Power law distribution of wealth in population based on a modified Equíluz-Zimmermann model

Yan-Bo Xie, Bing-Hong Wang,* Bo Hu, and Tao Zhou
Department of Modern Physics and The Nonlinear Science Center, University of Science and Technology of China, Hefei 230026, China
(Received 16 May 2004; revised manuscript received 3 January 2005; published 22 April 2005)

We propose a money-based model for the power law distribution (PLD) of wealth in an economically interacting population. It is introduced as a modification of the Equíluz and Zimmermann (EZ) model for crowding and information transmission in financial markets. Still, it must be stressed that in the EZ model a PLD without exponential correction is obtained only for a particular parameter, while our pattern will give the exact PLD within a wide range. The PLD exponent depends on the model parameters in a nontrivial way and is exactly calculated in this paper. The numerical results are in excellent agreement with the analytic prediction, and also comparable with empirical data of wealth distribution.

DOI: 10.1103/PhysRevE.71.046135 PACS number(s): 89.90.+n, 02.50.Le, 64.60.Cn, 87.10.+e

I. INTRODUCTION

Many real life distributions, including wealth allocation in individuals, sizes of human settlements, website popularity, and words ranked by frequency in a random corpus of text, observe the Zipf law. Empirical evidence of the Zipf distribution of wealth [1–9] has recently attracted a lot of interest of economists and physicists. To understand the micromechanism of this challenging problem, various models have been proposed. One type is based on a so-called multiplicative random process [10–21]. In this approach, individual wealth is updated multiplicatively by a random and independent factor. A very nice power law is given; however, this approach essentially does not contain interactions among individuals, which are also responsible for the economic structure and aggregate behavior. Another pattern takes into account an interaction between two individuals that results in a redistribution of their assets [22–25]. Unfortunately, some attempts only give a Boltzmann-Gibbs distribution of assets [24,25], while some others [23], though exhibiting power law distributions, fail to provide a stationary state.

In this paper, we shall introduce a different perspective to understand this problem. Our model is based on the following observations. (i) In order to minimize costs and maximize profits, two corporations or economic entities may combine into one. This phenomenon occurs frequently in the real economic world. Simply fixing our attention on money movements, we can equally say that two amounts of capital combine into one. (ii) The dissociation of an economic entity into many small fractions is commonplace, too. The bankruptcy of a corporation, for instance, can be effectively classified into this category. Allocating a fraction of assets for the employee's salary also serves as a good example. Under some appropriate assumptions, we shall establish a simple money-based model which is essentially a modification of the Eguíluz and Zimmermann (EZ) model for crowding and information transmission in financial markets [26,27]. The size of a cluster there is now identified as the wealth of an economic entity here. However, the analytical results will show that our model is quite different from EZ's [27]. The EZ model gives a power law distribution (PLD) with an exponential cutoff that vanishes only for a particular parameter. Here, a PLD of wealth is obtained within a wide range and without exponential correction. The PLD exponent can be analytically calculated and is found to have a nontrivial dependence on our model parameters.

It may be beneficial to notice that only two types of money movements among economic entities are discussed in the above paragraph, i.e., money aggregation due to the combination of two entities and money dispersion due to the dissociation of an entity. These two types of money movement have not been considered in the previous literature [10–25]. On the other hand, there are other important money movements in real economic activities. For instance, Refs. [14–16] discussed the money fluctuations of an individual as a result of the interaction between the individual and the environment. Also, Refs. [22–25] discussed the money exchange between two individuals. These two types of money movement are of course important too. However, we do not attempt to include all types of money movement in the present model. Instead, we shall only concentrate on the money aggregation and dispersion mentioned in the last paragraph. We are most interested in what type of distribution of wealth could emerge if these two opposite movements of money are considered together.

This paper is organized as follows. Section II describes the money-based model in detail. In Sec. III, we shall provide the master equation of n_s and present our analytical calculation of the PLD exponent. Next, we give numerical studies for the master equation, which are in excellent agreement with the analytic prediction. In Sec. V, the relevance of our model to the real world is mainly discussed.

II. THE MODEL

The money-based model contains N units of money, where N is conserved. Then the total wealth is allocated to M economic entities (or corporations), where M is variable. For simplicity, we may choose the initial state containing just N

*Electronic address: bhwang@ustc.edu.cn

corporations, each with one unit of money. The state of this system is mainly described by n_s, which denotes the number of cooperations owning s units of money. At each time step, we randomly select a unit of money from the wealth pool. Since it must belong to a certain corporation, we in this way select an economic entity too. Corporations with more wealth are of course chosen with a larger possibility; and this could be interpreted as the fact that larger companies have more chances for economic activities. The evolution of the system is under the following rules.

(1) With probability $1-a$, another unit of money is randomly selected. If the two selected units are occupied by different corporations, then the two corporations with all their money combine into one entity; otherwise, no combination. Thus, $1-a$ in our model is a factor reflecting the incorporation possibility of economic entities at a macroscopic level.

