin^2\phi + (D^r_{xy} + D^r_{yx})\sin\phi\cos\phi]\sin^2\theta + D^r_{zz}\cos^2\theta. \tag{51}$$

Using Eq. (47), it reads

$$D^r_L(\theta,\phi) = \left[\frac{D_{xx}(\beta\cos\phi - \Omega\sin\phi)^2 + D_{yy}(\Omega\cos\phi + \beta\sin\phi)^2}{(\beta^2+\Omega^2)^2} + \frac{D_{xy}[2\beta\Omega\cos 2\phi + (\beta^2 - \Omega^2)\sin 2\phi]}{(\beta^2+\Omega^2)^2}\right]\sin^2\theta + \frac{D_{zz}}{\beta^2}\cos^2\theta. \tag{52}$$

Equations (34), (49), and (50) make it possible to find simple analytic formulas for correlation functions of particle density fluctuations. In the case of individual-particle fluctuations ($k \gg k_\sigma$)

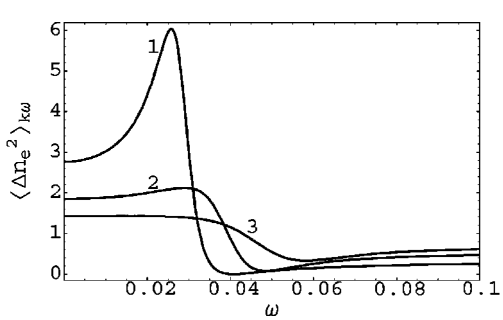

FIG. 11. Fluctuation spectra in nonisothermal plasma for different values of k/k_e: (1) $k/k_e=0.6$, (2) $k/k_e=0.9$, (3) $k/k_e=1.2$; $\Delta=0.3$; $T_e/T_i=2$; $\theta=10°$, $\phi=45°$.

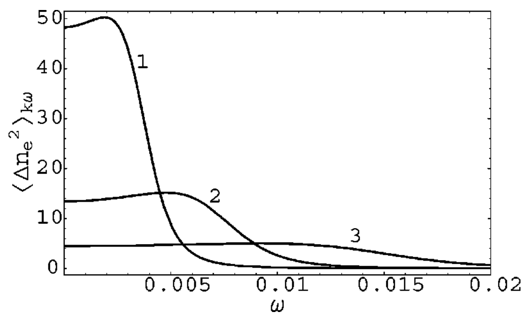

FIG. 12. Fluctuation spectra in isothermal plasma for different values of k/k_e: (1) $k/k_e=0.05$, (2) $k/k_e=0.1$, (3) $k/k_e=0.2$; $\Delta=0.3$; $T_e/T_i=1$; $\theta=10°$, $\phi=45°$.

$$\langle \delta n_e^2 \rangle_{\mathbf{k}\omega} \simeq \langle \delta n_e^{(0)2} \rangle_{\mathbf{k}\omega} = \frac{2n_e k^2 D^r_{Le}}{|\omega + ik^2 D^r_{Le}|^2}, \tag{53}$$

i.e., correlations are exhausted by those associated with the Langevin sources and collective effects do not contribute to the fluctuation spectra.

In the collective region ($k \ll k_\sigma$, i.e., $R \gg \lambda_D$, where λ_D is the Debye length),

$$\langle \delta n_e^2 \rangle_{\mathbf{k}\omega} = 2n_e \frac{k^2 D^r_A}{|\omega + ik^2 D^r_A|^2}\, \frac{k_i^4 D^r_{Li} + k_e^4 D^r_{Le}}{(k_e^2 + k_i^2)(k_e^2 D^r_{Le} + k_i^2 D^r_{Li})}, \tag{54}$$

where

$$D^r_A = \frac{(k_e^2 + k_i^2) D^r_{Le} D^r_{Li}}{k_e^2 D^r_{Le} + k_i^2 D^r_{Li}}.$$

In the equilibrium case ($T_i = T_e$), Eq. (54) reduces to

$$\langle \delta n_e^2 \rangle_{\mathbf{k}\omega} = 2n_e \frac{k^2 D^r_A}{|\omega + ik^2 D^r_A|^2}. \tag{55}$$

