# Two-parameter deformations of logarithm, exponential, and entropy: A consistent framework for generalized statistical mechanics

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A consistent generalization of statistical mechanics is obtained by applying the maximum entropy principle to a trace-form entropy and by requiring that physically motivated mathematical properties are preserved. The emerging differential-functional equation yields a two-parameter class of generalized logarithms, from which entropies and power-law distributions follow: these distributions could be relevant in many anomalous systems. Within the specified range of parameters, these entropies possess positivity, continuity, symmetry, expansibility, decisivity, maximality, concavity, and are Lesche stable. The Boltzmann-Shannon entropy and some one-parameter generalized entropies already known belong to this class. These entropies and their distribution functions are compared, and the corresponding deformed algebras are discussed.

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## I. INTRODUCTION

In recent years the study of an increasing number of natural phenomena that appear to deviate from standard statistical distributions has kindled interest in alternative formulations of statistical mechanics.

Among the large class of phenomena which show asymptotic power-law behaviors, we recall anomalous diffusion [1,2], turbulence [3,4], transverse-momentum distribution of hadron jets in $e^+e^-$ collisions [5], thermalization of heavy quarks in collisional process [6], astrophysics with long-range interaction [7], and others [8,9]. Typically, anomalous systems have multifractal or hierarchical structure, long-time memory, and long-range interaction [10,11].

The success of the Boltzmann-Gibbs (BG) theory has suggested that new formulations of statistical mechanics should preserve most of the mathematical and epistemological structure of the classical theory, while reproducing the emerging phenomenology of anomalous systems. To this end, new entropic forms have been introduced, which would generalize the classical one introduced by Boltzmann and Gibbs and, successively, by Shannon in a different context (BGS entropy)

$$S_{\rm BGS} = -\sum_{i=1}^{N} p_i \ln p_i. \tag{1.1}$$

There is no systematic way of deriving the "right" entropy for a given dynamical system. Among the many generalizations of the BGS entropy, one can find the entropies considered by Rényi [12], by Tsallis ($q$ entropy) [13], by Abe [14], by Tsallis, Mendes, and Plastino (escort entropy) [15], by Landsberg and Vedral [16], and recently by Kaniadakis ($\kappa$ entropy) [17,18]. For a historical outline see Ref. [19].

Generalizations lead to abandoning part of the original mathematical structure and properties. For instance, it is known that the BGS entropy is of trace form [20,21]

$$S = -\sum_{i=1}^{N} p_i \Lambda(p_i) = -\langle \Lambda(p_i)\rangle, \tag{1.2}$$

with $\Lambda(p_i)=\ln(p_i)$; on the contrary the Rényi entropy, the Landsberg-Vedral entropy, and the escort entropy are not of trace form. The Rényi entropy and the Landsberg-Vedral entropy are concave only for $0<q<1$, while the escort entropy is concave only for $q>1$ [22].

A fundamental test for a statistical functional $\mathcal{O}(p)$ of the probability distribution to be physically meaningful is given by the Lesche stability condition [23]: the relative variation $|[\mathcal{O}(p)-\mathcal{O}(p')]/\sup[\mathcal{O}(p)]|$ should go to zero in the limit that the probabilities $p_i \to p_i'$. This stability condition for a functional is a necessary but not sufficient condition for the existence of an associated observable. Lesche [24] showed that, adopting the measure $\|p-p'\|_1=\Sigma_i|p_i-p_i'|$ as estimator of the closure of the two distributions, the BGS entropy is stable, while the Rényi entropy is unstable (except for the limiting case $q=1$ corresponding to the BGS entropy) [25–27]. Also the Landsberg-Vedral entropy and the escort entropy do not satisfy the Lesche criterion [22]. On the other hand it is already known that the Abe entropy [23], the $q$ entropy [28], and the $\kappa$ entropy are stable [29].

In the present paper, a natural continuation of the work in Ref. [20], we consider the trace-form entropy given by Eq. (1.2), where $\Lambda(x)$ is an arbitrary analytic function that represents a generalized version of the logarithm, while its inverse function is the corresponding generalized exponential [30–33]. A consistent framework is maintained with the use of the maximum entropy (MaxEnt) principle. This approach yields a two-parameter class of nonstandard entropies intro-

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duced a quarter of a century ago by Mittal [34] and Sharma and Taneja [35] and successively studied by Borges and Roditi in Ref. [36]. It automatically unifies the entropies introduced by Tsallis [13], Abe [14], and Kaniadakis [17,18].

The paper is organized as follows. In Sec. II the differential-functional equation for the deformed logarithm $\Lambda(x)$ proposed in [18] is briefly reconsidered within the canonical ensemble formalism. In Sec. III we solve this equation obtaining the more general explicit form of $\Lambda(x)$. The properties of $\Lambda(x)$, and the consequent constraints on the allowed range for the deformation parameters, are discussed in Sec. IV. The deformed algebra arising from the deformed logarithms and exponential is discussed in Sec. V. Section VI is reserved to studying specific members of this class: the entropies considered by Tsallis, by Abe, and by Kaniadakis. Other up-to-now overlooked cases are also discussed. The generalized entropies and distributions related to the deformed logarithms are studied in Sec. VII. In Sec. VIII we show that this two-parametric entropy is stable according to Lesche. We summarize the results in the final section IX.

## II. CANONICAL FORMALISM

Guided by the form of the BGS entropy Eq. (1.1), we consider the following class of trace-form entropies:

$$S(p) = -\sum_{i=1}^{N} p_i \Lambda(p_i), \tag{2.1}$$

with $p \equiv \{p_i\}_{i=1,\ldots,N}$ a discrete probability distribution; one may think of $\Lambda(x)$ as a generalization of the logarithm.

We introduce the entropic functional

$$\mathcal{F}[p] = S(p) - \beta'\left(\sum_{i=1}^{N} p_i - 1\right) - \beta\left(\sum_{i=1}^{N} E_i p_i - U\right) \tag{2.2}$$

with $\beta'$ and $\beta$ Lagrange multipliers. Imposing that $\mathcal{F}[p]$ be stationary for variations of $\beta'$ and $\beta$ yields

$$\sum_{i=1}^{N} p_i = 1, \quad \sum_{i=1}^{N} E_i p_i = U, \tag{2.3}$$

which fix the normalization and mean energy for the canonical ensemble. In addition, if $\mathcal{F}[p]$ in Eq. (2.2) is stationary for variations of the probabilities $p_j$,

$$\frac{\delta}{\delta p_j}\mathcal{F}[p] = 0, \tag{2.4}$$

one finds

$$\frac{d}{dp_j}[p_j \Lambda(p_j)] = -\beta(E_j - \mu), \tag{2.5}$$

where $\mu = -\beta'/\beta$.

Without loss of generality, we can express the probability distribution $p_j$ as

$$p_j = \alpha \mathcal{E}\left(-\frac{\beta}{\lambda}(E_j - \mu)\right), \tag{2.6}$$

where $\alpha$ and $\lambda$ are two arbitrary, real and positive constants, and $\mathcal{E}(x)$ a still unspecified invertible function; we have in mind that $\mathcal{E}(x)$ be a generalization of, and in some limit reduce to, the exponential function.

Inverting Eq. (2.6) and plugging it into Eq. (2.5), one finds

$$\frac{d}{dp_j}[p_j \Lambda(p_j)] = \lambda \mathcal{E}^{-1}\left(\frac{p_j}{\alpha}\right). \tag{2.7}$$

Up to this point, $\Lambda(x)$ and $\mathcal{E}(x)$ are two unrelated functions and our only assumption has been that the entropy has trace form. Now if we require, by analogy with the relation between the exponential and the logarithm functions, that $\mathcal{E}(x)$ be the inverse function of $\Lambda(x)$, $\mathcal{E}\boldsymbol{(}\Lambda(x)\boldsymbol{)} = \Lambda\boldsymbol{(}\mathcal{E}(x)\boldsymbol{)} = x$, we obtain the following differential-functional equation for $\Lambda(x)$:

$$\frac{d}{dp_j}[p_j \Lambda(p_j)] = \lambda \Lambda\left(\frac{p_j}{\alpha}\right), \tag{2.8}$$

previously introduced in [18]. A simple and important example in this class of equations is obtained with the choice $\lambda = 1$ and $\alpha = e^{-1}$. In this case it is trivial to verify that the solution of Eq. (2.8) that satisfies the boundary conditions $\Lambda(1) = 0$ and $d\Lambda(x)/dx|_{x=1} = 1$ is $\Lambda(p_j) = \ln p_j$ and the entropy Eq. (2.1) reduces to the BGS entropy (1.1).

In this paper we will study the deformed logarithms $\Lambda(x)$ that are solutions of Eq. (2.8), the corresponding inverse functions (deformed exponentials), and the entropies that can be expressed using these deformed logarithms through Eq. (2.1).