(2) With probability $a\gamma/s$, the economic entity that owns the selected money is dissociated; here s is the amount of capital owned by this corporation, and $a\gamma$ reflects the dissociative (bankruptcy) possibility of any economic entity. After disassociation, these s units of money are simply assumed to be redistributed to s new companies, each with just one unit.

(3) With probability $a(1-\gamma/s)$, nothing is changed. This can control the frequency of economic occurrences.

This model is like an investing game, where the total wealth involved in this game is supposed to be conserved. Each entity should have a minimal requirement of wealth ($s=1$) to play the game. Hence, the game participants may increase or decline. They can combine to maximize their profits, and all entities confront the risk of bankruptcy. Thus, it is a money-exchange model. Analysis of some extreme cases may be helpful to understand it. One may find that as a is close to 1 and γ is not small (i.e., bankruptcy is prevailing), wealth is hard to aggregate and a financial oligarch could hardly emerge in the model evolution. When a is slightly above zero (i.e., combination is prevailing), all the capital is inclined to converge. Therefore, our model can generate a broad range of economic cases, by concentrating on two typical kinds of money movement.

One may relate our model to other types of stochastic process models. For instance in the zero range process model [28], the diffusion mechanism, which describes the combination of k_i particles on site i with k_j particles on site j, is similar to the combination of two corporations in our model. However, the dissociation process in our model has no correspondence in the zero range process model. Indeed, a power law distribution of particle number is observed only at a critical number density in the zero range process model. In contrast, a PLD of wealth can be obtained for a wide range of parameters in our model.

III. ANALYTIC RESULTS

Following Refs. [27,29,30] in the case of $N\gg 1$, we give the master equation for n_s:

$$\frac{\partial n_s}{\partial t} = \frac{1-a}{N}\sum_{r=1}^{s-1} r n_r (s-r) n_{s-r} - 2(1-a)s n_s - a s n_s \frac{\gamma}{s} \qquad (1)$$

for $s>1$ and

$$\frac{\partial n_1}{\partial t} = -2(1-a)n_1 + a\sum_{s=2}^{\infty} s^2 n_s \frac{\gamma}{s} = -2(1-a)n_1 + a\gamma(N-n_1) \qquad (2)$$

where the identity

$$\sum_{s=1}^{\infty} s n_s = N \qquad (3)$$

has been used. It may be helpful to explain the physical meaning of the three terms at the right hand side of Eq. (1). The first term represents the net gain of n_s from a combination of economic entities with sizes r and $s-r$. The second term represents the net loss of n_s due to the combination of an entity of size s with another entity. The third term represents the net loss of n_s due to the dissociation of an entity of size s. Equation (2) has a similar physical explanation. The first term represents the net loss of n_1 due to the combination of an entity of size 1 with another entity. The second term represents the net gain of n_1 coming from dissociation of the entities with size $s>1$. Notice that the validity of this master equation is based on the mean field approximation which can be justified as in Ref. [30] for the EZ model. In Appendix A, we explicitly show the validity of Eqs. (1) and (2) by assuming that the mean field approximation is correct. We must point out that Eq. (1) is almost the same as the master equation derived in Ref. [27] for the EZ model. The only difference is the third term on the right hand side of Eq. (1) and the second term in Eq. (2). The third term of the EZ model is $-asn_s$, while the third term of our model is $-asn_s\gamma/s$. The factor γ/s in our model greatly reduces the frequency of disintegration for large s entities. Without this reduction, the frequency of disintegration for large s entities would be too high, which is unreasonable in the real economic world (see Sec. V). It must be stressed that the mathematical structure of our model is completely different from that of the EZ model.

For facility of the analytical discussion, we introduce $\alpha = a\gamma/2(1-a)$ and $h_s = sn_s/N$, which indicates the ratio of the wealth owned by economic entities in rank s to the total wealth. Then, one can give the equations for the stationary state in a terse form:

$$h_s = \frac{s}{2(s+\alpha)}\sum_{r=1}^{s-1} h_r h_{s-r} \qquad (4)$$

and

$$h_1 = \frac{\alpha}{1+\alpha}. \qquad (5)$$

According to the definition of h_s, it should satisfy the normalization condition Eq. (3):

$$\sum_{s=1}^{\infty} h_s = 1. \qquad (6)$$

When α is less than a critical value $\alpha_c=4$ which will be determined numerically in Sec. IV, one can show that Eqs. (4) and (5) does not satisfy the normalization condition Eq. (3). This inconsistency implies that when $\alpha<\alpha_c$ the state with one agent who has all N units of money becomes important [29,30]. In this case, the finite-size effect and the fluctuation effect become nontrivial and the master equations (1)–(3) are no longer applicable to describe the system [29,30]. In this paper, we shall restrict our discussion to the case $\alpha > \alpha_c$.