Similarly, it is possible to find the ion correlations

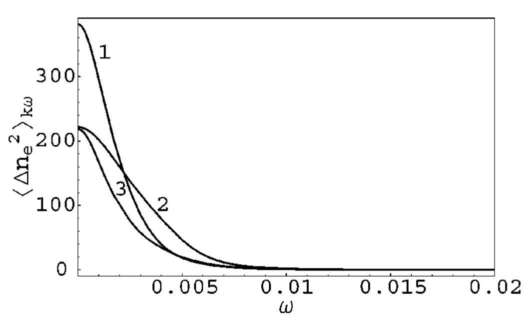

FIG. 13. Fluctuation spectra in isothermal plasma for anisotropic case and different angles ϕ ($k/k_e=0.05$): (1) $\phi=0°$, (2) $\phi=45°$, (3) $\phi=90°$; $\Delta_{xx}=0.1$; $\Delta_{yy}=1.9$; $\Delta_{zz}=0.1$; $\Delta_{xy}=0$; $T_e/T_i=1$; $\theta=85°$.

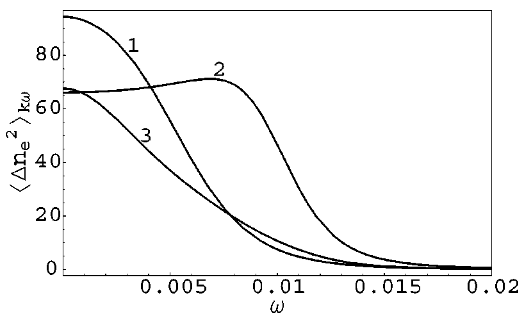

FIG. 14. Fluctuation spectra in isothermal plasma for anisotropic case and different angles ϕ (k/k_e=0.1): (1) ϕ=0°, (2) ϕ=45°, (3) ϕ=90°; Δ_{zz}=0.1; Δ_{xy}=0; T_e/T_i=1; θ=85°.

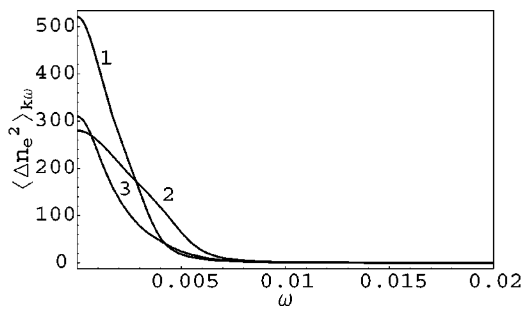

FIG. 16. Fluctuation spectra in nonisothermal plasma for anisotropic case and different angles ϕ (k/k_e=0.05): (1) ϕ=0°, (2) ϕ=45°, (3) ϕ=90°; Δ_{xx}=0.1; Δ_{yy}=1.9; Δ_{zz}=0.1; Δ_{xy}=0; T_e/T_i=2; θ=85°.

$$\langle \delta n_i^2 \rangle_{\mathbf{k}\omega} = 2n_i \frac{k^2 D_A^r}{|\omega + ik^2 D_A^r|^2} \, \frac{k_i^4 D_{Li}^r + k_e^4 D_{Le}^r}{(k_e^2 + k_i^2)(k_e^2 D_{Le}^r + k_i^2 D_{Li}^r)}. \tag{56}$$

Obviously in the case of a strong magnetic field ($|\Omega_\sigma| \gg \beta_\sigma$),

$$D_A \sim 2D_e^r \ll D_i^r,$$

i.e., collective effects result in considerable reduction of ion diffusion.

The analytical expressions presented were obtained in the asymptotic limits. In the general case, however, a detailed description of fluctuation spectra requires numerical analysis of Eqs. (30)–(33).

The results of calculations of electron density fluctuation spectra are presented in Figs. 6–18. In these figures, frequency is normalized by the electron cyclotron frequency and the fluctuation spectral intensity $\langle \delta n_\sigma^2 \rangle_{\mathbf{k}\omega}$ is normalized by n/Ω_{ce}, where $n=n_e=n_i$ is the averaged particle density. The diffusion coefficient is determined through the dimensionless parameter $\Delta \equiv D^r k_e^2/\omega_{pe}$ with the simplified relation $D^r = D_\sigma/\Omega_{c\sigma}^2$ set to be the same for electrons and ions. For all calculations, we set the ratio $\omega_{pe}/\Omega_{ce}=1.1$ and $m_i/m_e=10^3$. The direction of the vector $\mathbf{k}$ with respect to the main axes is determined by two angles θ and ϕ. The external magnetic field has only a z component $\mathbf{B}_0=(0,0,B_0)$.