### A. A counterexample

Since Eq. (2.8) imposes a strict condition on the form of the function $\Lambda(x)$, it is natural to ask what happens if this condition is relaxed and more general forms of deformed logarithms are considered. It should be clear from the derivation of Eq. (2.8) that, if such more general logarithms, which do not satisfy Eq. (2.8), are used to define the entropy by means of Eq. (2.1), the corresponding distributions cannot be written as Eq. (2.6) with the "exponential" $\mathcal{E}(x)$ the inverse of $\Lambda(x)$. Alternatively, if one wants that the distribution be of the form in Eq. (2.6) with the "exponential" $\mathcal{E}(x)$ the inverse of $\Lambda(x)$, the entropy cannot be Eq. (2.1).

For instance, let us consider the following family of generalized logarithms:

$$\ln_{(\kappa,\xi)}(x) = \mathrm{sgn}(x-1)|\ln_{\{\kappa\}}(x)|^{\xi}, \tag{2.9}$$

where $\ln_{\{\kappa\}}(x)$ is the $\kappa$ logarithm [17], which we discuss in Sec. VI C. This family depends on two real parameters $\kappa \in (-1,1)$ and $\xi > 0$. We observe that Eq. (2.9) is a solution of Eq. (2.8), for suitable constants $\alpha$ and $\lambda$, only for the case $\xi = 1$, with $\ln_{(\kappa,1)}(x) \equiv \ln_{\{\kappa\}}(x)$.

The inverse function of Eq. (2.9) is

$$\exp_{(\kappa,\xi)}(x) = \exp_{\{\kappa\}}[\mathrm{sgn}(x)|x|^{1/\xi}]. \tag{2.10}$$

For $\xi=1$ Eq. (2.10) reduces to the $\kappa$ exponential [17] $\exp_{\{\kappa\}}(x)$, while it reduces to the stretched exponential for $\kappa=0$. Moreover, this family of logarithms and exponentials inherits from the $\kappa$ logarithm and the $\kappa$ exponential the properties $\ln_{(\kappa,\xi)}(1/x)=-\ln_{(\kappa,\xi)}(x)$ and $\exp_{(\kappa,\xi)}(-x)\exp_{(\kappa,\xi)}(x)=1$.

Introducing the entropy

$$S_{\kappa,\xi}(p) = -\lambda\sum_{i=1}^{N}\int_0^{p_i}\ln_{(\kappa,\xi)}\left(\frac{x}{\alpha}\right)dx + \lambda\int_0^1 \ln_{(\kappa,\xi)}\left(\frac{x}{\alpha}\right)dx, \tag{2.11}$$

the variational principle yields

$$p_j = \alpha\exp_{(\kappa,\xi)}\left(-\frac{\beta}{\lambda}(E_j-\mu)\right). \tag{2.12}$$

Equation (2.12) becomes the $\kappa$ distribution [17] for $\xi=1$ and the stretched exponential distribution for $\kappa=0$; correspondingly Eq. (2.11) reduces to the $\kappa$ entropy

$$S_\kappa(p) = -\sum_{i=1}^{N} p_i\ln_{\{\kappa\}}(p_i) \tag{2.13}$$

in the $\xi\to 1$ limit [18], whereas in the $\kappa\to 0$ limit ($\alpha=\lambda=1$) it reduces to the stretched exponential entropy [37,38]

$$S_\xi(p) = \sum_{i=1}^{N}\Gamma(1+\xi,-\ln p_i) - \Gamma(1+\xi), \tag{2.14}$$

where $\Gamma(\mu,x)$ is the incomplete gamma function of the second kind and $\Gamma(\mu)=\Gamma(\mu,0)$ is the gamma function [39]. Equation (2.14) demonstrates that the entropy (2.11) has in general a form different from Eq. (2.1): $S_{\kappa,\xi}\neq -\Sigma_i p_i\ln_{(\kappa,\xi)}(p_i)$.

### B. Integrals of the functions $\Lambda(x)$ and $\mathcal{E}(x)$

The fact that the generalized logarithm $\Lambda(x)$ is a solution of the differential equation (2.8) is sufficient to calculate its integral

$$\int_{x_1}^{x_2}\Lambda(x)dx = \frac{x_2\Lambda(\alpha x_2) - x_1\Lambda(\alpha x_1)}{\lambda}. \tag{2.15}$$

From the definition of the generalized exponential $\mathcal{E}(x)$ as the inverse of the generalized logarithm $\Lambda(x)$ it is also simple to calculate the integral of $\mathcal{E}(x)$ with the change of variable $x=\mathcal{E}^{-1}(s)=\Lambda(s)$,

$$\int_{x_1}^{x_2}\mathcal{E}(x)dx = x_2\mathcal{E}(x_2) - x_1\mathcal{E}(x_1) - \frac{\mathcal{E}(x_2)\Lambda(\alpha\mathcal{E}(x_2)) - \mathcal{E}(x_1)\Lambda(\alpha\mathcal{E}(x_1))}{\lambda}. \tag{2.16}$$

## III. SOLUTIONS OF THE DIFFERENTIAL-FUNCTIONAL EQUATION

In this section we study the solutions of Eq. (2.8), which we rewrite in the following form:

$$\frac{d}{dx}[x\Lambda(x)] - \lambda\Lambda\left(\frac{x}{\alpha}\right) = 0. \tag{3.1}$$

We shall select the solutions of Eq. (3.1) that satisfy appropriate boundary conditions and that keep those properties of the standard logarithms that we judge important even for a generalized logarithm.

By performing the change of variable

$$x = \exp\left(\frac{t}{\lambda\alpha}\right) \tag{3.2}$$

and introducing the function

$$\Lambda(x) = \frac{1}{x}f(\lambda\alpha\ln x), \tag{3.3}$$

the homogeneous differential-functional equation of the first order shown in Eq. (3.1) becomes

$$\frac{d\,f(t)}{dt} - f(t-t_0) = 0, \tag{3.4}$$

with $t_0=\lambda\alpha\ln\alpha$. The most general solution of Eq. (3.4), a differential-difference equation belonging to the class of delay equations [40], can be written in the form

$$f(t) = \sum_{i=1}^{n}\sum_{j=0}^{m_i-1} a_{ij}(s_1,\ldots,s_n)t^j e^{s_i t}, \tag{3.5}$$

where $n$ is the number of independent solutions $s_i$ of the characteristic equation

$$s_i - e^{-t_0 s_i} = 0, \tag{3.6}$$

$m_i$ their multiplicity [$s_i$ is a solution not only of Eq. (3.6), but also of its first $m_i-1$ derivatives], and $a_{ij}$ multiplicative coefficients that depend on the parameters $s_i$. In terms of the original function and variable the general solution and the characteristic equation are

$$\Lambda(x) = \sum_{i=1}^{n}\sum_{j=0}^{m_i-1} a_{ij}(s_1,\ldots,s_n)[\lambda\,\alpha\ln(x)]^j x^{\lambda\,\alpha\,s_i-1} = \sum_{i=1}^{n}\sum_{j=0}^{m_i-1} a'_{ij}(\kappa_1,\ldots,\kappa_n)[\ln(x)]^j x^{\kappa_i}, \tag{3.7}$$

$$1+\kappa_i = \lambda\,\alpha^{-\kappa_i}, \tag{3.8}$$

where $\kappa_i=\lambda\,\alpha\,s_i-1$ and $a'_{ij}=(\lambda\,\alpha)^j a_{ij}$.

In the present work, we are interested in nonoscillatory solutions for Eq. (3.4): this kind of solution maintains a closer relation with the standard logarithm. Therefore, we consider only real solutions of Eq. (3.6). There exist four different cases depending on the value of $t_0$: (a) for $t_0\geqslant 0$ we have one solution, $n=1$ and $m=1$; (b) for $-1/e<t_0<0$, we have two nondegenerate solutions, $n=2$ and $m_i=1$; (c) for

$t_0=-1/e$ we have two degenerate solutions, $n=1$ and $m=2$; (d) for $t_0<-1/e$ there exist no solutions.

We discuss in order the three cases (a), (b), and (c) that yield solutions of the delay equation (3.4) and, therefore, of the corresponding Eq. (3.1).

The case (a) is the least interesting:

$$\Lambda(x) = a x^{\kappa} \tag{3.9}$$

is just a single power and cannot change sign as one would require from a logarithm.

In the case (b), we obtain a binomial solution:

$$\Lambda(x) = A_1(\kappa_1,\kappa_2)x^{\kappa_1} + A_2(\kappa_1,\kappa_2)x^{\kappa_2}, \tag{3.10}$$

where $A_1$ and $A_2$ are the integration constants.