When $\alpha > \alpha_c$, one can show that (see Appendix B)

$$h_s \to A/s^{\eta} \tag{7}$$

for sufficiently large s with

$$\eta = \frac{\alpha}{\sum_{r=1}^{\infty} r h_r}. \tag{8}$$

Notice that this equation is consistent only when $\eta > 2$ because otherwise the sum $\Sigma_{r=1}^{\infty} r h_r$ would be divergent, and thus $h_s \to A/s^{\eta}$ would become an inconsistent formula.

$\Sigma_{r=1}^{\infty} r h_r$ can be further evaluated. Introducing the generating function

$$G(x) = \sum_{r=1}^{\infty} x^r h_r \tag{9}$$

one can rewrite Eq. (4) as

$$x(G' - h_1) + \alpha(G - h_1 x) = xG' + \alpha(G - x) = xG'G$$

or

$$G' x(G - 1) = \alpha(G - x) \tag{10}$$

with the initial condition

$$G(0) = 0. \tag{11}$$

Since $h_s \to A/s^{\eta}$ as $s \to \infty$, G is defined only in the interval $|x| \leqslant 1$. From Eq. (6), we also have $G(1)=1$. What we need to calculate is just

$$G'(1) = \sum_{r=1}^{\infty} r h_r.$$

Since the left and the right hand sides of Eq. (10) are both zero at $x=1$, we differentiate both sides by x and obtain

$$G'' x(1 - G) + G'(1 - G) - xG'^2 = \alpha(1 - G').$$

Let $x \to 1$ and one finds that $G''(1-G)$ vanishes in this limit provided $\eta > 2$; thus

$$G'^2(1) - \alpha G'(1) + \alpha = 0. \tag{12}$$

One immediately obtains that

$$\sum_{r=1}^{\infty} r h_r = \frac{\alpha - \sqrt{\alpha^2 - 4\alpha}}{2} \tag{13}$$

and the exponent

$$\eta = \frac{2}{1 - \sqrt{1 - 4/\alpha}} \tag{14}$$

which is a positive real number for $\alpha \geqslant 4$. Notice that when $\alpha=4$, the exponent $\eta=2$. This implies that our calculation is self-consistent, provided Eq. (6) is satisfied. In summary, we find from the master equation that h_s obeys a PLD when s is sufficiently large and $\alpha > 4$. It may be important to point out that when s is small, h_s also approximately obeys the PLD, and the restriction $\alpha > 4$, introduced for the sake of discussing the master equation, can be actually relaxed. This argument has been tested by a simulator investigation, which supplies the gap in analytical tools and verifies the analytical outcome.

TABLE I. The results of H for various values of α. When $\alpha > 4.2$, $H=1$.

α	H
3.0	0.9940886
3.5	0.9997818
3.6	0.9999214
3.7	0.9999743
3.8	0.9999922
3.9	0.9999977
4.0	0.9999995
4.1	0.9999999

IV. NUMERICAL RESULTS

We have numerically calculated the number

$$H = \sum_{r=1}^{\infty} h_r$$

based on the recursion formula Eq. (4) with the initial condition Eq. (5). Table I lists the results of H for various value of α. From Table I, one immediately finds that the normalization condition is satisfied for $\alpha > \alpha_c = 4$, which, again, indicates the consistency of the related equations.

Figures 1 and 2 show h_s as a function of s in a log-log scale for $\alpha=10$ and 4.5, respectively. From Fig. 1, one can

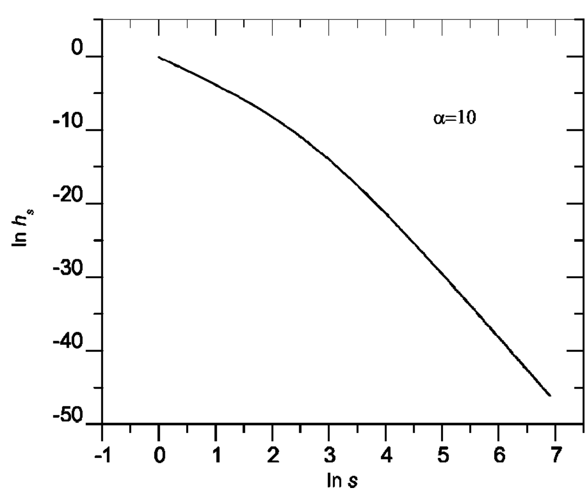

FIG. 1. The dependence of h_s on s in a log-log scale for $\alpha=10$.

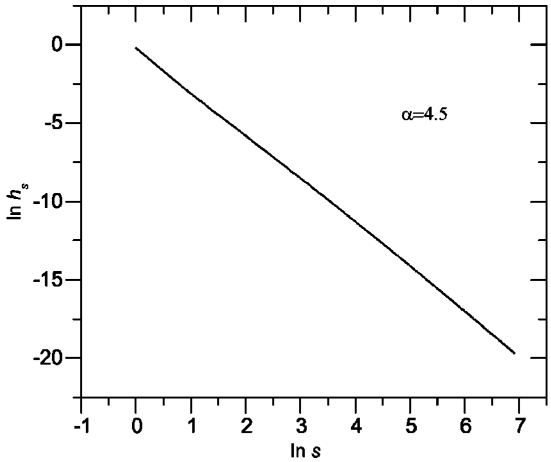

FIG. 2. The dependence of h_s on s in a log-log scale for $\alpha = 4.5$.

see that h_s conforms to a PLD for $s \gg 1$ with the exponent η given by Eq. (15). Figure 2 indicates that h_s observes the PLD for nearly all s with $\eta = 3.0$.