The range of Δ which we use here is chosen to demonstrate the most pronounced features of spectra. However, it corresponds to realistic plasma parameters ($D^r \sim 1$ m^2/s for fusion devices and $D^r \sim 100$ m^2/s for Aurora [15]).

Calculations show that the details of spectral distributions in the low-frequency domain ($\omega \ll \omega_{pe}, \Omega_{ce}$) are considerably dependent on the values of $\mathbf{k}$ and its direction. The influence of particle friction and diffusion can also be pronounced.

For small values of θ ($\theta < 30°$), the electron density fluctuation spectra are quite similar to those for plasmas with no external magnetic field (Figs. 6 and 7). If diffusion is weak ($\Delta \lesssim 0.1$), the appropriate curves on these figures recover the dependencies obtained for collisionless plasmas (curves No. 1). Namely, in the case of isothermal plasma, one observes a Gaussian profile with deformation near zero frequency.

In nonisothermal plasmas, the resonancelike maximum at $\omega \simeq kc_s \cos\theta$ [where $c_s=(T_e/m_i)^{1/2}$ is the ion sound velocity] is a well-pronounced feature of the spectrum. The increase of Δ leads to a transformation of a Gaussian curve into a Lorentzian-like one. Regarding nonisothermal plasmas, a broadening of the collective maximum associated with the fluctuating wave excitation is observed.

At large values of θ, the influence of the external magnetic field can be dominant. It results (for $\Delta \lesssim 10^{-3}$) in a resonant series at $\omega \simeq N\Omega_{ci}$ generated by excitation of

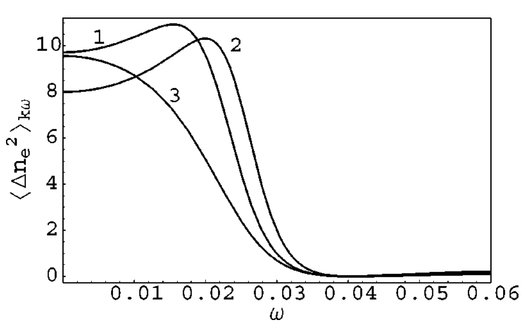

FIG. 15. Fluctuation spectra in isothermal plasma for anisotropic case and different angles ϕ (k/k_e=0.3): (1) ϕ=0°, (2) ϕ=45°, (3) ϕ=90°; Δ_{xx}=0.1; Δ_{yy}=1.9; Δ_{zz}=0.1; Δ_{xy}=0; T_e/T_i=1; θ=85°.

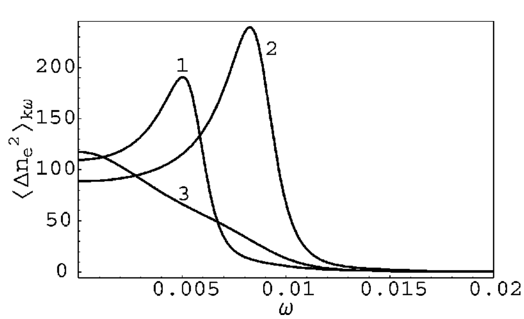

FIG. 17. Fluctuation spectra in nonisothermal plasma for anisotropic case and different angles ϕ (k/k_e=0.1): (1) ϕ=0°, (2) ϕ=45°, (3) ϕ=90°; Δ_{xx}=0.1; Δ_{yy}=1.9; Δ_{zz}=0.1; Δ_{xy}=0; T_e/T_i=2; θ=85°.