The characteristic equation (3.8) can be solved for the two constants $\alpha$ and $\lambda$,

$$\alpha = \left(\frac{1+\kappa_2}{1+\kappa_1}\right)^{1/(\kappa_1-\kappa_2)}, \tag{3.11}$$

$$\lambda = \frac{(1+\kappa_2)^{\kappa_1/(\kappa_1-\kappa_2)}}{(1+\kappa_1)^{\kappa_2/(\kappa_1-\kappa_2)}}. \tag{3.12}$$

The two arbitrary coefficients $A_1$ and $A_2$ correspond to the freedom of scaling $x$ and $\Lambda(x)$ in Eq. (3.1). We fix these integration constants using the two boundary conditions

$$\Lambda(1) = 0, \tag{3.13}$$

$$\left.\frac{d\,\Lambda(x)}{d\,x}\right|_{x=1} = 1. \tag{3.14}$$

The first condition implies $A_1=-A_2\equiv A$ while from the second one has $A=1/(\kappa_1-\kappa_2)$. Equation (3.10) assumes the final expression

$$\Lambda(x) = \frac{x^{\kappa_1} - x^{\kappa_2}}{\kappa_1 - \kappa_2}, \tag{3.15}$$

a two-parameter function which reduces to the standard logarithm in the $(\kappa_1,\kappa_2)\to(0,0)$ limit.

After introducing the notation $\Lambda(x)=\ln_{\{\kappa,r\}}(x)$ and the two auxiliary parameters $\kappa=(\kappa_1-\kappa_2)/2$ and $r=(\kappa_1+\kappa_2)/2$, Eq. (3.15) becomes

$$\ln_{\{\kappa,r\}}(x) = x^r\frac{x^{\kappa}-x^{-\kappa}}{2\,\kappa} = x^r \ln_{\{\kappa\}}(x). \tag{3.16}$$

The constants $\alpha$ and $\lambda$ expressed in terms of $\kappa$ and $r$

$$\alpha = \left(\frac{1+r-\kappa}{1+r+\kappa}\right)^{1/2\kappa}, \tag{3.17}$$

$$\lambda = \frac{(1+r-\kappa)^{(r+\kappa)/2\kappa}}{(1+r+\kappa)^{(r-\kappa)/2\kappa}}, \tag{3.18}$$

are symmetric for $\kappa\leftrightarrow-\kappa$ and satisfy the useful relations $(1+r\pm\kappa)\alpha^{r\pm\kappa}=\lambda$, which is the characteristic equation (3.8), and $1/\lambda=\ln_{\{\kappa,r\}}(1/\alpha)$, which is the differential equation (3.1) at $x=1$. In the following, we call the solution (3.16) the deformed logarithm or $(\kappa,r)$ logarithm.

Finally, we consider the case (c). This case can also be obtained as the limit of case (b) when the two distinct solutions $s_1$ and $s_2$ become degenerate. When $t_0=-1/e$, $s=e$ verifies not only Eq. (3.6), but also its first derivative $1+t_0\exp(-t_0 s)=0$: this solution is twice degenerate ($s_1=s_2=e$); Eq. (3.7) with the boundary conditions (3.13) and (3.14) becomes

$$\Lambda(x) = x^r \ln x, \tag{3.19}$$

where the parameter $r=\lambda\,\alpha\,e-1$. The standard logarithm $\Lambda(x)=\ln x$ is recovered for $r=0$; the same standard logarithm is actually recovered from the $(\kappa,r)$ logarithm in the limit $(\kappa,r)\to(0,0)$ independently of the direction.

## IV. PROPERTIES OF DEFORMED FUNCTIONS

The properties of the entropy (2.1), and of the corresponding distribution (2.6), follow from the properties of the deformed logarithm $\Lambda(x)\equiv\ln_{\{\kappa,r\}}(x)$, which is used in its definition. Naudts [30] gives a list of general properties that a deformed logarithm must satisfy in order that the ensuing entropy and distribution function be physical. In this section we determine the region of parameter space $(\kappa,r)$ where the logarithm (3.16) satisfies these properties and list the corresponding properties of its inverse, the $(\kappa,r)$ exponential.

### A. (κ,r)-deformed logarithm

The following properties for the $(\kappa,r)$ logarithm hold when $\kappa$ and $r$ satisfy the corresponding limitations:

$$\ln_{\{\kappa,r\}}(x) \in C^{\infty}(\mathbb{R}^+), \tag{4.1}$$

$$\frac{d}{dx}\ln_{\{\kappa,r\}}(x) > 0, \quad -|\kappa| \leqslant r \leqslant |\kappa|, \tag{4.2}$$

$$\frac{d^2}{dx^2}\ln_{\{\kappa,r\}}(x) < 0, \quad -|\kappa| \leqslant r \leqslant \frac{1}{2} - \left|\frac{1}{2}-|\kappa|\right|, \tag{4.3}$$

$$\ln_{\{\kappa,r\}}(1) = 0, \tag{4.4}$$

$$\int_0^1 \ln_{\{\kappa,r\}}(x)dx = -\frac{1}{(1+r)^2-\kappa^2}, \quad 1+r > |\kappa|, \tag{4.5}$$

$$\int_0^1 \ln_{\{\kappa,r\}}\left(\frac{1}{x}\right)dx = \frac{1}{(1-r)^2-\kappa^2}, \quad 1-r > |\kappa|. \tag{4.6}$$

Equation (4.1) states that the $(\kappa,r)$ logarithm is an analytical function for all $x\geqslant 0$ and for all $\kappa,r\in\mathbb{R}$; Eq. (4.2) that it is a strictly increasing function for $-|\kappa|\leqslant r\leqslant|\kappa|$; Eq. (4.3) that it is concave for $-|\kappa|\leqslant r\leqslant|\kappa|$, when $|\kappa|<1/2$, and for $-|\kappa|\leqslant r<1-|\kappa|$, when $|\kappa|\geqslant 1/2$; Eq. (4.4) states that the $(\kappa,r)$ logarithm satisfies the boundary condition (3.13); Eqs. (4.5) and (4.6) that it has at most integrable divergences for $x\to 0^+$ and $x\to+\infty$. These two last conditions (4.5) and (4.6) assure the normalization of the canonical ensembles distribu-

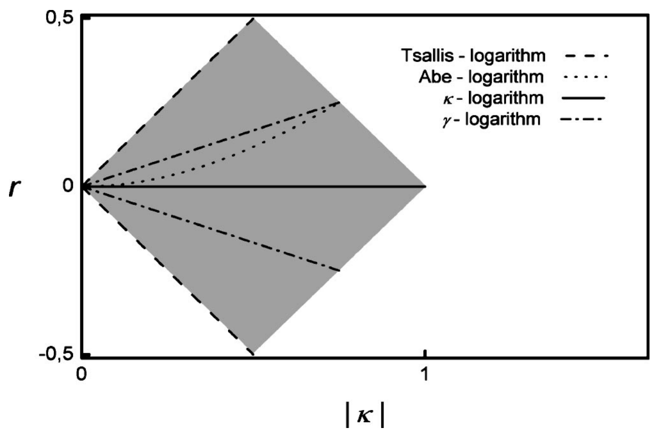


FIG. 1. Parameter space $(\kappa,r)$ for the logarithm (3.16). The shaded region represents the constraints of Eq. (4.7) on the parameters. The four lines, dashed, dotted, solid, and dash-dotted, correspond to the Tsallis (6.1), Abe (6.5), $\kappa$ (6.7), and $\gamma$ (6.12) logarithm, respectively.

tion arising from entropy (2.1). Conditions (4.2)–(4.6) select the following region:

$$\mathbb{R}^2 \supset \mathcal{R} = \begin{cases} -|\kappa| \leqslant r \leqslant |\kappa| & \text{if } 0 \leqslant |\kappa| < \frac{1}{2}, \\ |\kappa| - 1 < r < 1 - |\kappa| & \text{if } \frac{1}{2} \leqslant |\kappa| < 1, \end{cases} \tag{4.7}$$

which is shown in Fig. 1: every point in $\mathcal{R}$ selects a deformed logarithm that satisfies all properties (4.1)–(4.6). This region $\mathcal{R}$ include values of the parameters for which the logarithm is finite in the limit $x\to 0$ or $x\to+\infty$. Note that points in region (4.7) always satisfy $|\kappa|<1$, the condition obtained in Ref. [18] for the case $r=0$. In addition, $\kappa\to 0$ implies $r\to 0$. Following Ref. [30] we introduce the dual logarithm

$$\ln^*_{\{\kappa,r\}}(x) = -\ln_{\{\kappa,r\}}\left(\frac{1}{x}\right), \tag{4.8}$$

which is related to the original $(\kappa,r)$ logarithm by

$$\ln^*_{\{\kappa,r\}}(x) = \ln_{\{\kappa,-r\}}(x) = x^{-2r}\ln_{\{\kappa,r\}}(x). \tag{4.9}$$

Note that this last result implies that Eq. (4.6) is equivalent to Eq. (4.5) on exchanging $r\leftrightarrow -r$. Let us define that a logarithm is self-dual when

$$\ln_{\{\kappa,r\}}(x) = \ln^*_{\{\kappa,r\}}(x). \tag{4.10}$$

Then Eq. (4.9) shows that $\ln_{\{\kappa,r\}}(x)$ is self-dual if and only if $r=0$: this case coincides with the $\kappa$ logarithm [17,18].

