Scaling laws for noise-induced superpersistent chaotic transients

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A superpersistent chaotic transient is characterized by the following scaling law for its average lifetime: $\tau \sim \exp[C(p-p_c)^{-\alpha}]$, where $C>0$ and $\alpha>0$ are constants, $p \geqslant p_c$ is a bifurcation parameter, and p_c is its critical value. As p approaches p_c from above, the exponent in the exponential dependence diverges, leading to an extremely long transient lifetime. Historically the possibility of such transient raised the question of whether asymptotic attractors are relevant to turbulence. Here we investigate the phenomenon of noise-induced superpersistent chaotic transients. In particular, we construct a prototype model based on random maps to illustrate this phenomenon. We then approximate the model by stochastic differential equations and derive the scaling laws for the transient lifetime versus the noise amplitude ε for both the subcritical ($p<p_c$) and the supercritical ($p>p_c$) cases. Our results are the following. In the subcritical case where a chaotic attractor exists in the absence of noise, noise-induced transients can be more persistent in the following sense of double-exponential and algebraic scaling: $\tau \sim \exp[K_0 \exp(K_1 \varepsilon^{-\gamma})]$ for small noise amplitude ε, where $K_0>0$, $K_1>0$, and $\gamma>0$ are constants. The longevity of the transient lifetime in this case is striking. For the supercritical case where there is already a superpersistent chaotic transient, noise can significantly reduce the transient lifetime. These results add to the understanding of the interplay between random and deterministic chaotic dynamics with surprising physical implications.

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I. INTRODUCTION

Transient chaos is ubiquitous in nonlinear dynamical systems [1,2]. In such a case, dynamical variables of the system behave chaotically for a finite amount of time before settling into a final state that is usually not chaotic. A common situation for transient chaos to arise is where the system undergoes a *crisis* at which a chaotic attractor collides with the basin boundary separating it and another coexisting attractor [1]. Consider, for example, a three-dimensional flow or equivalently, a two-dimensional invertible map. After the crisis, the chaotic attractor is destroyed and converted into a nonattracting chaotic invariant set (chaotic saddle). Dynamically, a trajectory then wanders in the vicinity of the chaotic saddle for a period of time before approaching asymptotically to the other attractor. Chaotic transients of this sort are usually not "superpersistent" in the sense that their average lifetimes scale with the system parameter only algebraically. Specifically, let p be a system parameter and assume that as p is increased a crisis occurs at the critical parameter value p_c. There is thus transient chaos for $p>p_c$. It is well established both theoretically [3] and experimentally [4] that the average lifetime τ of the chaotic transients scales with the parameter variation algebraically:

$$\tau \sim (p-p_c)^{-h}, \quad p > p_c, \tag{1}$$

where $h>0$ is the algebraic scaling exponent. Since noise can trigger the occurrence of a crisis [5], transient chaos which is "regular" in the sense of the scaling law (1), can also be induced by noise.

There exists, however, another distinct class of transient chaos–superpersistent chaotic transients. They are characterized by the following scaling law for their lifetime [6]:

$$\tau \sim \exp[C(\Delta p)^{-\chi}], \tag{2}$$

where $\Delta p = p - p_c$, $C>0$ and $\chi>0$ are constants. As p approaches the critical value p_c from above, the transient lifetime τ becomes superpersistent in the sense that the exponent in the exponential dependence diverges. This type of chaotic transients was conceived to occur through the dynamical mechanism of unstable-unstable pair bifurcation, in which an unstable periodic orbit in the boundary of a chaotic attractor collides with another unstable periodic orbit preexisted outside the set [6]. The same mechanism was believed to cause the riddling bifurcation [7] that creates a riddled basin [8], so superpersistent chaotic transients can be expected at the onset of riddling. Unstable-unstable pair bifurcation is also key to the dynamical phenomenon of bubbling [9]. Earlier the transients were also identified in a class of coupled-map lattices, leading to the speculation that asymptotic attractors may not be relevant for turbulence [10]. Recently, noise-induced superpersistent chaotic transients were demonstrated [11] in phase synchronization [12,13] of weakly coupled chaotic oscillators. In addition, signatures of noise-induced superpersistent chaotic transients were found [14] in the advective dynamics of inertial particles in open fluid flows [15].

In this paper, we investigate the scaling laws for the lifetimes of noise-induced superpersistent chaotic transients. Consider, in the noiseless case, a chaotic attractor and its basin of attraction. The attractor, by being chaotic, has naturally embedded within itself an infinite number of unstable periodic orbits. A subset of these orbits can be accessible to the basin boundary in the sense that a path of finite length can be found which connects a periodic-orbit point to some point on the basin boundary. Likewise, there can be a subset of periodic orbits on the basin boundary that are accessible to the attractor. When noise is present, there can be a nonzero probability that two periodic orbits, one belonging to the accessible set on the attractor and another to the set on the basin boundary, can get close and coalesce temporally, giving rise to a nonzero probability that a trajectory on the chaotic attractor crosses the basin boundary and moves to the basin of another attractor. Transient chaos thus arises. Due to noise, the channels through which trajectory escapes from the chaotic attractor open and close intermittently in time. Because of chaos, the probability of escape is extremely small, as escaping through the channel requires staying of the trajectory in a small vicinity of the opening of the channel for a finite amount of time, which is an event with extremely small probability. In this sense, the channel must be "super" narrow [6,7], leading to a superpersistent chaotic transient. The creation of the channel by noise and the noisy dynamics in the channel are thus the key to understanding the noise-induced transient behavior.

To make analytic derivations and numerical computations possible, we construct a prototype class of two-dimensional, noninvertible maps as paradigmatic models for superpersistent chaotic transients. We will argue that the chaotic transient lifetime is determined by the probability that some escaping channel becomes open and, more importantly, the dynamics in the channel. In the presence of small noise, for both the subcritical and supercritical cases, there is a finite probability for the escaping channels to open, and the dynamics in the channel can be approximated by stochastic different equations, the solutions to which can be obtained by using the Fokker-Planck equation under appropriate initial and boundary conditions.

Our principal results of amplitude ε are the following. (1) In the critical case ($p=p_c$), a small noise induces transient chaos with average lifetime obeying the normal superpersistent scaling law, in the following sense:

$$\tau \sim \exp(C\varepsilon^{-\alpha}), \tag{3}$$

where $C>0$ is a constant and $\alpha>0$ is the algebraic scaling exponent in the exponential dependence of τ. While we expect the scaling law to be general, these constants are system-dependent. (2) In the subcritical case ($p<p_c$) where a chaotic attractor exists in the absence of noise, for relatively large noise (say $\varepsilon>\varepsilon_c$, where ε_c depends on $|p_c-p|$ and $\varepsilon_c\rightarrow 0$ as $p\rightarrow p_c$), the average lifetime of the noise-induced chaotic transients obeys the normal scaling law (3). However, for small noise ($\varepsilon<\varepsilon_c$), the average lifetime scales with the noise amplitude ε according to the following double exponential and algebraic law:

$$\tau \sim \exp[K_0 \exp(K_1\varepsilon^{-\gamma})] \quad \text{for } p < p_c, \tag{4}$$

where $K_0>0$, $K_1>0$, and $\gamma>0$ are constants. Because of the double exponential dependence and the algebraic divergence for small noise, the resulted transient lifetime can be *significantly longer* than that given by the normal superpersistent scaling law (2). We call such transients *extraordinarily-superpersistent chaotic transients* [16]. (3) For the supercritical case ($p>p_c$) where there is already a chaotic transient, the lifetime has no dependence on the noise amplitude if it is small. However, for relatively large noise, the lifetime decreases following the normal superpersistent scaling law (3). Thus, in this case the chaotic transient lifetime in the presence of noise can be *significantly shorter* than that in the absence of noise. These findings should be useful for better understanding the interplay between noise and chaotic dynamics.