The fitted exponents for various values of α are plotted in Fig. 3. They are given by

$$\frac{\ln(h_{900}/h_{1000})}{\ln(1000/900)}.$$

Figure 3 also exhibits the analytic results from Eq. (15). The analytic outcome fits the exponents calculated from recursion quite well for $\alpha > 4.2$. However, when $\alpha \to 4.0$, a discrepancy is obvious, since the convergence of h_s to the correct power law is then very slow. We have also performed a computer simulation, which gives excellent agreement with theoretical results derived from Eqs. (4) and (5) for $\alpha = 8$ and $s \leqslant 10$, (see Fig. 4).

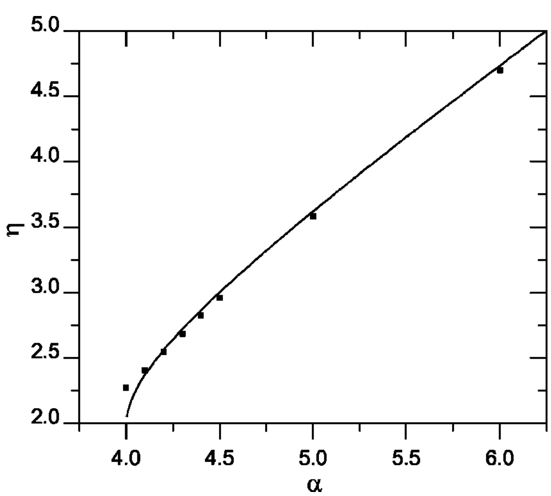

FIG. 3. The calculated exponent η for different values of α. Black squares represent the numerical results for η obtained from h_s using the extrapolation method (see text). The solid line represents the analytic result Eq. (14).

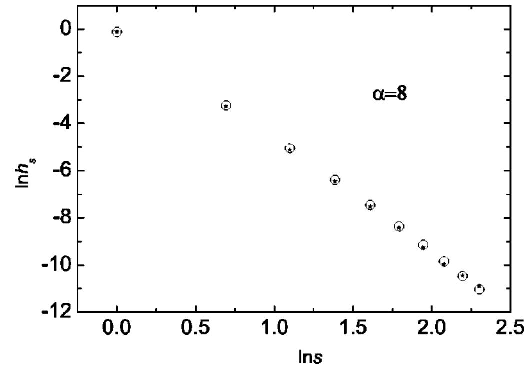

FIG. 4. h_s for $\alpha = 8$ from both numerical calculation and computer simulation. Black stars represent the outcome of a computer simulation for $N = 2.5 \times 10^5$, $\gamma = 2$, and $a = 0.888\,89$. A total 2×10^6 time steps were run and the final 5×10^5 time steps were used to count n_s statistically. The circles represent the theoretical results derived from Eqs. (4) and (5).

V. DISCUSSIONS

In this paper, we have introduced a money-based model to mimic and study the wealth allocation process. We find for a wide range of model parameters the wealth distribution $n_s \sim A/s^{\eta+1}$ with η given by Eq. (14) for sufficiently large s. The major difference between our model and the EZ model is that the dissociative probability Γ_d of an economic entity, after being chosen, is proportional to $1/s$ in our model. However, the corresponding probability in the EZ model is simply proportional to 1. This difference gives rise to distinct behaviors of n_s. In the EZ model, $n_s \sim A/s^{2.5}\exp(-\beta s)$ for large s [27]. Specifically, the corresponding Eq. (4) can be written as

$$h_s = D \sum_{r=1}^{s-1} h_r h_{s-r} \qquad (15)$$

in the EZ model. In fact, Eq. (15) is much easier to solve than Eq. (4). When n_s is interpreted as the number of corporations that own s units of money, the choice of $\Gamma_d \sim O(1/s)$ is reasonable and sound. Actually, because at the first step we randomly choose one unit of money, the entity with s units is picked out with a probability proportional to s. According to observation in real economic life, large companies or rich men are not more fragile than small or poor ones when they confront the same economic impact and fierce competition. If $\Gamma_d \sim O(1)$, the overall disassociation frequency would be proportional to s, implying that larger companies or richer men would be much weaker.