The asymptotic behavior of $\ln_{\{\kappa,r\}}(x)$ for $x$ approaching zero is

$$\ln_{\{\kappa,r\}}(x) \underset{x\to 0^+}{\sim} -\frac{1}{2|\kappa|}\frac{1}{x^{|\kappa|-r}}; \tag{4.11}$$

in particular it results that $\ln_{\{\kappa,r\}}(0^+)=-\infty$ for $r<|\kappa|$, while, if $r=|\kappa|$, $\ln_{\{\kappa,|\kappa|\}}(0^+)=-1/2|\kappa|$ is finite. Consequently, its inverse $\exp_{\{\kappa,|\kappa|\}}(x)$ goes to zero at finite $x$: the distribution function has a cutoff at finite energy, $\exp_{\{\kappa,|\kappa|\}}(-1/2|\kappa|)=0$.

Analogously, the behavior of $\ln_{\{\kappa,r\}}(x)$ for large values of $x$ is

$$\ln_{\{\kappa,r\}}(x) \underset{x\to+\infty}{\sim} \frac{x^{|\kappa|+r}}{2|\kappa|}, \tag{4.12}$$

which implies $\ln_{\{\kappa,r\}}(+\infty)=+\infty$ for $r>-|\kappa|$, while again, if $r=-|\kappa|$, then $\ln_{\{\kappa,-|\kappa|\}}(+\infty)=1/2|\kappa|$ and $\exp_{\{\kappa,-|\kappa|\}}(1/2|\kappa|)=+\infty$.

The generalized logarithm verifies the following scaling law

$$a\ln_{\{\kappa,r\}}(x) = \ln_{\{\kappa/a,r/a\}}(x^a), \tag{4.13}$$

of which Eq. (4.9) is a particular case for $a=-1$, and that becomes $a\ln x=\ln(x^a)$ when $(\kappa,r)\to(0,0)$.

In the following we give some useful relations. First the relation

$$\ln_{\{\kappa,r\}}(xy) = \frac{1}{2}(x^{r+\kappa}+x^{r-\kappa})\ln_{\{\kappa,r\}}(y) + \frac{1}{2}(y^{r+\kappa}+y^{r-\kappa})\ln_{\{\kappa,r\}}(x) \tag{4.14}$$

is easily proved taking into account the definition (3.16). By using the identity $y^{r-\kappa}=y^{r+\kappa}-2\kappa\ln_{\{\kappa,r\}}(y)$, Eq. (4.14) becomes

$$\ln_{\{\kappa,r\}}(xy) = x^{r+\kappa}\ln_{\{\kappa,r\}}(y) + y^{r+\kappa}\ln_{\{\kappa,r\}}(x) - 2\kappa\ln_{\{\kappa,r\}}(x)\ln_{\{\kappa,r\}}(y). \tag{4.15}$$

Moreover, using

$$\frac{1}{2}(x^{r+\kappa}+x^{r-\kappa}) = -(1+r)\ln_{\{\kappa,r\}}(x) + \lambda\ln_{\{\kappa,r\}}(x/\alpha), \tag{4.16}$$

Eq. (4.14) can be rewritten as

$$\ln_{\{\kappa,r\}}(xy) = -2(1+r)\ln_{\{\kappa,r\}}(x)\ln_{\{\kappa,r\}}(y) + \lambda\ln_{\{\kappa,r\}}(x)\ln_{\{\kappa,r\}}\left(\frac{y}{\alpha}\right) + \lambda\ln_{\{\kappa,r\}}\left(\frac{x}{\alpha}\right)\ln_{\{\kappa,r\}}(y). \tag{4.17}$$

### B. $(\kappa,r)$-deformed exponential

The deformed logarithm is a strictly increasing function for $-|\kappa|\leqslant r\leqslant|\kappa|$; therefore, it can be inverted for $(\kappa,r)\in\mathcal{R}$. We call its inverse the deformed exponential $\exp_{\{\kappa,r\}}(x)$, whose analytical properties follow from those of the deformed logarithm:

$$\exp_{\{\kappa,r\}}(x) \in C^\infty(\mathbb{I}), \tag{4.18}$$

$$\frac{d}{dx}\exp_{\{\kappa,r\}}(x) > 0, \tag{4.19}$$

$$\frac{d^2}{dx^2}\exp_{\{\kappa,r\}}(x) > 0, \tag{4.20}$$

$$\exp_{\{\kappa,r\}}(0) = 1, \tag{4.21}$$

$$\int_{-\infty}^{0} \exp_{\{\kappa,r\}}(x)dx = \frac{1}{(1+r)^2 - \kappa^2}, \tag{4.22}$$

$$\int_{-\infty}^{0} \frac{dx}{\exp_{\{\kappa,r\}}(-x)} = \frac{1}{(1-r)^2 - \kappa^2}. \tag{4.23}$$

Equation (4.18) states that the deformed exponential $\exp_{\{\kappa,r\}}(x)$ is a continuous function for all $x \in \mathbb{I}$, where $\mathbb{I} = \mathbb{R}^+$, when $-|\kappa| < r < |\kappa|$, $\mathbb{I} = (-1/2|\kappa|, \infty)$, when $r = |\kappa|$, and $\mathbb{I} = (-\infty, 1/2|\kappa|)$, when $r = -|\kappa|$. Equations (4.19)–(4.23) state that $\exp_{\{\kappa,r\}}(x)$ is a strictly increasing and convex function, normalized according to Eq. (4.21), and which goes to zero fast enough to be integrable for $x \to \pm\infty$.

Introducing the dual of the exponential function

$$\exp^*_{\{\kappa,r\}}(x) = \frac{1}{\exp_{\{\kappa,r\}}(-x)}, \tag{4.24}$$

Eq. (4.9) implies

$$\exp^*_{\{\kappa,r\}}(x) = \exp_{\{\kappa,-r\}}(x), \tag{4.25}$$

which means

$$\exp_{\{\kappa,r\}}(x)\exp_{\{\kappa,-r\}}(-x) = 1. \tag{4.26}$$

Only when $r=0$ does this relation reproduce that of the standard exponential [18].

The asymptotic behaviors (4.11) and (4.12) of $\ln_{\{\kappa,r\}}(x)$ imply

$$\exp_{\{\kappa,r\}}(x) \underset{x\to\pm\infty}{\sim} |2\kappa x|^{1/(r\pm|\kappa|)}; \tag{4.27}$$

in particular

$$\exp_{\{\kappa,r\}}(-\infty) = 0^+ \quad \text{for } r < |\kappa|, \tag{4.28}$$

$$\exp_{\{\kappa,r\}}(+\infty) = +\infty \quad \text{for } r > -|\kappa|, \tag{4.29}$$

while

$$\exp_{\{\kappa,r\}}(-1/2|\kappa|) = 0^+ \quad \text{when } r = |\kappa|, \tag{4.30}$$

$$\exp_{\{\kappa,r\}}(+1/2|\kappa|) = +\infty \quad \text{when } r = -|\kappa|. \tag{4.31}$$

Finally, the scaling law

$$[\exp_{\{\kappa,r\}}(x)]^a = \exp_{\{\kappa/a,r/a\}}(ax) \tag{4.32}$$

reduces to Eq. (4.26) for $a=-1$ and reproduces the property $[\exp(x)]^a = \exp(ax)$ in the $(\kappa,r)\to(0,0)$ limit.

## V. DEFORMED ALGEBRA

Using the definition of the deformed logarithm and its inverse function, we can introduce two composition laws, the deformed sum $x \overset{\kappa,r}{\oplus} y$ and product $x \underset{\kappa,r}{\otimes} y$.

Let us define the deformed sum:

$$x \overset{\kappa,r}{\oplus} y = \ln_{\{\kappa,r\}}[\exp_{\{\kappa,r\}}(x)\exp_{\{\kappa,r\}}(y)], \tag{5.1}$$

which reduces, in the $(\kappa,r)\to(0,0)$ limit, to the ordinary sum $x \overset{0,0}{\oplus} y = x+y$. Its definition implies that the deformed sum satisfies the following properties: (a) it is associative; (b) it is commutative; (c) its neutral element is 0; (d) the opposite of $x$ is $\ln_{\{\kappa,r\}}[1/\exp_{\{\kappa,r\}}(x)]$.