In Sec. II, we introduce our prototype model for superpersistent chaotic transients. In Sec. III, we analyze the tunneling time based on the corresponding class of stochastic differential equations and derive noisy scaling laws for the transient lifetimes for all three cases: critical, supercritical, and subcritical. In Sec. IV, we present numerical verification. A discussion is presented in Sec. V.

II. MODEL FOR NOISE-INDUCED SUPERPERSISTENT CHAOTIC TRANSIENTS

A. Dynamical mechanism for noise-induced superpersistent chaotic transients

There are two essential features that lead to a superpersistent chaotic transient: (1) the finite time it takes for a trajectory to escape through a stochastic "channel" from a chaotic attractor, and (2) the extremely small probability for a chaotic trajectory to spend this finite amount of time in a small location around the opening of the channel on the attractor. The location of an opening can be a fixed point, or a component of an unstable periodic orbit in the chaotic attractor. In the subcritical case, there is a chaotic attractor and no escaping channel exists in the absence of noise. In this case, the channel is induced by noise and it opens and closes randomly in time. In the supercritical case, the channel is open and there is already a superpersistent chaotic transient. The presence of noise affects the deterministic dynamics in the channel. In either case, the dynamics in the channel can be regarded as being driven by a stochastic force and, hence, it can be modeled by a stochastic differential equation, the solution to which gives the tunneling time through the channel. Apparently, this time depends on the noise amplitude. The dependence, in combination with the small probability for a trajectory to move to the opening of the channel and to stay there for the duration of the tunneling time, gives the scaling of the average lifetime of the superpersistent chaotic transients with the noise amplitude.

Previous works suggested unstable-unstable pair bifurcation as the generic mechanism for superpersistent chaotic transients [6,7]. One can imagine two unstable periodic orbits of the same periods, one on the chaotic attractor and another on the basin boundary, as shown in Fig. 1(a). In a noiseless situation, as a bifurcation parameter passes through its critical value, the two orbits *coalesce* and disappear simultaneously, leaving behind a "channel" in the phase space through which trajectories on the chaotic attractor can escape. The chaotic attractor is thus converted into a chaotic transient, but the channel created by this mechanism is typically super narrow [6,7]. From Fig. 1(a), we see that if the phase space is two dimensional, the periodic orbit on the attractor must be a saddle and the one on the basin boundary must be a repeller. This can arise if the map is noninvertible. Thus, the bifurcation can occur in invertible maps of at least

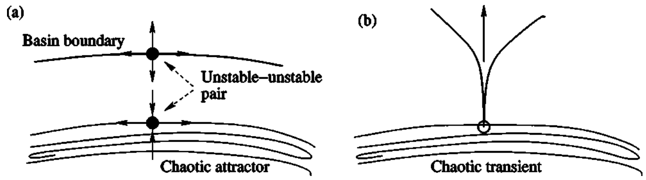

FIG. 1. (a) In the absence of noise, a chaotic attractor, the basin boundary, and the pair of unstable periodic orbits. (b) Escaping channel induced by noise through the mechanism of unstable-unstable pair bifurcation, converting the originally attracting motion into a chaotic transient.

dimension three, or in flows of dimension of at least four. Our interest is in transient chaos induced by noise. If the attractor is close to the basin boundary, noise can induce an unstable-unstable pair bifurcation, creating a narrow channel through which trajectories can escape, as shown schematically in Fig. 1(b). For small noise, the probability for the channel to open is small, but there can be a nonzero probability for trajectories to move to the location of the channel for some amount of time to escape through the channel while it is open. Suppose on average, it takes time T for a trajectory to travel through the channel in the phase space so that it is no longer on the attractor, we expect T to increase as the noise amplitude ε is decreased, because the probability for channel to remain open is smaller for weaker noise. In fact, as we will argue below, we expect T to increase at least algebraically as ε is decreased and, $T\to\infty$ as $\varepsilon\to 0$.

B. A prototype model for superpersistent chaotic transients

Let $\lambda>0$ be the largest Lyapunov exponent of the chaotic attractor. After an unstable-unstable pair bifurcation the opened channel is locally transverse to the attractor. In order for a trajectory to escape, it must spend at least time T at the location of the opening on the attractor centered around an unstable periodic orbit—the *mediating* orbit involved in the unstable-unstable pair bifurcation. The trajectory must come to within distance of about $\exp[-\lambda T(\varepsilon)]$ from this orbit. The probability for this to occur is proportional to $\exp[-\lambda T(\varepsilon)]$. The average time for the trajectory to remain on the attractor, or the average transient lifetime, is thus

$$\tau \sim \exp[\lambda T(\varepsilon)]. \tag{5}$$

We see that the dependence of $T(\varepsilon)$ on ε, which is the average time that trajectories spend in the escaping channel, or the *tunneling time*, is the key quantity determining the scaling of the average chaotic transient lifetime τ.

To obtain the scaling dependence of the tunneling time $T(\epsilon)$ on ϵ, we note that, since the escaping channel is extremely narrow, for typical situations where $\lambda>0$ and $T(\epsilon)$ large, the dynamics in the channel is approximately one dimensional along which the periodic orbit on the attractor is stable but the orbit on the basin boundary is unstable for $p<p_c$ [Fig. 1(a)]. This feature can thus be captured through the following simple one-dimensional map:

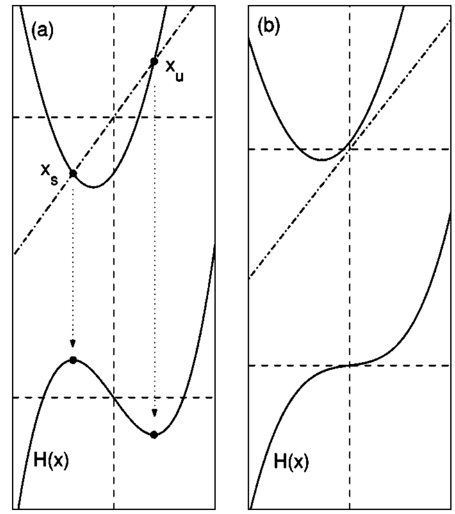

FIG. 2. For the prototype model Eq. (6), (a) the stable and unstable fixed points for the subcritical cases ($p<0$), and (b) the supercritical case ($p>0$). The function $H(x)$ will be defined in Eq. (13).

$$x_{n+1} = x_n^2 + x_n + p + \epsilon\xi(n), \tag{6}$$

where x denotes the dynamical variable in the channel, p is a bifurcation parameter with critical point $p_c=0$, ϵ is the noise amplitude, and $\xi(n)$ is a Gaussian random variable of zero mean and unit variance. For $p<p_c=0$, the map has a stable fixed point $x_s=-\sqrt{-p}$ and an unstable fixed point $x_u=\sqrt{-p}$. These two collide at p_c and disappear for $p>p_c$, mimicking an unstable-unstable pair bifurcation.