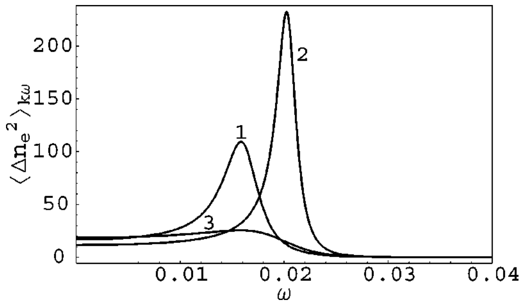

FIG. 18. Fluctuation spectra in nonisothermal plasma for anisotropic case and different angles ϕ ($k/k_e=0.3$): (1) $\phi=0°$, (2) $\phi=45°$, (3) $\phi=90°$; $\Delta_{xx}=0.1$; $\Delta_{yy}=1.9$; $\Delta_{zz}=0.1$; $\Delta_{xy}=0$; $T_e/T_i=2$; $\theta=85°$.

Bernstein waves. At $\Delta \gtrsim 0.1$, these resonances are strongly damped (Fig. 9).

The increase of k leads to a weakening of collective effects; spectral density considerably decreases (Figs. 10–12).

The influence of diffusion anisotropy ($\Delta_{xx}=0.1$, $\Delta_{yy}=1.9$, which corresponds to 95% anisotropy in the x-y plane) leads to the dependence of the spectrum on the angle ϕ (Figs. 13–18). This dependence is observed in both isothermal (Figs. 13–15) and nonisothermal (Figs. 16–18) cases. As is seen, anisotropy of the spectrum for the case under consideration can be explained by a transition from the weakly collisional ($\phi=0°$) to the diffusive regime ($\phi=90°$).

VI. SUMMARY AND CONCLUSIONS

Using the generalized Liouville equation, we formulated the kinetic equation for the test particle distribution function in the presence of an external magnetic field and obtained an explicit solution of the initial-value problem for particle transition probability in the phase space. The Fokker-Planck collision term in the most general form (anisotropic friction coefficient and the presence of off-diagonal elements of the diffusion coefficient in velocity space) was used to obtain the solution.

The mean-square displacements in configuration and velocity spaces were calculated. It was shown that in the case of small friction coefficients ($\beta/\Omega \ll 1$), particle diffusion is accompanied by oscillations of mean-square displacements. A transition from a cubic time dependence at the initial stage to the classical diffusion regime (linear dependence) was observed. The mean-square velocity displacement in such cases manifests a linear time dependence and saturation, respectively.

It was shown that in the case of an anisotropic spectrum of random forces (anisotropic diffusion coefficients with off-diagonal components), the mean-square displacements can be considerably different from those for the symmetric diffusion.

We have formulated general relations for the description of large-scale fluctuations in the system under consideration. Their reduction to the drift-diffusion and collisionless limits is done.

We have also presented a detailed analysis of electron density fluctuation spectra for various sets of parameters. It is shown that particle diffusion influences fluctuations considerably. In particular, it leads to a change of the spectral shape and its broadening. Anisotropy of the diffusion coefficients in velocity space generates an angular dependence of the spectrum in the plane perpendicular to the external magnetic field. This dependence is particularly important if different regimes of fluctuation propagation (weakly collisional, or diffusive) are dominant for different directions.

Although the obtained results can be treated to some extent as qualitative due to assumptions we made, they give a better understanding of low-frequency plasma turbulence. As an example of practical importance, we should mention the application of the theory presented to the collective scattering diagnostics of fusion and ionospheric plasmas [15].

APPENDIX

Coefficients $a_{ij}(\tau)$ are the following:

$$a_{11}(\tau) = e^{(\beta_x+\beta_y)\tau}\Bigg\{ D_{xx}\cos^2\tilde{\Omega}\tau + \frac{2}{\tilde{\Omega}}\left(\frac{\beta_x-\beta_y}{2}D_{xx} - \Omega D_{xy}\right) \times\cos\tilde{\Omega}\tau\sin\tilde{\Omega}\tau + \frac{1}{\tilde{\Omega}^2}\left[\left(\frac{\beta_x-\beta_y}{2}\right)^2 D_{xx} - (\beta_x-\beta_y)\Omega D_{xy} + \Omega^2 D_{yy}\right]\sin^2\tilde{\Omega}\tau\Bigg\},$$

$$a_{22}(\tau) = e^{(\beta_x+\beta_y)\tau}\Bigg\{ D_{yy}\cos^2\tilde{\Omega}\tau + \frac{2}{\tilde{\Omega}}\left(-\frac{\beta_x-\beta_y}{2}D_{yy} + \Omega D_{xy}\right) \times\cos\tilde{\Omega}\tau\sin\tilde{\Omega}\tau + \frac{1}{\tilde{\Omega}^2}\left[\left(\frac{\beta_x-\beta_y}{2}\right)^2 D_{yy} - (\beta_x-\beta_y)\Omega D_{xy} + \Omega^2 D_{xx}\right]\sin^2\tilde{\Omega}\tau\Bigg\},$$