It may be interesting to compare our theoretical results with empirical data. For instance, Dragulescu and Yakovenko discussed the wealth distribution in the United Kingdom [5]. They found that for the top 10% of the population the wealth distribution observes a power law (the PLD exponent is 1.9), but for the bottom 90% the distribution is exponential. Meanwhile the exponent predicted in our model is greater than 2 [31]. The agreement for the top 10% would be good if one chose the parameter $\alpha \sim 4$. Nevertheless, our model does not explain the wealth distribution for the bottom

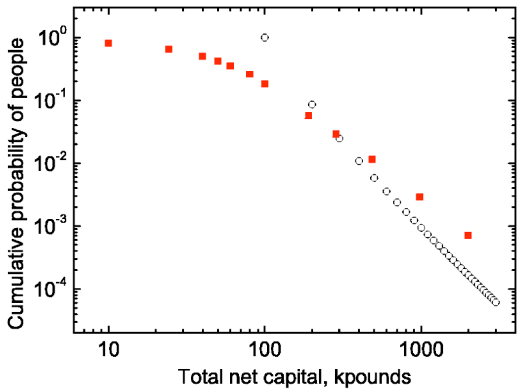

FIG. 5. The cumulative probability distribution of people as function of total net capital (wealth) in United Kingdom. The squares are the empirical data for 1996 [5]. The open circles are the numerical results for $\alpha=4$. We have assumed that $s=1$ corresponds to the net capital 100 k£.

90%. This indicates that our model is only applicable to economic entities with wealth above a certain threshold, which can be just $s=1$ in our model. For those under the threshold, their economic activities cannot be described by the present model. Some other ingredients must be integrated into consideration then. In Fig. 5, the empirical data taken from Ref. [5] are compared with the numerical results obtained from our model for $\alpha=4$ with the cutoff $s=1$ corresponding to 100 k£. From the figure, one may find that the agreement between the empirical data and our model is not very bad when the net capital is greater than 100 k£. Still, the agreement is not excellent, indicating the relevance of other possible mechanisms [14–16,22–25] in the explanation of the empirical data. It is still interesting to discuss the wealth distribution of the bottom 90% of people, though our model is no longer applicable in this regime. This distribution cannot follow an exact power law because otherwise the cumulative percent of people would not be convergent to 100% when the wealth approaches zero. The explanation of the exponential law found for the bottom 90% of people in the empirical data [5] requires different money exchange mechanisms.

In a real economic environment, capitals and corporations behave similarly at some point. For instance, they both constantly display integration and disintegration phenomena, driven by the motivation to maximize profits and efficiency. This mechanism updates the system every time, and gives rise to clusters and herd behaviors. Furthermore, in an agent-based model, it is usually indispensable to consider the individual diversity that is all too often impossible to deal with. When it comes to the money-based model, this microcomplexity may be considerably simplified. Finally, the conceptual movement and interaction among capitals is not as restricted by space and time as between agents. Therefore, when econophysics is much more interested in the behaviors of money than that of agents, it is recommended to adopt such a money-based perspective. The methodology to fix our attention on capital movement, instead of interactions among individuals, will bring a lot of facility for analysis; moreover, using such random variables as γ and a to represent the macroscopic level of the micromechanism also help us find a possible bridge between the evolution of the system and the protean behaviors of individuals. Whether the bridge is steady or not can only be tested by further investigation.

ACKNOWLEDGMENTS

This work has been partially supported by the State Key Development Program of Basic Research (973 Project) of China, the National Natural Science Foundation of China (under Grants No. 70271070 and No. 10472116), and the Specialized Research Fund for the Doctoral Program of Higher Education (SRFDP Grant No. 20020358009).

APPENDIX A: DERIVATION OF EQS. (1) and (2) FROM MEAN FIELD APPROXIMATION

Following Ref. [29], we describe the dynamics of our model by considering the partition of N units of money $[l_1,l_2,\ldots,l_N]$. Here l_s is the number of entities that own s units of money. It follows that

$$\sum_{i=1}^{N} il_i = N. \tag{A1}$$

Since any state of our model can be characterized by a partition $[l_1,\ldots,l_N]$, the system can be described by the probability function $P[l_1,\ldots,l_N]$. The time evolution of $P[l_1,\ldots,l_N]$ is governed by the dynamics for entity combination and dissociation as follows:

$$\begin{aligned}\frac{dP[l_1,l_2,\ldots,l_N]}{dt} = &-\frac{1-a}{N(N-1)}\left(\sum_{i=1}^{N} il_i i(l_i-1)+\sum_{i<j}2il_ijl_j\right)P[l_1,l_2,\ldots,l_N]\\ &+\frac{1-a}{N(N-1)}\left(\sum_{i=1}^{N} i(l_i+2)i(l_i+1)P[\ldots,l_i+2,\ldots,l_{2i}-1,\ldots]\right.\\ &\left.+\sum_{i<j}2i(l_i+1)j(l_j+1)P[\ldots,l_i+1,\ldots,l_j+1,\ldots,l_{i+j}-1,\ldots]\right)\\ &-\frac{a\gamma}{N}\sum_{i=2}^{N} l_iP[l_1,\ldots,l_N]+\frac{a\gamma}{N}\sum_{i=2}^{N}(l_i+1)P[l_1-i,\ldots,l_i+1,\ldots,l_N].\end{aligned} \tag{A2}$$