We are thus motivated to analyze the tunneling time using the following general class of one-dimensional random maps:

$$x_{n+1} = x_n^{k-1} + x_n + p + \varepsilon\xi(n), \tag{7}$$

where $k\geqslant 3$ is an odd integer so as to generate a pair of fixed points with different unstable dimension. If the tunneling time is $T\gg 1$, Eq. (7) can be approximated by

$$\frac{dx}{dt} = x^{k-1} + p + \varepsilon\xi(t). \tag{8}$$

For $p<0$, the deterministic system for Eq. (8) has a stable fixed point $x_s=-|p|^{1/(k-1)}$ and an unstable fixed point $x_u=|p|^{1/(k-1)}$, but there are no more fixed points for $p>0$, as shown in Fig. 2. Let $x_r=x_s$ for $p<0$ and $x_r=0$ for $p\geqslant 0$, and let T_p^k be the tunneling time. We will show that a properly formulated first-passage-time problem for this one-dimensional stochastic process yields the scaling of T_p^k with the noise amplitude ε.

III. SCALING THEORY FOR AVERAGE TUNNELING TIME AND AVERAGE CHAOTIC TRANSIENT LIFETIME

Let $P(x,t)$ be the probability density function of the stochastic process governed by Eq. (8). This density function satisfies the Fokker-Planck equation [17]:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}[(x^{k-1}+p)P(x,t)] + \frac{\varepsilon^2}{2}\frac{\partial^2 P}{\partial x^2}. \tag{9}$$

Let l be the effective length of the channel in the sense that a trajectory with $x > l$ is considered to have escaped the channel. The time required for a trajectory to travel through the channel is equivalent to the mean first passage time T from x_r to l. Our interest is in the trajectories that do escape. For such a trajectory, we assume that once it falls into the channel through x_r, it will eventually exit the channel at $x = l$ without even going back to the original chaotic attractor. This is reasonable considering that the probability for a trajectory to fall in the channel and then to escape is already exponentially small [Eq. (5)] and, hence, the probability for any "second-order" process to occur, where a trajectory falls in the channel, moves back to the original attractor, and falls back in the channel again, is negligible. For trajectories in the channel there is thus a reflecting boundary condition at $x = x_r$:

$$\left[P(x,t) - \frac{\partial P}{\partial x}\right]\bigg|_{x=x_r} = 0. \tag{10}$$

That trajectories exit the channel at $x = l$ indicates an absorbing boundary condition at $x = l$:

$$P(l,t) = 0. \tag{11}$$

Assuming that trajectories initially are near the opening of the channel (but in the channel), we have the initial condition

$$P(x,x_r) = \delta(x - x_r^+). \tag{12}$$

Under these boundary and initial conditions, the solution to the Fokker-Planck equation yields the following mean first-passage time [17] for the stochastic process (8):

$$T_p^k(\varepsilon) = \frac{2}{\varepsilon^2}\int_{x_r}^{l} dy \exp[-bH(y)]\int_{x_r}^{y} \exp[bH(y')]dy', \tag{13}$$

where $H(x) = (x^k + kpx)$ and $b = 2/(k\varepsilon^2)$.

A. The critical case $p = p_c = 0$

In the critical case, we have $H(y) = y^k$. The series expansion

$$\exp(bw^k) = \sum_{n=0}^{\infty}\frac{(bw^k)^n}{n!}$$

yields

$$\int_0^y \exp[bH(w)]dw = \sum_{n=0}^{\infty}\frac{b^n y^{kn+1}}{n!(kn+1)}.$$

Letting $q = by^k$, we have

$$\begin{aligned} T_0^k(\varepsilon) &= \frac{2}{\varepsilon^2}\int_0^l \sum_{n=0}^{\infty}\frac{(by^k)^n}{n!(kn+1)} y \exp(-by^k)dy \\ &= \frac{2}{\varepsilon^2}\sum_{n=0}^{\infty}\frac{1}{n!(kn+1)}\int_0^{l'} q^n \exp(-q)\frac{1}{kb}\left(\frac{q}{b}\right)^{-(k-2)/k} dq \\ &\sim \varepsilon^{-2+4/k}\sum_{n=0}^{\infty}\frac{1}{n!(kn+1)k}\int_0^{l'} q^{n-(k-2)/k}\exp(-q)dq \\ &\sim \varepsilon^{-2+4/k}\sum_{n=0}^{\infty}\frac{1}{n!(kn+1)k}\Gamma(n+2/k), \end{aligned} \tag{14}$$

where $l' = bl^k$ and $\Gamma(x) = \int_0^\infty t^{x-1}\exp(-t)dt$ is the gamma function. In order to prove that the infinite series

$$\sum_{n=0}^{\infty}\frac{1}{n\,!\,(kn+1)k}\Gamma(n+2/k) \tag{15}$$

is convergent, we make use of the double-inequality lemma for the gamma function [18],

$$\frac{b^{b-1}}{a^{a-1}}\exp(a-b) < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}\exp(a-b)\ (b > a \geqslant 1).$$

We thus obtain the upper bound of $\Gamma(n+2/k)$:

$$\begin{aligned} \Gamma(n+2/k) &< \Gamma(n+1)\exp(1-2/k)\frac{(n+2/k)^{n-1+2/k}}{(n+1)^n} \\ &< \Gamma(n+1)\exp(1-2/k)(n+2/k)^{-1+2/k}. \end{aligned} \tag{16}$$

Since $\Gamma(n+1) = n!$, we have

$$\begin{aligned} \sum_{n=0}^{\infty}\frac{1}{n!(kn+1)k}\Gamma(n+2/k) &< \frac{\exp(1-2/k)}{k^2}\sum_{n=0}^{\infty}\frac{1}{(n+1/k)^{2-2/k}} \\ &< \infty. \end{aligned} \tag{17}$$

The convergence of the infinite series (15) implies the following scaling relation between the tunneling time and the noise amplitude for the critical case:

$$T_0^k(\varepsilon) \sim \varepsilon^{-(2-4/k)}. \tag{18}$$

Clearly we have $T_0^k(\varepsilon) \to \infty$ as $\varepsilon \to 0$, which is consistent with the dynamics in the deterministic case. Substituting this result, for instance for $k = 3$, in Eq. (5) gives the following scaling law for noise-induced superpersistent chaotic transient in the normal sense:

$$\tau_{p=0} \sim \exp(C_0\varepsilon^{-2/3}). \tag{19}$$

B. Supercritical regime ($p > p_c = 0$)

In the supercritical regime, we have

$$
\begin{aligned}
T_p^k &= \frac{2}{\varepsilon^2}\int_0^l dy \exp[-bH(y)]\int_0^y \exp[bH(w)]dw \\
&= \frac{2}{\varepsilon^2}\int_0^l dy \exp[-b(y^k+kpy)]\int_0^y \exp[b(w^k+kpw)]dw \\
&\leqslant \frac{2}{\varepsilon^2}\int_0^l dy \exp[-b(y^k+kpy)]\exp(kbpy)\int_0^y \exp[bw^k]dw \\
&= T_0^k.
\end{aligned}
$$