$$a_{33}(\tau) = \frac{\beta_y^2 D_{xx} + 2\Omega\beta_y D_{xy} + \Omega^2 D_{yy}}{(\beta_x\beta_y + \Omega^2)^2},$$

$$a_{44}(\tau) = \frac{\beta_x^2 D_{yy} - 2\Omega\beta_x D_{xy} + \Omega^2 D_{xx}}{(\beta_x\beta_y + \Omega^2)^2},$$

$$a_{12}(\tau) = e^{(\beta_x+\beta_y)\tau}\Bigg\{ D_{xy}\cos^2\tilde{\Omega}\tau + \frac{1}{\tilde{\Omega}}(D_{xx}-D_{yy})\cos\tilde{\Omega}\tau\sin\tilde{\Omega}\tau - \frac{1}{\tilde{\Omega}^2}\left[\left(\frac{\beta_x-\beta_y}{2}\right)^2 D_{xy} - 2\Omega\frac{\beta_x-\beta_y}{2}\frac{D_{xx}+D_{yy}}{2} + \Omega^2 D_{xy}\right]\sin^2\tilde{\Omega}\tau\Bigg\},$$

$$a_{13}(\tau) = \frac{e^{(1/2)(\beta_x+\beta_y)\tau}}{\beta_x\beta_y + \Omega^2}\left\{(\beta_y D_{xx} + \Omega D_{xy})\cos\tilde{\Omega}\tau + \frac{1}{\tilde{\Omega}}\left[\beta_y\left(\frac{\beta_x-\beta_y}{2}D_{xx} - \Omega D_{xy}\right) + \Omega\left(\frac{\beta_x-\beta_y}{2}D_{xy} - \Omega D_{yy}\right)\right]\sin\tilde{\Omega}\tau\right\},$$

$$a_{14}(\tau) = \frac{e^{(1/2)(\beta_x+\beta_y)\tau}}{\beta_x\beta_y + \Omega^2}\left\{(\beta_x D_{xy} - \Omega D_{xx})\cos\tilde{\Omega}\tau + \frac{1}{\tilde{\Omega}}\left[\beta_x\left(\frac{\beta_x-\beta_y}{2}D_{xy} - \Omega D_{yy}\right) - \Omega\left(\frac{\beta_x-\beta_y}{2}D_{xx} - \Omega D_{xy}\right)\right]\sin\tilde{\Omega}\tau\right\},$$

$$a_{23}(\tau) = \frac{e^{(1/2)(\beta_x+\beta_y)\tau}}{\beta_x\beta_y + \Omega^2}\left\{(\beta_y D_{xy} + \Omega D_{yy})\cos\tilde{\Omega}\tau + \frac{1}{\tilde{\Omega}}\left[-\beta_y\left(\frac{\beta_x-\beta_y}{2}D_{xy} - \Omega D_{xx}\right) - \Omega\left(\frac{\beta_x-\beta_y}{2}D_{yy} - \Omega D_{xy}\right)\right]\sin\tilde{\Omega}\tau\right\},$$

$$a_{24}(\tau) = \frac{e^{(1/2)(\beta_x+\beta_y)\tau}}{\beta_x\beta_y + \Omega^2}\left\{(\beta_x D_{yy} - \Omega D_{xy})\cos\tilde{\Omega}\tau + \frac{1}{\tilde{\Omega}}\left[-\beta_x\left(\frac{\beta_x-\beta_y}{2}D_{yy} - \Omega D_{xy}\right) + \Omega\left(\frac{\beta_x-\beta_y}{2}D_{xy} - \Omega D_{xx}\right)\right]\sin\tilde{\Omega}\tau\right\},$$

$$a_{34}(\tau) = \frac{\beta_x\beta_y D_{xy} + \Omega(\beta_x D_{yy} - \beta_y D_{xx} - \Omega D_{xy})}{(\beta_x\beta_y + \Omega^2)^2}.$$

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