The first four terms on the right hand side of the above equation describe the combination of entities. The first term describes the reduction in $P[l_1,\ldots,l_N]$ due to the change from the partition $[\ldots,l_i,\ldots,l_{2i},\ldots]$ to the partition $[\ldots,l_i-2,\ldots,l_{2i}+1,\ldots]$ when two different entities that own the same amount of money i are combined to form a larger entity that owns the money $2i$. The factor $il_ii(l_i-1)/N(N-1)$ is the probability of selecting two units of money belonging to two different entities that own the same amount of money i. Similarly, the second term describes the change from the partition $[\ldots,l_i,\ldots,l_j,\ldots,l_{i+j},\ldots]$ to the partition $[\ldots,l_i-1,\ldots,l_j-1,\ldots,l_{i+j}+1,\ldots]$ when an entity that owns i units of money combines with an entity that owns j units of money to form an entity that owns $i+j$ units of money. The factor $2il_ijl_j/N(N-1)$ is the probability of selecting a unit of money from an entity that owns i units of money and another unit of money from an entity that owns j units of money. The third term describes the increase in $P[l_1,\ldots,l_N]$ due to the change from the partition $[\ldots,l_i+2,\ldots,l_{2i}-1,\ldots]$ to $[\ldots,l_i,\ldots,l_{2i},\ldots]$. Similarly, the fourth term describes the change from the partition $[\ldots,l_i+1,\ldots,l_j+1,\ldots,l_{i+j}-1,\ldots]$ to $[\ldots,l_i,\ldots,l_j,\ldots,l_{i+j},\ldots]$. The last two terms describe the change in $P[l_1,\ldots,l_N]$ due to dissociations of entities. The fifth term describes the change from the partition $[l_1,\ldots,l_i,\ldots]$ to $[l_1+i,\ldots,l_i-1,\ldots]$ when an entity that owns i units of money dissolves. The factor $il_i/N\times a\gamma/i$ comes from two facts in our model: the factor il_i/N is the probability of selecting a unit of money from an entity that owns i units of money, while the factor $a\gamma/i$ represents the probability that the entity dissolves. The last term describes the change from the partition $[l_1-i,\ldots,l_i,\ldots]$ to $[l_1,\ldots,l_i,\ldots]$.

Since $d/dt\Sigma_{[l_1,\ldots,l_N]}P[l_1,\ldots,l_N]=0$, a normalization condition can be introduced as

$$\sum_{[l_1,\ldots,l_N]} P[l_1,\ldots,l_N] = 1. \tag{A3}$$

In the stationary state,

$$\frac{d}{dt}P[l_1,\ldots,l_N] = 0.$$

Now, introducing

$$\langle n_1^{m_1}\cdots n_i^{m_i}\cdots\rangle = \sum_{[l_1,\ldots,l_N]} P[l_1,\ldots,l_N]l_1^{m_1}\cdots l_i^{m_i}\cdots . \tag{A4}$$

From Eq. (A1), one obtains that

$$\sum_{i=1}^{N} i\langle n_iW\rangle = N\langle W\rangle \tag{A5}$$

where $W=n_1^{m_1}\cdots n_N^{m_N}$. Multiplying Eq. (A2) by $l_1^{m_1}\cdots l_i^{m_i}\cdots$ and summing over all possible partitions $[l_1,\ldots,l_N]$, one obtains the following exact equations:

$$\begin{aligned}\frac{1-a}{N-1}\Bigg(&-\sum_{i=1}^{N} i^2\langle\cdots n_i(n_i-1)n_i^{m_i}\cdots n_{2i}^{m_{2i}}\cdots\rangle + \sum_{i=1}^{N} i^2\langle\cdots n_i(n_i-1)(n_i-2)^{m_i}\cdots(n_{2i}+1)^{m_{2i}}\cdots\rangle\\ &-\sum_{i<j}2ij\langle\cdots n_in_i^{m_i}\cdots n_jn_j^{m_j}\cdots n_{i+j}^{m_{i+j}}\cdots\rangle + \sum_{i<j}2ij\langle\cdots n_i(n_i-1)^{m_i}\cdots n_j(n_j-1)^{m_j}\cdots\times\cdots(n_{i+j}+1)^{m_{i+j}}\cdots\rangle\Bigg)\\ &-a\gamma\sum_{i=2}^{N}\langle n_1^{m_1}\cdots n_i^{m_i+1}\cdots\rangle + a\gamma\sum_{i=2}^{N}\langle(n_1+i)^{m_1}\cdots n_i(n_i-1)^{m_i}\cdots\rangle = 0\end{aligned} \tag{A6}$$

for the stationary state. Now let us consider the limit $N\gg 1$. When i is finite, $\langle n_i\rangle\sim N\gg 1$. Assuming the mean field approximation is correct, one has

$$\langle n_1^{m_1}\cdots n_i^{m_i}\cdots\rangle = \langle n_1\rangle^{m_1}\cdots\langle n_i\rangle^{m_i}\cdots \sim N^{\Sigma_i m_i} \tag{A7}$$

when m_i is nonzero only for finite i. From the above equation, one obtains if $\Sigma_i m_i' > \Sigma_i m_i$,