That is, T_0^k is an upper bound for T_p^k. Using binomial expansion, we obtain, for $0 \leqslant y \leqslant l$, the following inequality:

$$\frac{y(y^k+kpy)^n}{kn+1} \leqslant \int_0^y (w^k+kpw)^n dw \leqslant \frac{y(y^k+kpy)^n}{n+1}.$$

Using the series expansion

$$\int_0^y \exp[b(w^k+kpw)]dw \sim \sum_{n=0}^{\infty}\frac{1}{(n+1)!}b^n y(y^k+kpy)^n,$$

letting $r=b(y^k+kpy)\geqslant 0$, and noticing that $0 \leqslant y/(y^{k-1}+p) \leqslant p^{-(k-2)/(k-1)}$ for $y \geqslant 0$, we obtain

$$
\begin{aligned}
&\frac{2}{\varepsilon^2}\int_0^l \exp[-bH(y)]y[b(y^k+kpy)]^n dy \\
&\qquad \leqslant \int_0^{l'} \exp(-r)r^n p^{-(k-2)/(k-1)}dr.
\end{aligned}
$$

It implies that

$$
\begin{aligned}
T_p^k(\varepsilon) &\leqslant \sum_{n=0}^{\infty}\frac{1}{(n+1)!}\int_0^{l'} \exp(-r)r^n p^{-(k-2)/(k-1)}dr \\
&\sim p^{-(k-2)/(k-1)}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\int_0^{l'} \exp(-r)r^n dr \\
&\sim p^{-(k-2)/(k-1)}.
\end{aligned}
$$

Utilizing the upper bounds for T_p^k, i.e., $T_p^k(\varepsilon) < \varepsilon^{-(2-4/k)}$ and $T_p^k(\varepsilon) < p^{-(k-2)/(k-1)}$, we obtain the scaling relation between the critical noise strength and the bifurcation parameter:

$$\varepsilon_c \sim p^{k/[2(k-1)]}. \tag{20}$$

For the large noise regime $\varepsilon \gg \varepsilon_c$, since $pb \to 0$ as $k \to \infty$, we have $\exp(bw^k+kpbw)) \sim \exp(bw^k)$, which implies

$$T_p^k \sim T_0^k.$$

Thus, in the supercritical regime $(p>0)$, we have the following scaling results for the average tunneling time:

$$T_p^k(\varepsilon) \sim \begin{cases} \varepsilon^{-(2-4/k)}, & \varepsilon \gg \varepsilon_c \\ p^{-(k-2)/(k-1)}, & \varepsilon \ll \varepsilon_c, \end{cases} \tag{21}$$

where $\varepsilon_c \sim p^{k/(2(k-1))}$.

Note that in the supercritical regime, an unstable-unstable pair bifurcation has already occurred so that the corresponding dynamical system in the absence of noise already exhibits a superpersistent transient behavior. The average transient lifetime is determined by the average tunneling time as $\exp[A \cdot T_p^k(\varepsilon)]$, where A is a constant. The scaling result in Eq. (21) implies that the average transient time becomes constant in the small noise regime and decreases in the large noise regime as the noise amplitude is increased. That is, the role of noise is to reduce the chaotic transient lifetime.

C. Subcritical regime $p < p_c = 0$

In the subcritical regime, we have

$$
\begin{aligned}
T_p^k &= \frac{2}{\varepsilon^2}\int_{x_s}^l dy \exp[-b(y^k+kpy)]\int_{x_s}^y \exp[b(w^k+kpw)]dw \\
&\geqslant \frac{2}{\varepsilon^2}\int_0^l dy \exp[-b(y^k+kpy)]\exp(kpyb)\int_0^y \exp[bw^k]dw \\
&= \frac{2}{\varepsilon^2}\int_0^l dy \exp(by^k)\int_0^y \exp[bw^k]dw = T_0^k.
\end{aligned} \tag{22}
$$

That is, T_0^k is now a lower bound of T_p^k, in contrast to the supercritical regime. For $\varepsilon \gg |p|^{-k/[2(k-1)]}$, we have $\exp(-by^k+kbpy) \sim \exp(by^k)$. This implies $T_p^k \sim T_0^k$ for $\varepsilon \gg |p|^{-k/[2(k-1)]}$. That is, in the large noise regime, the scaling of the average transient lifetime is normally superpersistent.

Note that the function H in the subcritical regime changes from being positive to negative and then to positive on the interval $[x_s, l]$, versus the supercritical case where H is always positive and an increasing function on the interval $[0, l]$, as shown in Fig. 2. The technical consequence is that, for the supercritical regime, $\exp[-bH(x)]$ is a decreasing function and $\int_0^y \exp[bH(w)]dw$ is an increasing function on the integration interval $[0, l]$, while for the subcritical regime, $\exp(-bH(x))$ changes from being decreasing to increasing on the integration interval $[x_s, l]$ and $\int_{x_s}^y \exp[bH(w)]dw$ is an increasing function with varying rate. In order to calculate the mean first-passage time, it is convenient to partition the integral interval and write the integral in Eq. (22) as

$$
\begin{aligned}
T_p^k &= \frac{2}{\varepsilon^2}\int_{x_s}^l dy \exp[-bH(y)]\int_{x_s}^y \exp[bH(w)]dw \\
&= \frac{2}{\varepsilon^2}\int_{x_s}^0 dy \exp[-bH(y)]\int_{x_s}^y \exp[bH(w)]dw \qquad (23) \\
&\quad + \frac{2}{\varepsilon^2}\int_0^l dy \exp[-bH(y)]\int_{x_s}^0 \exp[bH(w)]dw \qquad (24) \\
&\quad + \frac{2}{\varepsilon^2}\int_0^{x_u} dy \exp[-bH(y)]\int_0^y \exp[bH(w)]dw \qquad (25) \\
&\quad + \frac{2}{\varepsilon^2}\int_{x_u}^l dy \exp[-bH(y)]\int_0^{x_u} \exp[bH(w)]dw \qquad (26)
\end{aligned}
$$

$$+\frac{2}{\varepsilon^2}\int_{x_u}^{l} dy \exp[-bH(y)]\int_{x_u}^{y} \exp[bH(w)]dw. \tag{27}$$

Since the function H is locally quadratic at the stable and the unstable points, we can approximate it near $x=x_s$ as

$$H(x)=H_1(x)\approx H(x_s)+[H''(x_s)/2](x-x_s)^2 = (k-1)|p|^{k/(k-1)}-\frac{k(k-1)}{2}|p|^{(k-2)/(k-1)}(x-x_s)^2.$$

Near $x=x_u$, we have

$$H(x)=H_2(x)\approx H(x_u)+[H''(x_u)/2](x-x_u)^2 = -(k-1)|p|^{k/(k-1)} + \frac{k(k-1)}{2}|p|^{(k-2)/(k-1)}(x-x_u)^2.$$

Using the approximations for H_1 and H_2, the five integrals in Eqs. (23)–(27) can be carried out (see Appendix). We obtain, for the small noise regime ($\varepsilon \ll |p|^{k/[2(k-1)]}$), the average tunneling time:

$$T_p^k(\varepsilon)\sim\frac{1}{|p|^{(k-2)/(k-1)}}\exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right). \tag{28}$$

The tunneling time at the critical noise amplitude ϵ_c is

$$T_p^k(\varepsilon_c(p))\sim|p|^{-(k-2)/(k-1)}. \tag{29}$$

D. Summary of scaling laws

The scaling laws of the average tunneling time $T_p^k(\varepsilon)$ with noise can be summarized, as follows.