If $x$ and $y$ are positive Eq. (5.1) yields

$$\ln_{\{\kappa,r\}}(xy) = \ln_{\{\kappa,r\}}(x) \overset{\kappa,r}{\oplus} \ln_{\{\kappa,r\}}(y), \tag{5.2}$$

which, when $(\kappa,r)\to(0,0)$, reduces to the well-known property $\log(xy) = \log x + \log y$.

In the same way, let us introduce the deformed product between positive $x$ and $y$:

$$x \underset{\kappa,r}{\otimes} y = \exp_{\{\kappa,r\}}[\ln_{\{\kappa,r\}}(x) + \ln_{\{\kappa,r\}}(y)], \tag{5.3}$$

which reduces, for $(\kappa,r)\to(0,0)$, to the ordinary product $x \underset{0,0}{\otimes} y = xy$. This product satisfies the following properties: (a) it is associative; (b) it is commutative; (c) its neutral element is 1; (d) the inverse element of $x$ is $\exp_{\{\kappa,r\}}[-\ln_{\{\kappa,r\}}(x)]$.

According to Eq. (5.3) we have

$$\exp_{\{\kappa,r\}}(x+y) = \exp_{\{\kappa,r\}}(x) \underset{\kappa,r}{\otimes} \exp_{\{\kappa,r\}}(y), \tag{5.4}$$

which reproduces in the $(\kappa,r)\to(0,0)$ limit the well-known property of the exponential $\exp(x)\exp(y) = \exp(x+y)$.

Note that the algebraic structures $A_1 \equiv (\mathbb{R}, \overset{\kappa,r}{\oplus})$ and $A_2 \equiv (\mathbb{R}^+, \underset{\kappa,r}{\otimes})$ are two Abelian groups. The deformed sum (5.1) and product (5.3) are not distributive and the structure $A_3 \equiv (\mathbb{R}^+, \overset{\kappa,r}{\oplus}, \underset{\kappa,r}{\otimes})$ is not an Abelian field. In any case, following Ref. [18] it is possible to define a deformed product $\overset{\kappa,r}{\otimes}$ and sum $\underset{\kappa,r}{\oplus}$ which are distributive with respect to $\overset{\kappa,r}{\oplus}$ and $\underset{\kappa,r}{\otimes}$, respectively, so that the structures $\mathcal{A}_1 \equiv (\mathbb{R}^+, \overset{\kappa,r}{\oplus}, \overset{\kappa,r}{\otimes})$ and $\mathcal{A}_2 \equiv (\mathbb{R}^+, \underset{\kappa,r}{\oplus}, \underset{\kappa,r}{\otimes})$ are Abelian.

Finally, from Eqs. (4.17) and (5.2), we obtain

$$x \overset{\kappa,r}{\oplus} y = x[\exp_{\{\kappa,r\}}(y)]^{r+\kappa} + y[\exp_{\{\kappa,r\}}(x)]^{r+\kappa} - 2\kappa xy. \tag{5.5}$$

From the practical point of view this last expression, like all the expressions involving the $(\kappa,r)$ exponentials, are more useful for those particular values of $r$ and $\kappa$ for which an explicit closed form of the $(\kappa,r)$ exponential can be given. In the next section we shall see some examples.

## VI. EXAMPLES OF ONE-PARAMETER DEFORMED LOGARITHMS

The two-parameter class of deformed logarithms (3.16) includes an infinity of one-parameter deformed logarithms that can be specified by selecting a relation between $\kappa$ and $r$. In this section we discuss a few specific one-parameter logarithms that are already known in the literature and have been used to define entropies in the context of generalizations of statistical mechanics and thermodynamics: we show that they are in fact members of the same two-parameter class; we also introduce a few different examples of one-parameter logarithms.

### A. Tsallis logarithm

The first example is obtained with the choice $r=-\kappa$ for $-1/2<\kappa<1/2$. After introducing the parameter $q=1+2\kappa$ $[0<q<2]$ we obtain the Tsallis logarithm $\ln_q(x) \equiv \ln_{\{(q-1)/2,(1-q)/2\}}(x)$ and the Tsallis exponential $\exp_q(x)$ as follows:

$$\ln_q(x) = \frac{x^{1-q}-1}{1-q}, \tag{6.1}$$

$$\exp_q(x) = [1+(1-q)x]^{1/(1-q)}. \tag{6.2}$$

The relation (4.9) reads

$$\ln_q(x) = -\ln_{2-q}\left(\frac{1}{x}\right), \tag{6.3}$$

while Eq. (4.15) becomes

$$\ln_q(xy) = \ln_q(x) + \ln_q(y) + (1-q)\ln_q(x)\ln_q(y). \tag{6.4}$$

The $q$-deformed algebra already discussed in Refs. [41,42] results as a particular case with $r=-\kappa=(1-q)/2$ of the deformed algebra discussed in Sec. V.

### B. Abe logarithm

As a second example we consider the constraint $(r+1)^2 =1+\kappa^2$ and define $q_A=r+\kappa+1$. Then the two-parameter logarithm in Eq. (3.16) becomes the logarithm associated with the entropy introduced by Abe [14],

$$\ln_{q_A}(x) = \frac{x^{(q_A^{-1})-1} - x^{q_A-1}}{q_A^{-1}-q_A}, \tag{6.5}$$

which reduces to the standard logarithm for $q_A \to 1$. The invariance of Eq. (3.16) for $\kappa \to -\kappa$ results in $\ln_{q_A}(x)$ being invariant for $q_A \to 1/q_A$. In this case the inverse function of the Abe logarithm (6.5), which exists because $\ln_{q_A}(x)$ is monotonic for $1/2<q_A<2$, cannot be expressed in terms of elementary functions, since Eq. (6.5) is not invertible algebraically. We remark that Eq. (4.15) in the present case reads

$$\ln_{q_A}(xy) = x^{q_A-1}\ln_{q_A}(y) + y^{q_A-1}\ln_{q_A}(x) + (q_A^{-1}-q_A)\ln_{q_A}(x)\ln_{q_A}(y). \tag{6.6}$$

### C. $\kappa$ logarithm

Our third example is obtained with the constraint $r=0$. Introducing the notation $\ln_{\{\kappa\}}(x) \equiv \ln_{\{\kappa,0\}}(x)$ from Eq. (3.16) we obtain the $\kappa$ logarithm and consequently its inverse function, namely, the $\kappa$ exponential introduced in [17,18]:

$$\ln_{\{\kappa\}}(x) = \frac{x^{\kappa}-x^{-\kappa}}{2\kappa}, \tag{6.7}$$

$$\exp_{\{\kappa\}}(x) = (\sqrt{1+\kappa^2x^2}+\kappa x)^{1/\kappa}, \tag{6.8}$$

with $\kappa \in (-1,1)$. We remind the reader that, because of property (4.9), the $\kappa$ logarithm is the only member of the family that is self-dual,

$$\ln_{\{\kappa\}}(x) = -\ln_{\{\kappa\}}\left(\frac{1}{x}\right). \tag{6.9}$$

The function $\exp_{\{\kappa\}}(x)$ increases at the same rate that the function $\exp_{\{\kappa\}}(-x)$ decreases,

$$\exp_{\{\kappa\}}(x)\exp_{\{\kappa\}}(-x) = 1. \tag{6.10}$$

The $\kappa$ deformed sum is obtained from the more general Eq. (5.1) by setting $r=0$:

$$x \stackrel{\kappa}{\oplus} y = x\sqrt{1+\kappa^2y^2} + y\sqrt{1+\kappa^2x^2}, \tag{6.11}$$

which reduces to the ordinary sum for $\kappa \to 0$. The opposite approach, i.e., starting from the $\kappa$ deformed sum (6.11) to obtain the $\kappa$ logarithm and $\kappa$ exponential has been taken in Refs. [17,18], where it is shown that the $\kappa$ deformed sum is the additivity law of relativistic momenta.

### D. Other examples

If we define the parameter $w=r/|\kappa|$, we observe that when $w=0, \pm 1/3, \pm 1/2, \pm 1, \pm 5/3, \pm 2, \pm 3, \pm 5$, and $\pm 7$, the inverse function of the deformed logarithm can be found by solving an algebraic equation of degree not larger than 4; the corresponding deformed exponential can be written explicitly. In particular, the cases $w=0$ and $\pm 1$ correspond, respectively, to the $\kappa$ logarithm and to the $q$ logarithm; the remaining cases are different. Among these additional logarithms and corresponding exponentials, only the cases $w=\pm 1/3$ and $\pm 1/2$ satisfy all the requirements discussed in Sec. IV.

We consider explicitly the case $r=\pm|\kappa|/3$. Introducing the parameter $\gamma=\pm 2|\kappa|/3$, Eq. (3.16) defines a generalized logarithm

$$\log_{\gamma}(x) = \frac{x^{2\gamma}-x^{-\gamma}}{3\gamma}, \tag{6.12}$$

which reduces to the standard ones in the $\gamma \to 0$ limit. This logarithm is an analytical, concave, and increasing function for all $x \geqslant 0$, when $-1/2<\gamma<1/2$.