$$\langle\cdots n_i^{m_i'}\cdots\rangle \gg \langle\cdots n_i^{m_i}\cdots\rangle \tag{A8}$$

where m_i' and m_i are nonzero only for finite i. Expanding Eq. (A6) and keeping the leading term and using $N-1\sim N$, one obtains

$$\frac{1-a}{N}\left(-2\sum_{i=1}^{N} i^2 m_i\langle\cdots n_i^{m_i+1}\cdots n_{2i}^{m_{2i}}\cdots\rangle+\sum_{i=1}^{N} i^2 m_{2i}\langle\cdots n_i^{m_i+2}\cdots n_{2i}^{m_{2i}-1}\cdots\rangle-2\sum_{i<j} ij[m_i\langle\cdots n_i^{m_i}\cdots n_j^{m_j+1}\cdots n_{i+j}^{m_{i+j}}\cdots\rangle\right.$$
$$\left.+m_j\langle\cdots n_i^{m_i+1}\cdots n_j^{m_j}\cdots n_{i+j}^{m_{i+j}}\cdots\rangle]+2\sum_{i<j} ijm_{i+j}\langle\cdots n_i^{m_i+1}\cdots n_j^{m_j+1}\cdots n_{i+j}^{m_{i+j}-1}\cdots\rangle\right)$$
$$+a\gamma\left(-\sum_{i=2}^{N} m_i\langle n_1^{m_1}\cdots n_i^{m_i}\cdots\rangle+\sum_{i=2}^{N} im_1\langle n_1^{m_1-1}\cdots n_i^{m_i+1}\cdots\rangle\right)=0. \tag{A9}$$

Using the identity Eq. (A5), one can rewrite the above equation as

$$\left(2(1-a)\sum_{i=1}^{N} im_i+a\gamma\sum_{i=2}^{N} m_i\right)\langle\cdots n_j^{m_j}\cdots\rangle$$
$$=\frac{1-a}{N}\sum_{s=2}^{N} m_s\sum_{r=1}^{s-1} r(s-r)$$
$$\times\langle\cdots n_r^{m_r+1}\cdots n_{s-r}^{m_{s-r}+1}\cdots n_s^{m_s-1}\cdots\rangle$$
$$+a\gamma\sum_{i=2}^{N} m_1 i\langle n_1^{m_1-1}\cdots n_i^{m_i+1}\cdots\rangle. \tag{A10}$$

Now, taking $m_1=0$ and $m_i=\delta_{is}$ for $s>1$ and using the mean field approximation $\langle n_r n_{s-r}\rangle=\langle n_r\rangle\langle n_{s-r}\rangle$, one obtains the master equation (1) for the stationary state. Taking $m_1=1$ and $m_i=0$ for $i>1$, one obtains the master equation (2) for the stationary state.

APPENDIX B: MORE DETAILS ABOUT EQ. (7)

We shall first assume that

$$h_s \approx Af(s)/s^{\eta} \tag{B1}$$

where A is a constant. When $s\gg 1$, we first assume that $f(s)$ is a smooth function of s and $f(s)\leqslant 1$, and $\eta>2$. Choosing

$$1>\delta>\frac{\eta+2}{2\eta}$$

for sufficiently large s one can rewrite Eq. (4) as

$$h_s=\frac{s}{2(s+\alpha)}\left(2\sum_{r=1}^{s^{\delta}} h_r h_{s-r}\right)+B_1 \tag{B2}$$

where

$$B_1=\frac{s}{2(s+\alpha)}\sum_{r=s^{\delta}+1}^{s-s^{\delta}-1} h_{s-r}h_r\leqslant A^2\frac{s}{s^{2\delta\eta}}\ll\frac{1}{s^{\eta+1}}\sim\frac{h_s}{s}. \tag{B3}$$

Using Taylor series, h_{s-r} can be expanded around s as

$$h_{s-r}=h_s-rh_s'+\cdots .$$

We shall assume $h_s''\ll h_s'\ll h_s$ and neglect the higher order terms in the above expansion. The validness of this assumption is based on the slow variance of h_s when s is large. This assumption will be justified when the asympotic behavior of h_s is obtained. Then

$$h_s=\frac{s}{s+\alpha}\sum_{r=1}^{s^{\delta}} h_r(h_s-rh_s')+B_1. \tag{B4}$$

From Eq. (6), we have

$$\sum_{r=1}^{s^{\delta}} h_r=1-\sum_{r=s^{\delta}+1}^{\infty} h_r=1-b_2, \tag{B5}$$

$$\sum_{r=1}^{s^{\delta}} rh_r=\sum_{r=1}^{\infty} rh_r-\sum_{r=s^{\delta}+1}^{\infty} rh_r=C-b_3, \tag{B6}$$