(1) For the small noise regime ($\varepsilon \ll \varepsilon_c \sim |p|^{k/[2(k-1)]}$), we have

$$T_p^k(\varepsilon)\sim\begin{cases} p^{-(k-2)/(k-1)}, & p>0\\ \varepsilon^{-(2-4/k)}, & p=0\\ |p|^{-(k-2)/(k-1)}\exp\left(\dfrac{|p|^{k/(k-1)}}{\varepsilon^2}\right), & p<0. \end{cases} \tag{30}$$

(2) For the large noise regime ($\varepsilon \gg \varepsilon_c$), we have

$$T_p^k\sim\varepsilon^{-(2-4/k)}. \tag{31}$$

These laws imply the following scaling laws for the average lifetime of the chaotic transients in various regimes [substituting the expressions of T_p^k in Eq. (5)].

(1) For the small noise regime ($\varepsilon \ll \varepsilon_c \sim |p|^{k/[2(k-1)]}$), we have

$$\tau_p^k(\varepsilon)\sim\begin{cases} \exp(D_1 p^{-(k-2)/(k-1)}), & p>0\\ \exp(D_2\varepsilon^{-(2-4/k)}), & p=0\\ \exp(D_3|p|^{-(k-2)/(k-1)}\exp[|p|^{k/(k-1)}/\varepsilon^2]), & p<0, \end{cases} \tag{32}$$

where D_1, D_2, and D_3 are constants.

(2) For the large noise regime ($\varepsilon \gg \varepsilon_c$), we have

$$\tau_p^k\sim\exp(D_4\varepsilon^{-(2-4/k)}), \tag{33}$$

where D_4 is a constant. The general observation is that for large noise ($\varepsilon \gg \varepsilon_c$), the transient is normally superpersistent. For small noise, three behaviors arise depending on the bifurcation parameter p: constant (independent of noise) for the supercritical regime, normally superpersistent for the critical case, and extraordinarily superpersistent for the subcritical regime.

E. Remark: relation to Kramer's law

The scaling law (33), which holds generally for all three regimes (subcritical, critical, and supercritical) and for $\varepsilon \gg \varepsilon_c$, resembles the Kramers' law [19] in statistical physics. This law states that, for a heavily damped particle in a potential well with barrier height ΔE, under Gaussian noise of amplitude ε, in the weak noise regime $\varepsilon^2 \ll \Delta E$ the rate of transition for the particle to cross over the potential barrier is

$$K\sim\exp(-\Delta E/\varepsilon^2), \tag{34}$$

so the average dwelling time of the particle in the potential well is

$$\langle t\rangle\sim\exp(\Delta E/\varepsilon^2). \tag{35}$$

The similarity between Eqs. (33) and (35) suggests that our noise-induced transient chaos problem, particularly in the subcritical regime where it is essentially a problem of noise-induced escape from a chaotic attractor, can be understood based on the picture of particle escape from a potential well. This further suggests that in the regime $\varepsilon \gg \varepsilon_c$ where the scaling law (33) is valid, the theory of quasipotential [20–23] may be applicable to the problem of superpersistent chaotic transients.

Notice, however, in the extremely small noise regime $\varepsilon \ll \varepsilon_c$, the scaling law (32) deviates from that for the Kramers time, except for the critical case ($p=0$). (The deviation can also be seen clearly in numerical experiments, as in Fig. 9.) This suggests that the picture of particle escaping from potential well may not be applicable. At the present we do not understand the failure of the Kramers theory in this case.

IV. NUMERICAL SUPPORT

A. Average tunneling time

We use the prototype model Eq. (7) to numerically verify the scaling laws of the average tunneling time in different regimes.

1. The critical case

Figure 3 shows, for $p=p_c=0$, scaling of the average tunneling time $T_0^k(\varepsilon)$ with noise on a logarithmic scale, where the lower, middle and upper curves correspond to $k=3,5,7$, respectively. The solid lines represent the theoretical prediction. There is a good agreement between the numerical computation and the theoretical result (18).

2. Supercritical regime

Figure 4 shows, for $p=\exp(-15)>0$, scaling of the average tunneling time $T_0^k(\varepsilon)$ with noise on a logarithmic scale,

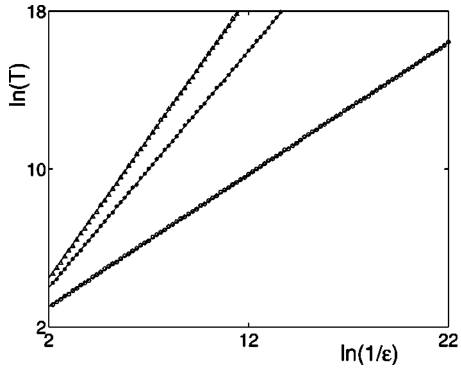

FIG. 3. Verification of the scaling law (18) for the critical case for three choices of the power k in the prototype model Eq. (8): $k=3$ (the lower trace), $k=5$ (the middle trace), and $k=7$ (the upper trace).

where the lower, middle and upper curves correspond to $k=3,5,7$, respectively. The critical noise levels ε_c that can be used to distinguish small and large noise regimes for these cases are $\varepsilon_c \approx p^{3/4} = \exp(11.25)$, $\varepsilon_c \approx p^{5/8} = \exp(9.375)$, and $\varepsilon_c \approx p^{7/12} = \exp(8.75)$, which are indicated by the vertical solid, dashed, and dot-dashed lines, respectively. We observe that the average tunneling time $T_p^k(\varepsilon)$ scales algebraically for $\varepsilon < \varepsilon_c$ but for $\varepsilon > \varepsilon_c$ the time is approximately constant, as predicted by Eq. (21).

3. Subcritical regime

Figure 5(a) shows, for $p = -\exp(-15) < 0$, scaling of the average tunneling time $T_p^k(\varepsilon)$ with noise on a logarithmic scale, where the lower, middle, and upper curves correspond to $k=3,5,7$, respectively. The vertical solid, dashed, and dot-dashed lines indicate the critical noise level for these cases, respectively. We see that for $\varepsilon > \varepsilon_c$, the scaling is algebraic, as in the critical case. However, for $\varepsilon < \varepsilon_c$, the average tunneling time increases much faster than that given by the algebraic scaling as the noise amplitude is decreased. Figure 5(b) shows the same cases but the vertical axis is on a double-logarithmic scale. The approximately linear behavior for $\varepsilon < \varepsilon_c$ indicates that the tunneling time itself is superpersistent, as suggested by the theoretical prediction (28). Note that the linear regions are narrow. This is due to computational constraints as the average transient time is already of the order of $e^{24} \sim 10^{13}$ iterations in the small noise regime. The scaling relation (28) predicts that the slope of the linear fit between $\ln[\ln(T_p^k)]$ and $\ln(1/\varepsilon)$ should be 2, regardless of the value of k. While the linear fits for different values of k in Fig. 5(b) indeed appear to be parallel to each other, the slope is about 1.2, which is below the theoretical value. Since the scaling relation (28) is derived under the assumption $\varepsilon \ll \varepsilon_c$, we suspect that going into smaller ε regime may yield slopes closer to that predicted by theory. However, this is computationally infeasible at the present.