If $\gamma$ is positive the asymptotic behaviors for $x \to 0$ and $x \to \infty$ are

$$\ln_\gamma(x) \underset{x\to 0^+}{\sim} -\frac{x^{-\gamma}}{3\gamma}, \quad \ln_\gamma(x) \underset{x\to +\infty}{\sim} \frac{x^{2\gamma}}{3\gamma}. \tag{6.13}$$

Since the logarithm (6.12) satisfies the duality relation $\ln_\gamma(x)=-\ln_{-\gamma}(1/x)$, the asymptotic behaviors for $x\to 0^+$ and $x\to\infty$ in Eq. (6.13) are exchanged when $\gamma<0$.

The corresponding $\gamma$ exponential is

$$\exp_\gamma(x) = \left[\left(\frac{1+\sqrt{1-4\gamma^3x^3}}{2}\right)^{1/3} + \left(\frac{1-\sqrt{1-4\gamma^3x^3}}{2}\right)^{1/3}\right]^{1/\gamma}, \tag{6.14}$$

which is an analytic, monotonic, and convex function for all $x\in\mathbb{R}$ when $-1/2<\gamma<1/2$, and reduces to the standard exponential in the limit $\gamma\to 0$. The asymptotic power-law behaviors for $\gamma$ positive are

$$\exp_\gamma(x) \underset{x\to -\infty}{\sim} (3\gamma|x|)^{-1/\gamma}, \quad \exp_\gamma(x) \underset{x\to +\infty}{\sim} (3\gamma x)^{1/2\gamma}, \tag{6.15}$$

where it is clear that the asymptotic behaviors for $x\to+\infty$ and $x\to-\infty$ are exchanged when $\gamma$ changes sign, coherently with the property $\exp_\gamma(x)\exp_{-\gamma}(-x)=1$. Finally the deformed sum given by Eq. (5.1) becomes

$$x \overset{\gamma}{\oplus} y = -3\gamma xy + x\left[\left(\frac{1+\sqrt{1-4\gamma^3y^3}}{2}\right)^{1/3} + \left(\frac{1-\sqrt{1-4\gamma^3y^3}}{2}\right)^{1/3}\right]^2 + y\left[\left(\frac{1+\sqrt{1-4\gamma^3x^3}}{2}\right)^{1/3} + \left(\frac{1-\sqrt{1-4\gamma^3x^3}}{2}\right)^{1/3}\right]^2 \tag{6.16}$$

and reduces to the ordinary sum for $\gamma\to 0$.

## VII. ENTROPIES AND DISTRIBUTIONS

Having obtained the deformed logarithm as a solution of the differential equation (3.1), the corresponding generalized entropy follows from Eq. (2.1):

$$S_{\kappa,r}(p) = -\sum_{i=1}^N p_i \ln_{\{\kappa,r\}}(p_i) = -\sum_{i=1}^N p_i^{1+r}\frac{p_i^\kappa - p_i^{-\kappa}}{2\kappa}. \tag{7.1}$$

We observe that this class of entropies coincides with the one introduced by Mittal [34] and Sharma and Taneja [35] (MST) and successively derived by Borges and Roditi in Ref. [36] by using an approach based on a biparametric generalization of the Jackson derivative.

Since the entropy is defined in terms of the deformed logarithm (3.16), the properties (4.1)–(4.6) of $\ln_{\{\kappa,r\}}(x)$ assure that the entropy (7.1) satisfies many of the properties satisfied by the standard BGS entropy (1.1). In particular, it is (a) positive definite, $S_{\kappa,r}(p)\geqslant 0$ for $p\in[0,1]$; (b) continuous; (c) symmetric, $S_{\kappa,r}(p)=S_{\kappa,r}(q)$ with $p\equiv(p_1,\dots,p_N)$ and $q\equiv(p_{\pi(1)},\dots,p_{\pi(N)})$ where $\tau$ is any permutation from 1 to $N$; (d) expansible, which means that $S_{\kappa,r}(p)=S_{\kappa,r}(q)$ for $p\equiv(p_1,\dots,p_N)$ and $q\equiv(p_1,\dots,p_N,0,\dots,0)$; (e) decisive, in the sense that $S_{\kappa,r}(p^{(0)})=0$ where $p^{(0)}\equiv(0,\dots,1,\dots,0)$ is a completely ordered state; (f) maximal, which means that entropy reaches its maximal value when the distribution is uniform, $\max[S(p)]$ for $p=p^{(U)}$, with $p^{(U)}\equiv(1/N,\dots,1/N)$; and, finally, (g) concave. Moreover, it will be shown in the next section that the whole family of entropies (7.1) satisfies the Lesche inequality.

We observe that for a uniform distribution $p^{(U)}$, the entropy (7.1) assumes the expression

$$S_{\kappa,r}(p^{(U)}) = -\ln_{\{\kappa,r\}}\left(\frac{1}{N}\right), \tag{7.2}$$

and only for the case $r=0$, according to Eq. (4.9), does it become

$$S_\kappa(p^{(U)}) = \ln_{\{\kappa\}}(N), \tag{7.3}$$

which is the generalization of the well-known Boltzmann formula and gives the entropy of a nonextensive microcanonical system as the deformed logarithm of the number of accessible states of the system. If the alternative form

$$S_{\kappa,r}(p) = \sum_i p_i \ln_{\{\kappa,r\}}\left(\frac{1}{p_i}\right) \tag{7.4}$$

is adopted, the entropy of a uniform distribution reduces to Eq. (7.3) for any values of $r$ and $\kappa$.

From a mathematical point of view, the properties of the entropy (7.1) follow from those of $\ln_{\{\kappa,r\}}(x)$ in the range $x\in[0,1]$, while the properties of the entropy (7.4) follow from those of $\ln_{\{\kappa,r\}}(x)$ in the range $x\in[1,+\infty)$. This justifies our study of the properties of $\ln_{\{\kappa,r\}}(x)$ in the whole range $x\in[0,+\infty]$.

Regarding the relationship between the entropy of a system and the entropies of its subsystems, additivity and extensivity do not hold, in general. However, it is possible to show that any entropy belonging to the family (7.1) satisfies an extended version of the additive and extensive property [18].

In fact Eq. (7.1) can be written as

$$S_{\kappa,r}(p) = -\langle \ln_{\{\kappa,r\}}(p)\rangle, \tag{7.5}$$

which expresses the entropy $S_{\kappa,r}(p)$ as the mean value of $\ln_{\{\kappa,r\}}(p)$. Given two systems $A$ and $B$, with probability distributions $p_i^A$ and $p_i^B$, we can define a joint system $A\cup B$ with distribution $p_{ij}^{A\cup B}=p_i^A \underset{\kappa,r}{\otimes} p_j^B$, where the deformed product $\underset{\kappa,r}{\otimes}$ is discussed in Sec. V. From Eqs. (5.2) and (7.5) it follows that

$$S_{\kappa,r}(A\cup B) = S_{\kappa,r}(A) + S_{\kappa,r}(B). \tag{7.6}$$

In Fig. 2 we plot four one-parameter entropies belonging to the family of the MST entropy as functions of $p$ for system with two states of probabilities $p$ and $1-p$. (a) is the Tsallis entropy

$$S_q(p) = \sum_{i=1}^N \frac{p_i^q - p_i}{1-q}. \tag{7.7}$$

Notice that the entropy (7.7) is expressed in terms of the Tsallis logarithm as $S_q(p)=\Sigma_i p_i \ln_q(1/p_i)$, which is different