with

$$C=\sum_{r=1}^{\infty} rh_r. \tag{B7}$$

It is easy to find that

$$b_2\leqslant\frac{2A}{\eta-1}\frac{1}{s^{\delta(\eta-1)}}\ll\frac{1}{s},$$

$$b_3\leqslant\frac{2A}{\eta-2}\frac{1}{s^{\delta(\eta-2)}}\ll 1.$$

Therefore,

$$h_s=\frac{s}{s+\alpha}[h_s(1-b_2)-h_s'(C-b_3)]+B_1. \tag{B8}$$

From the asympotic behavior of B_1, b_2, and b_3, one knows that their contributions can be neglected when s is large. Accordingly, we have

$$h_s=\frac{s}{s+\alpha}(h_s-Ch_s') \tag{B9}$$

and

$$h_s'=-\alpha h_s/C \tag{B10}$$

and Eqs. (7) and (8) are obtained. Notice that the solution Eq. (7) indicates $f(s)=1$ and all assumptions used in this appendix are justified provided $\eta>2$.

[1] G. K. Zipf, *Human Behavior and the Principle of Least Effort* (Addison-Wesley, Cambridge, MA, 1949).
[2] V. Pareto, *Cours d'Economique Politique* (Macmillan, Paris, 1897), Vol. 2.
[3] B. Mandelbrot, Econometrica **29**, 517 (1961).
[4] B. B. Mandelbrot, C. R. Hebd. Seances Acad. Sci. **232**, 1638 (1951).
[5] A. Dragulescu and V. M. Yakovenko, Physica A **299**, 213 (2001).
[6] A. B. Atkinson and A. J. Harrison, *Distribution of Total Wealth in Britain* (Cambridge University Press, Cambridge, U.K., 1978).
[7] H. Takayasu, A.-H. Sato, and M. Takayasu, Phys. Rev. Lett. **79**, 966 (1997).
[8] P. W. Anderson, in *The Economy as an Evolving Complex System II* edited by W. B. Arthur, S. N. Durlauf, and D. A. Lane (Addison-Wesley, Reading, MA, 1997).
[9] *The Theory of Income and Wealth Distribution* edited by Y. S. Brenner *et al.*, (St. Martin's Press, New York, 1988).
[10] H. A. Simon and C. P. Bonini, Am. Econ. Rev. **48**, 607 (1958).
[11] U. G. Yule, Philos. Trans. R. Soc. London, Ser. B **213**, 21 (1924).
[12] D. G. Champernowne, Econometrica **63**, 318 (1953).
[13] H. Kesten, Acta Math. **131**, 207 (1973).
[14] S. Solomon, in *Annual Reviews of Computational Physics II*, edited by D. Stauffer (World Scientific, Singapore, 1995), p. 243.
[15] M. Levy and S. Solomon, Int. J. Mod. Phys. C **7**, 595 (1996).
[16] S. Solomon and M. Levy, Int. J. Mod. Phys. C **7**, 745 (1996).
[17] O. Malcai, O. Biham, and S. Solomon, Phys. Rev. E **60**, 1299 (1999).
[18] B. B. Mandelbrot, Int. Econom. Rev. **1**, 79 (1960).
[19] E. W. Montroll and M. F. Shlesinger, in *Nonequilibrium Phenomena II. From Stochastics to Hydrodynamics*, edited by J. L. Lebowitz and E. W. Montroll (North-Holland, Amsterdam, 1984).
[20] D. Sornette and R. Cont, J. Phys. I **7**, 431 (1997).
[21] H. Takayasu and K. Okuyama, Fractals **6**, 67 (1998).
[22] Z. A. Melzak, *Mathematical Ideas, Modeling and Applications*, Companion to Concrete Mathematics Vol. 2 (Wiley, New York, 1976), p. 279.
[23] S. Ispolatov, P. L. Krapivsky, and S. Redner, Eur. Phys. J. B **2**, 267 (1998).
[24] A. Dragulescu and V. M. Yakovenko, Eur. Phys. J. B **17**, 723 (2000).
[25] C. B. Yang, Chin. Phys. Lett. (to be published).
[26] V. M. Eguíluz and M. G. Zimmermann, Phys. Rev. Lett. **85**, 5659 (2000).
[27] R. D'Hulst and G. J. Rodgers, Eur. Phys. J. B **20**, 619 (2001).
[28] For instance see E. Levine, D. Mukamel, and G. Ziv, J. Stat. Mech.: Theory Exp. P05001 (2004).
[29] Y. B. Xie, B. H. Wang, H. J. Quan, W. S. Yang, and P. M. Hui, Phys. Rev. E **65**, 046130 (2002).
[30] Y. B. Xie, B. H. Wang, H. J. Quan, W. S. Yang, and W. N. Wang, Acta Phys. Sin. **52**, 2399 (2003) (in Chinese).
[31] Notice that if $h_s \sim 1/s^{\eta}$, then $n_s \sim 1/s^{\eta+1}$. In Ref. [5], the cumulative probability distribution w_s is discussed. Since $w_s = \sum_{s'=s}^{\infty} n'_s \sim 1/s^{\eta}$, the exponent for w_s is also greater than 2 when $\alpha > 4$ in our model.