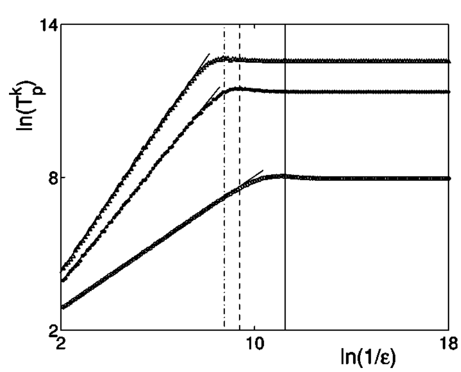

FIG. 4. Verification of the scaling law (21) in the supercritical regime for three choices of the power k in the prototype model Eq. (8): $k=3$ (the lower trace), $k=5$ (the middle trace), and $k=7$ (the upper trace).

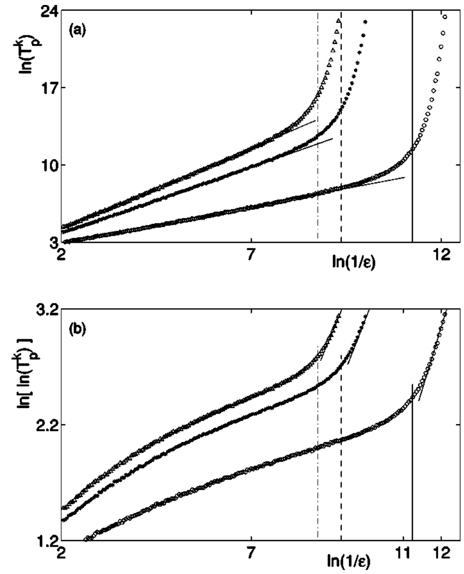

FIG. 5. Verification of the scaling laws of the average tunneling time for the subcritical case where $p = -\exp(-15) < 0$. Shown in each panel are the times for three choices of the power k in the prototype model Eq. (8): $k=3$ (the lower trace), $k=5$ (the middle trace), and $k=7$ (the upper trace). (a) $\ln(T_p^k)$ versus $\ln(1/\varepsilon)$ and (b) $\ln[\ln(T_p^k)]$ versus $\ln(1/\varepsilon)$. We see that for large noise, the scaling is algebraic but for small noise, the tunneling time itself appears to obey the superpersistent scaling law, as predicted by Eq. (28).

Figure 6 shows, for the subcritical regime, the relation between the average tunneling time at the critical noise level and the parameter p for $k=3$ (the lower trace), $k=5$ (the middle trace), and $k=7$ (the upper trace). The robust linear behaviors verify the algebraic scaling relation (29).

B. A two-dimensional map

We consider the class of two-dimensional maps that were used by Grebogi, Ott, and Yorke [6] to first report superpersistent chaotic transients:

$$\theta_{n+1} = 2\theta_n \bmod 2\pi, \tag{36}$$

$$z_{n+1} = az_n + z_n^2 + \beta \cos \theta_n,$$

where a and β are parameters. Because of the z_n^2 term in the z equation, for large $|z_n|$ we have $z_{n+1} > z_n$. There is thus an

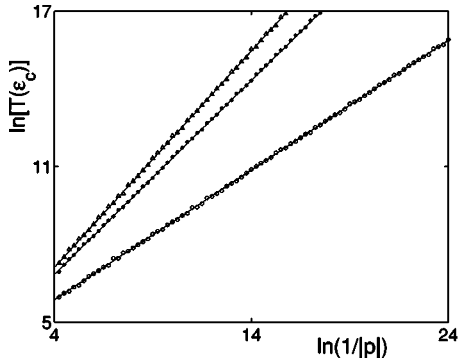

FIG. 6. Verification of the theoretical prediction (29) for the scaling of the average tunneling time at the critical noise level with the bifurcation parameter p. The three curves are for $k=3$ (the lower trace), $k=5$ (the middle trace), and $k=7$ (the upper trace) in the prototype model Eq. (8).

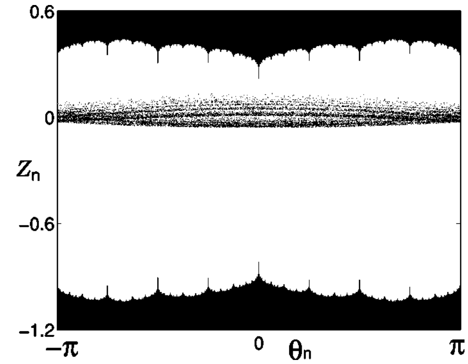

FIG. 7. For the two-dimensional map model (36), in the absence of noise, a chaotic attractor near $z=0$, its basin of attraction (blank), and the basin of the attraction of the attractor at $z=+\infty$ (black).

attractor at $z=+\infty$. Near $z=0$, depending on the choice of the parameters, there can be either a chaotic attractor or none. For instance, for $0<\beta\ll 1$, there is a chaotic attractor near $z=0$ for $a<a_c=1-2\sqrt{\beta}$ and the attractor becomes a chaotic transient for $a>a_c$ [6]. The transient is superpersistent for $a>a_c$, which can be argued [6], as follows.

For $a<a_c$ there are two fixed points: $(\theta_1,z_1)=(0,z_b)$ and $(\theta_2,z_2)=(0,z_c)$, where $z_{c,b}=(1-a\pm r)/2$ and $r=\sqrt{(1-a)^2-4\beta}$. The fixed points $(0,z_b)$ and $(0,z_c)$ are on the basin boundary and on the chaotic attractor, respectively. They coalesce at $a=a_c$. For $a>a_c$, a channel is created through which trajectories on the original attractor can escape to an attractor at infinity. At the location of the channel where $\theta=0$, the z mapping can be written as

$$z_{n+1}-z_n=(a-1)z_n+z_n^2+\beta.$$

Letting $\delta=z-z_*$, where z_* is the minimum of the quadratic function of z on the right-hand side, yields

$$\delta_{n+1}=\delta_n+\delta_n^2+b, \tag{37}$$

where $b=\sqrt{\beta}(a-a_c)-[(a-a_c)/2]^2$ and for $a>a_c$, we have $b\approx\sqrt{\beta}(a-a_c)$. In the continuous-time approximation, the dynamics in δ can be described by $d\delta/dt=\delta^2+b$. Thus the time T required to tunnel through the escaping channel is

$$T\approx\frac{1}{b^{1/2}}\int_0^\infty\frac{d\delta}{\delta^2+1}=\frac{\pi}{2b^{1/2}}.$$

Since the θ dynamics is uniformally chaotic with Lyapunov exponent $\lambda=\ln 2>0$, the probability for a trajectory to fall in the opening of the channel and to stay near there in the θ direction for consecutively T iterations is proportional to $e^{-T\ln 2}$. For $a>a_c$, the average chaotic transient time is thus given by

$$\tau\sim e^{T\ln 2}\approx e^{(\pi\ln 2/2)b^{-1/2}}\approx e^{C(a-a_c)^{-1/2}}, \tag{38}$$

where $C=\pi\ln 2\beta^{-1/4}/2$ is a positive constant. Thus in the noiseless situation, for $a>a_c$ the exponent appeared in the scaling of the average transient lifetime with the parameter variation assumes the value of $1/2$, which was verified numerically [6].