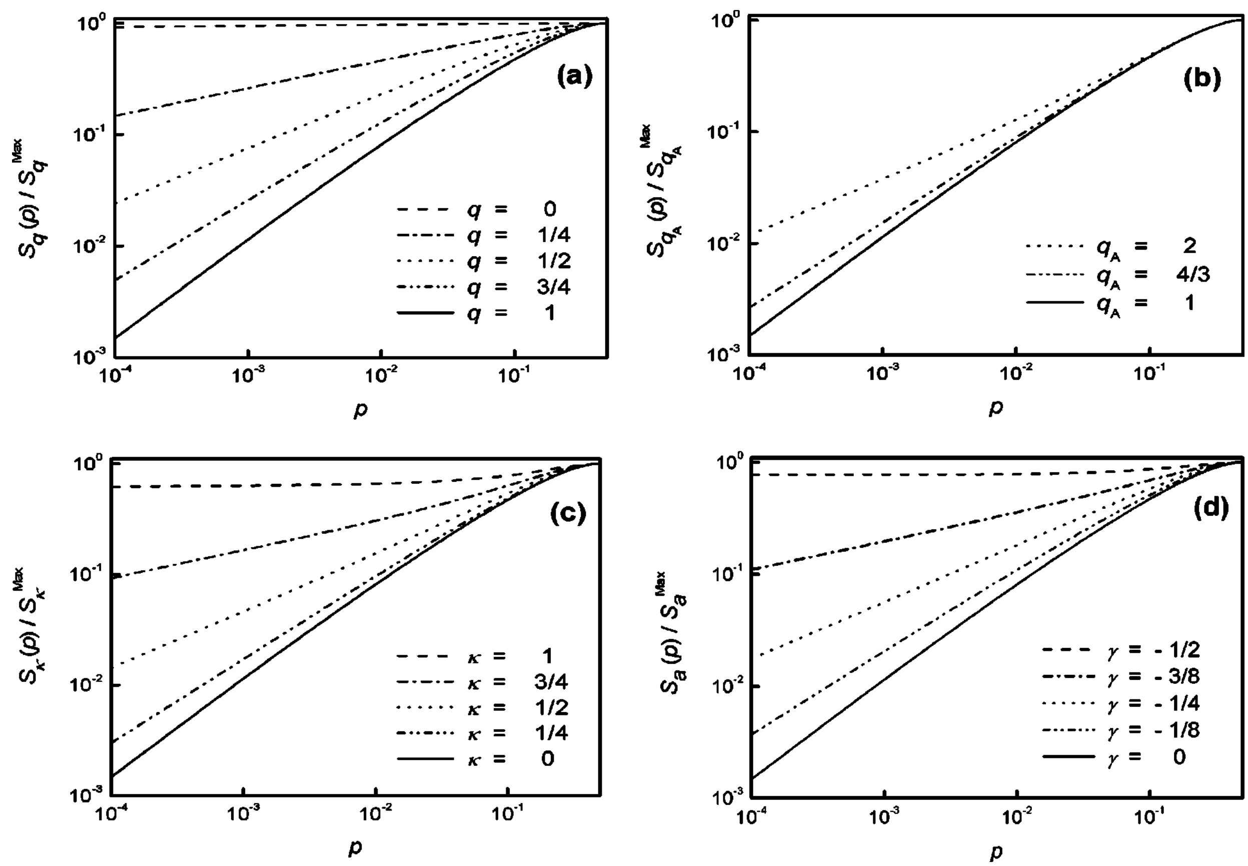


FIG. 2. Four one-parameter entropies for several values of the deformed parameter as a function of $p$ in a two-level system: (a) Tsallis entropy Eq. (7.7); (b) Abe entropy Eq. (7.8); (c) $\kappa$ entropy Eq. (7.9); and (d) $\gamma$ entropy Eq. (7.10). Broken curves with the same style show entropies whose corresponding distributions have the same power-law asymptotic decay $x^{-\nu}$, $\nu=1,4/3$, 2, and 4 from top to bottom; the solid curves show the Shannon entropy.

from our choice (2.1). The property $\ln_q(x)=-\ln_{2-q}(\frac{1}{x})$ shows that our choice $S_q(p)=-\Sigma_i p_i \ln_q(p_i)=\Sigma_i p_i \ln_{2-q}(1/p_i)$ corresponds to a different labeling of the entropy $q\to 2-q$.

(b) is the Abe entropy

$$S_{q_A}(p) = -\sum_{i=1}^{N} \frac{p_i^{q_A} - p_i^{(q_A^{-1})}}{q_A - q_A^{-1}}. \tag{7.8}$$

(c) is the $\kappa$ entropy

$$S_\kappa(p) = -\sum_{i=1}^{N} \frac{p_i^{1+\kappa} - p_i^{1-\kappa}}{2\kappa}. \tag{7.9}$$

Notice that the entropy (7.9) due to the property $\ln_{\{\kappa\}}(1/x) = -\ln_{\{\kappa\}}(x)$ can be written in the form $S_\kappa(p)=\Sigma_i p_i \ln_{\{\kappa\}}(1/p_i) = -\Sigma_i p_i \ln_{\{\kappa\}}(p_i)$ like the Boltzmann-Shannon entropy.

(d) is the $\gamma$ entropy

$$S_\gamma(p) = -\sum_{i=1}^{N} \frac{p_i^{1+2\gamma} - p_i^{1-\gamma}}{3\gamma}. \tag{7.10}$$

Entropies with the same broken-curve style yield distributions with the same-power asymptotic behavior $1/x^\nu$: $\nu=1,4/3$, 2, and 4 from top to bottom; the solid curve shows the Shannon entropy.

The distribution that optimizes the entropy (7.1) with the constraints of the canonical ensembles (2.3) is, by construction,

$$p_i = \alpha \exp_{\{\kappa,r\}}\left(-\frac{\beta}{\lambda}(E_i - \mu)\right), \tag{7.11}$$

where we recall that the deformed exponential $\exp_{\{\kappa,r\}}(x)$ is defined as the inverse of the deformed logarithm, which exists since $\ln_{\{\kappa,r\}}(x)$ is a monotonic function. The parameter $\mu$ is determined by $\Sigma_i p_i=1$.

In the $(\kappa,r)\to(0,0)$ limit, $\lambda=1$ and $\alpha=e^{-1}$, and Eq. (7.11) reduces to the well-known Gibbs distribution

$$p_i = Z(\beta)^{-1}\exp(-\beta E), \tag{7.12}$$

where the partition function is given by $Z(\beta)=\exp(1-\beta\mu) = \Sigma_i \exp(-\beta E_i)$. The quantity $\exp(-\beta E)$ is named the Boltzmann factor. Analogously we can call $\exp_{\{\kappa,r\}}(-\beta E/\lambda)$ the generalized Boltzmann factor.

We observe that the distribution (7.11) cannot be factorized as in Eq. (7.12): the normalization constraint is satisfied by fixing $\mu$ in the generalized Boltzmann factor.

Figure 3 shows the generalized Boltzmann factors corresponding to the four one-parameter entropies of Fig. 2. Curves with the same style have the same asymptotic behav-

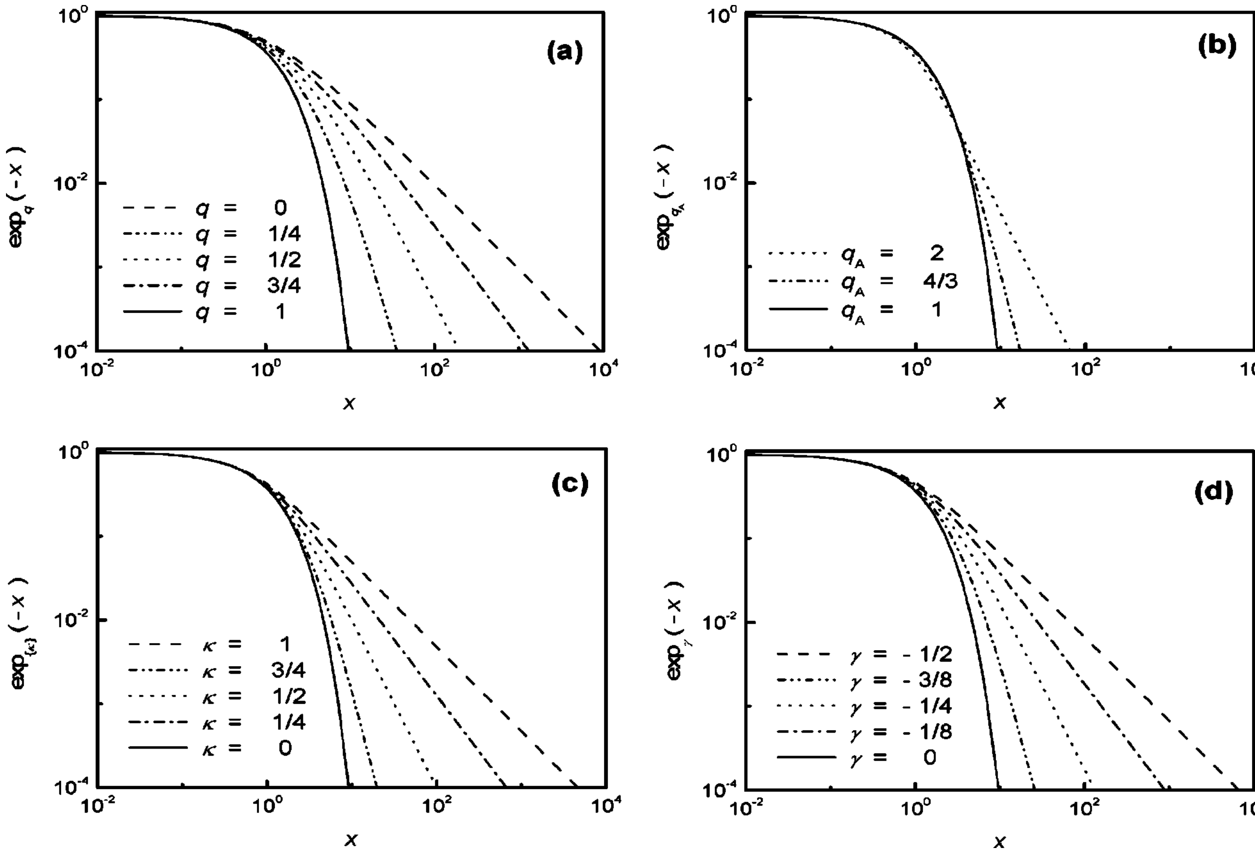


FIG. 3. The generalized Boltzmann factors that correspond to entropies in Fig. 2.

ior. Given this constraint and the normalization, the main difference between the distributions is in the middle region which joins the linear region ($x \ll 1$) and the Zip-Pareto region ($x \gg 1$).