The above heuristic analysis indicates that the dynamics in the escaping channel is governed approximately by the one-dimensional equation (37) with a single bifurcation parameter b. The critical point $a=a_c$ in the original two-dimensional map Eq. (36) corresponds to $b=b_c=0$. This justifies our use of Eq. (6) and its generalization Eq. (7) as paradigmatic model for analytically obtaining the scaling laws of the average tunneling lifetime with noise.

Figure 7 shows, for $\beta=0.04$ and $a=a_c=0.6$, the chaotic attractor near $z=0$, its basin of attraction (blank), and the basin of the attraction of the attractors at $z=+\infty$ (black). Signature that an unstable-unstable pair bifurcation is about to occur can be seen by the vertical, downward tip at $\theta=0$, which is close to the chaotic attractor. For convenience we write $p\equiv a-a_c$. Figure 8(a) shows the behaviors of the tunneling time for five cases in the subcritical regime ($p=-0.0001$, $p=-0.001$, $p=-0.01$, $p=-0.1$, and $p=-0.2$, and the five vertical dashed lines from right to left indicate the values of $\ln 1/\epsilon_c$ for these five cases, respectively), for the critical case ($p=p_c=0$), and for the supercritical case ($p=0.0001$). We observe a robust algebraic scaling with the $2/3$ exponent for the critical case. For large noise $\epsilon>\epsilon_c$, the scalings for the subcritical and supercritical cases coincide with that for the critical case. For the supercritical case, however, the tunneling time plateaus for $\varepsilon<\varepsilon_c$. Since the plateaued value of T is approximately the tunneling time in the absence of noise, we see that as the noise amplitude is increased, the tunneling time decreases, as predicted by our theory. For the subcritical cases, the tunneling time increases substantially for $\varepsilon<\varepsilon_c$ as compared with that in the critical case. Figure 8(b) shows, on a double-logarithmic versus logarithmic scale, the behaviors of the tunneling time for the five cases in the subcritical regime. The approximately linear fits in the small-noise range indicate that the *tunneling times themselves* for those cases obey the superpersistent scaling law. This can be considered as *indirect* evidence for extraordinarily superpersistent chaotic transient. Note that the slopes of the linear fits between $\ln[\ln(T_p)]$ and $\ln(1/\varepsilon)$ are approximately the same for different values of p, as predicted theoretically, but the values of the slopes are less than the theoretical value of 2, for the same reason that we specu-

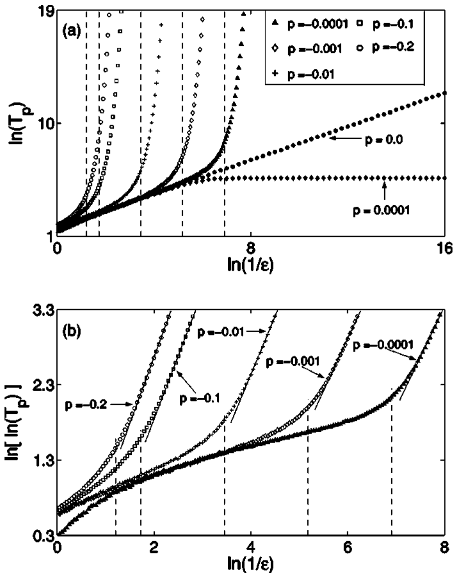

FIG. 8. For the two-dimensional map model (36): (a) scaling behaviors of the average tunneling times $T_p(\varepsilon)$ with noise on a logarithmic scale for five cases in the subcritical regime ($p=-0.0001$, $p=-0.001$, $p=-0.01$, $p=-0.1$, and $p=-0.2$), for the critical case ($p=p_c=0$), and for the supercritical case ($p=0.0001$), where $p\equiv a-a_c$. (b) Replots of the tunneling times on a double-logarithmic versus logarithmic scale for the five cases in the subcritical regime in (a).

lated for Fig. 5(b). That is, because of the extremely rapid increase in the tunneling time as ε is decreased from ε_c, it is difficult to extend the range of the noise variation in Fig. 8(b). [For instance, decreasing ε by one order of magnitude increases $T_p(\varepsilon)$ by a factor of e^{100}.]

Figure 9 shows, for the numerical model (36), the scaling of the average chaotic transient lifetime with the noise amplitude ε on a proper scale. The scaling in the critical case follows the normal superpersistent law (3). For the subcritical and supercritical cases, the scalings coincide with the normal superpersistent law only for large noise [$\varepsilon \gg \varepsilon_c \approx (0.1\sqrt{\beta})^{3/4}$]. In the supercritical regime, the transient lifetime deviates from the normal superpersistent scaling law for

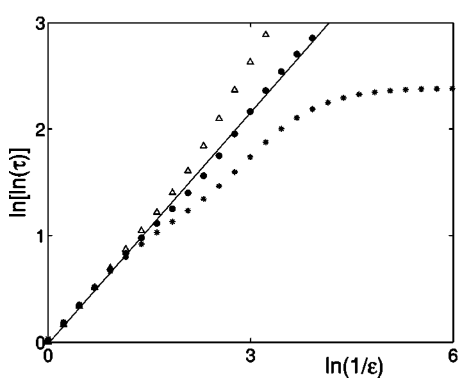

FIG. 9. For our numerical model, noisy scaling laws of the average chaotic transient lifetime for the subcritical ($a=0.5<a_c$, triangles), critical ($a=a_c$, filled circles), and supercritical ($a=0.7>a_c$, asterisks) cases.

$\varepsilon \ll \varepsilon_c$ and then levels off, approaching a constant as $\varepsilon \to 0$. In the subcritical regime, for small noise, the chaotic transient lifetime increases much faster as compared with the critical case. This behavior, in combination with Fig. 8(b), represents numerical evidence for our predicted extraordinarily superpersistent chaotic transients. Note that the ordinate of Fig. 9 is already on a double-logarithmic scale. Because of the scaling constants in Eq. (4), a triple-logarithmic scale plot would be inappropriate because such a plot still would not yield a linear behavior.

V. DISCUSSIONS

In summary, we have addressed the phenomenon of noise-induced superpersistent chaotic transients and derived scaling laws governing the dependence of the average transient lifetime on noise amplitude. To make analytic treatment possible, we constructed a prototype map that captures the essential dynamical features for superpersistent chaotic transients. We argue that noise can induce the opening of a narrow channel through which trajectories on a original chaotic attractor can escape. As a result, the average transient lifetime has an exponential dependence on the average time that escaping trajectories spend in the channel. We propose that the noisy dynamics in the channel can be modeled by a class of one-dimensional stochastic differential equations and the tunneling time can be approximated by the first-passage time of the underlying stochastic process. This picture holds in subcritical, critical, and supercritical parameter regimes. We find that at the critical point, the scaling of the average transient lifetime follows the superpersistent scaling law in the normal sense. However, in the subcritical regime, the lifetime can be substantially longer than that given by the normal superpersistent scaling law. In contrast, in the supercritical region where there is already a superpersistent chaotic transient, noise tends to reduce the transient lifetime.

These results can be relevant to physical situations. For instance, we recently discovered [14] that noise-induced superpersistent chaotic transients can occur in the physical space in the context of advection of inertial particles in open chaotic flows. In particular, it has been known that ideal particles with zero mass and size simply follow the velocity of the flow and, as such, the advective dynamics can be described as Hamiltonian [24–26] in the physical space for which chaos can arise but not attractors. Thus, in an open Hamiltonian flow, ideal particles coming from the upper stream cannot be trapped and they must necessarily go out of the region of interest in finite time. However, the inertia of the advective particles can alter the flow locally [27]. As a result, the dynamical system underlying the advection of such particles becomes dissipative for which attractors can arise. This means that, due to the presence of an attractor, particles can be trapped permanently in some region in the physical space [28]. This interesting phenomenon has been demonstrated recently in a model of two-dimensional flow past a cylindrical obstacle [15]. As the authors of Ref. [15] pointed out, this result has implications in environmental science where forecasting aerosol and pollutant transport is a basic task. The possibility that toxin particles can be trapped in physical space is particularly worrisome. We were thus motivated to study the structural stability of such attractors [29]. In particular, we ask, can chaotic attractors so formed be persistent under small noise? We find that, indeed, the attractor can be destroyed by small noise and replaced by a chaotic transient, whose lifetime versus noise appears to obey the superpersistent scaling law. An implication is that, for small noise, the extraordinarily long trapping time makes the transient particle motion practically equivalent to an attracting motion with similar physical or biological effects.

ACKNOWLEDGMENT

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APPENDIX

We calculate the five integrals in Eq. (23)–(27). Using the approximation H_1 of H, the integral in Eq. (23) can be computed as

$$\frac{2}{\varepsilon^2}\int_{x_s}^{0} dy\,\exp[-bH(y)]\int_{x_s}^{y}\exp[bH(w)]dw \sim \frac{2}{\varepsilon^2}\int_{x_s}^{0} dy\,\exp[-bH_1(y)]\int_{x_s}^{y}\exp[bH_1(w)]dw \sim \frac{2}{\varepsilon^2}\int_{x_s}^{0} dy\,\exp\left[\gamma\frac{(y-x_s)^2}{\varepsilon^2}\right]\int_{x_s}^{y}\exp\left[-\gamma\frac{(w-x_s)^2}{\varepsilon^2}\right]dw,$$

where $\gamma=[k(k-1)/2]|p|^{(k-2)/(k-1)}$. Let $(y-x_s)/\varepsilon=x$ and $(w-x_s)/\varepsilon=z$. We have

$$\frac{2}{\varepsilon^2}\int_{x_s}^{0} dy\,\exp\left[\gamma\frac{(y-x_s)^2}{\varepsilon^2}\right]\int_{x_s}^{y}\exp\left[-\gamma\frac{(w-x_s)^2}{\varepsilon^2}\right]dw = \int_{0}^{-x_s/\varepsilon} dx\,\exp[\gamma x^2]\int_{0}^{x}\exp[-\gamma z^2]dz \sim \int_{0}^{-x_s/\varepsilon} x\,\exp[\gamma x^2]dx \sim |p|^{-(k-2)/(k-1)}\exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right).$$

For calculating the integral in Eq. (24), we use the approximations H_1 and H_2 of H and the fact

$$\int_{-\infty}^{\infty}\exp[-(x-a)^2/b^2]dx=\sqrt{2\pi}b.$$

Since

$$\int_{-\infty}^{\infty}\exp[-bH_2(y)]dy \sim \exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right)\times\int_{-\infty}^{\infty}\exp[-\gamma(y-x_u)^2/\varepsilon^2]dy \sim \varepsilon\sqrt{2\pi/|p|^{(k-2)/(k-1)}}\exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right)$$

and similarly

$$\int_{-\infty}^{\infty}\exp[b(H_1(w)\ ]dw \sim \varepsilon\sqrt{2\pi/|p|^{(k-2)/(k-1)}}\exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right),$$

we obtain, for the integral in Eq. (24),

$$\frac{2}{\varepsilon^2}\int_{0}^{l} dy\,\exp[-bH(y)]\int_{x_s}^{0}\exp[bH(w)]dw \sim |p|^{-(k-2)/(k-1)}\exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right).$$

Since H is negative on $[0,x_u]$, we have $\exp[bH(y)]<1$ and $H\approx H_2$ on $[0,x_u]$. Thus

$$\int_{0}^{x_u} dy\,\exp[-bH(y)]\int_{0}^{y}\exp[bH(w)]dw \sim \exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right)\int_{0}^{x_u} y\,\exp[-\gamma(y-x_u)^2/\varepsilon^2]dy.$$

Since $-x_u^2\leqslant -2x_uy+x_u^2\leqslant x_u^2$ on $[0,x_u]$, we have $|-\gamma(-2x_uy+x_u^2)|<|p|^{k/(k-1)}$. This implies that

$$\exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right)\int_{0}^{x_u} y\,\exp[-\gamma(y^2-2x_uy+x_u^2)/\varepsilon^2]dy < \exp\left(\frac{2|p|^{k/(k-1)}}{\varepsilon^2}\right)\int_{0}^{x_u} y\,\exp(-\gamma y^2/\varepsilon^2)dy.$$

Letting $y/\varepsilon=z$, we have

$$\int_{0}^{x_u} y\,\exp(-\gamma y^2/\varepsilon^2)dy \sim \varepsilon^2\int_{0}^{\infty} z\,\exp(-\gamma z^2)dy \sim \varepsilon^2/|p|^{(k-2)/(k-1)}.$$

Thus the integral in Eq. (25) is

$$|p|^{-(k-2)/(k-1)}\exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right).$$

The integral in Eq. (26) can be calculated as follows:

$$\int_{x_u}^{l}\exp[-bH(y)]dy \sim \int_{x_u}^{l}\exp[-bH_2(y)]dy \sim \exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right)\times\int_{x_u}^{l}\exp[-\gamma(y-x_u)^2/\varepsilon^2]dy \sim \exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right)\times\int_{-\infty}^{\infty}\exp[-\gamma(y-x_u)^2/\varepsilon^2]dy \sim \frac{\varepsilon}{|p|^{(k-2)/(2k-2)}}\exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right)$$

and

$$\int_0^{x_u} \exp[bH(w)]dw \sim \int_0^{x_u} \exp[bH_2(w)]dw \sim \int_0^{x_u} \exp\{[-(k-1)|p|^{k/(k-1)} + \gamma(w-x_u)^2]/\varepsilon^2\}dw \sim \int_0^{x_u} \exp[(\gamma w^2 - 2\gamma x_u w + |p|^{k/(k-1)})/\varepsilon^2]dw \sim \exp\left(\frac{|p|^{k/(k-1)}}{\varepsilon^2}\right) \times \int_0^{x_u} \exp[(-2\gamma x_u w)/\varepsilon^2]dw.$$

Since

$$\int_0^{x_u} \exp[(-2\gamma x_u w)/\varepsilon^2]dw \sim \varepsilon^2/|p|,$$

the integral in Eq. (26) is approximately

$$\frac{\varepsilon}{|p|^{(3k-4)/(2k-1)}} \exp(|p|^{k/(k-1)}/\varepsilon^2).$$

Since H is increasing on (x_u,∞), we have $0 < H(y) - H(w)$ for $x_u \leqslant w \leqslant y$, which implies that the integral in Eq. (27) is

$$\int_{x_u}^{l} dy \int_{x_u}^{y} \exp\{-b[H(y)-H(w)]\}dw = C,$$

where C is a constant.

Combining the results of these five integrals and using $\varepsilon \ll 1$, we obtain the scaling relation (28).

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