## VIII. LESCHE INEQUALITY

An important issue is whether the entropies of the family under consideration are stable under small changes of the distribution [23,24,29,31]: we want to demonstrate that, if the two distributions are sufficiently close, the corresponding relative difference of entropies can be made as small as one wishes. To this end, we rewrite the entropy (2.1):

$$
\begin{aligned}
S(p) &= -\sum_{i=1}^{N} p_i \Lambda(p_i) \\
&= -\sum_{i=1}^{N} \int_0^{p_i} \frac{d}{dx}[x\Lambda(x)]dx \\
&= -\lambda \sum_{i=1}^{N} \int_0^{p_i} \Lambda\left(\frac{x}{\alpha}\right) dx \\
&= -\lambda \sum_{i=1}^{N} x\Lambda\left(\frac{x}{\alpha}\right)\Bigg|_0^{p_i} + \lambda \sum_{i=1}^{N} \int_0^{p_i} x \frac{d}{dx}\left[\Lambda\left(\frac{x}{\alpha}\right)\right] dx \\
&= -\lambda \sum_{i=1}^{N} p_i \Lambda\left(\frac{p_i}{\alpha}\right) - \sum_{i=1}^{N} \int_{-\lambda\Lambda(0^+)}^{-\lambda\Lambda(p_i/\alpha)} \alpha \Lambda^{-1}\left(-\frac{s}{\lambda}\right) ds,
\end{aligned}
\tag{8.1}
$$

where in the second equality we have used Eq. (3.1), in the last equality we have made the change of variables $s = -\lambda\Lambda(x/\alpha)$, and $\Lambda(0^+) \equiv \lim_{x\to 0^+}\Lambda(x)$. For the moment we use the notation $\Lambda(x)$ and $\mathcal{E}(x)$, since we do not need the specific form of the deformed logarithm and exponential. Using the fact that $\Lambda^{-1}(x) = \mathcal{E}(x)$ for the class of entropies under scrutiny, one finds

$$
\begin{aligned}
S(p) &= -\sum_{i=1}^{N} \int_{-\lambda\Lambda(p_i/\alpha)}^{-\lambda\Lambda(0^+)} \left[p_i - \alpha\mathcal{E}\left(-\frac{s}{\lambda}\right)\right] ds - \lambda\Lambda(0^+) \\
&= -\sum_{i=1}^{N} \int_{-1}^{-\lambda\Lambda(0^+)} \left[p_i - \alpha\mathcal{E}\left(-\frac{s}{\lambda}\right)\right]_+ ds - \lambda\Lambda(0^+) \\
&= \int_{-1}^{-\lambda\Lambda(0^+)} [1 - A(p,s)] ds - 1,
\end{aligned}
\tag{8.2}
$$

where in the second equality we used $\Sigma_i p_i = 1$, $\alpha\mathcal{E}(1/\lambda) \geqslant \alpha\mathcal{E}(-s/\lambda) > p_i$ for $-1 \leqslant s < -\lambda\Lambda(p_i/\alpha)$, and the definitions $[x]_+ \equiv \max(x,0)$ and

$$A(p,s) \equiv \sum_{i=1}^{N} \left[ p_i - \alpha\mathcal{E}\left(-\frac{s}{\lambda}\right)\right]_+ . \tag{8.3}$$

From now on we revert to the notation $\exp_{\{\kappa,r\}}(x)$ and $\ln_{\{\kappa,r\}}(x)$. We remark that the upper limit of the integral $s_m \equiv -\lambda\Lambda(0^+)$ in Eq. (8.2) is $s_m = +\infty$ for $r \neq |\kappa|$ and $s_m = (1/2|\kappa|)$ for $r = |\kappa|$; see Eq. (4.11).

The definition of $A(p,s)$, Eq. (8.3), implies that [28]

$$|A(p,s) - A(q,s)| \leqslant \sum_{i=1}^{N} |p_i - q_i| \equiv \|p - q\|_1, \tag{8.4}$$

and, for values of $s \geqslant -\lambda \ln_{\{\kappa,r\}}(1/N)$,

$$1 - N\alpha \exp_{\{\kappa,r\}}\left(-\frac{s}{\lambda}\right) = \left[\sum_{i=1}^{N}\left(p_i - \alpha \exp_{\{\kappa,r\}}\left(-\frac{s}{\lambda}\right)\right)\right]_+ < \sum_{i=1}^{N} p_i = 1, \tag{8.5}$$

from which it follows that

$$|A(p,s) - A(q,s)| < N\alpha \exp_{\{\kappa,r\}}\left(-\frac{s}{\lambda}\right). \tag{8.6}$$

From Eq. (8.2) the absolute difference of the entropies of two different distributions $p \equiv \{p_i\}_{i=1,\ldots,N}$ and $q \equiv \{q_i\}_{i=1,\ldots,N}$ satisfies

$$\begin{aligned}|S_{\kappa,r}(p) - S_{\kappa,r}(q)| &= \left| \int_{-1}^{s_m} [A(p,s) - A(q,s)] ds \right| \\ &\leqslant \int_{-1}^{s_m} |A(p,s) - A(q,s)| ds \\ &= \int_{-1}^{\ell} |A(p,s) - A(q,s)| ds \\ &\quad + \int_{\ell}^{s_m} |A(p,s) - A(q,s)| ds. \end{aligned} \tag{8.7}$$

Choosing $-\lambda \ln_{\{\kappa,r\}}(1/N) \leqslant \ell < s_m$, by using Eq (8.4) in the first integral and Eq. (8.6) in the second integral of Eq. (8.7), we obtain

$$|S_{\kappa,r}(p) - S_{\kappa,r}(q)| \leqslant \|p - q\|_1(\ell + 1) + N\alpha \int_{\ell}^{s_m} \exp_{\{\kappa,r\}}\left(-\frac{s}{\lambda}\right) ds. \tag{8.8}$$

In particular Eq. (8.8) holds for that value $\overline{\ell}$ that minimizes the right-hand side of Eq. (8.8),

$$\overline{\ell} = -\lambda \ln_{\{\kappa,r\}}\left(\frac{\|p - q\|_1}{\alpha N}\right), \tag{8.9}$$

as long as $\overline{\ell} \geqslant -\lambda \ln_{\{\kappa,r\}}(1/N)$, which is true when

$$\|p - q\|_1 \leqslant \alpha, \tag{8.10}$$

i.e., for sufficiently close distributions, according to the metric $\|\cdots\|_1$. Introducing Eqs. (8.9) and (8.10) in Eq. (8.8) and performing the integration using the result (2.16), we obtain

$$|S_{\kappa,r}(p) - S_{\kappa,r}(q)| \leqslant \|p - q\|_1 \left[1 - \ln_{\{\kappa,r\}}\left(\frac{\|p - q\|_1}{N}\right)\right], \tag{8.11}$$

and the relative difference of entropies can be written as

$$\left| \frac{S_{\kappa,r}(p) - S_{\kappa,r}(q)}{S_{\max}} \right| \leqslant F_{\kappa,r}(\|p - q\|_1, N), \tag{8.12}$$

with

$$F_{\kappa,r}(\|p - q\|_1, N) = \frac{\|p - q\|_1}{\ln_{\{\kappa,-r\}}(N)}\left[1 - \ln_{\{\kappa,r\}}\left(\frac{\|p - q\|_1}{N}\right)\right], \tag{8.13}$$

because $S_{\max} \equiv \ln_{\{\kappa,-r\}}(N)$.

This result demonstrates that if the two distributions are sufficiently close the corresponding absolute difference of entropies can be made as small as one wishes, since Eq. (4.5) implies that $\lim_{x\to 0^+} x \ln_{\{\kappa,r\}}(x) = 0$.

In particular, the Lesche inequality for the family of entropies under scrutiny is valid also in the thermodynamic limit $N \to \infty$

$$\lim_{\|p - p'\| \to 0^+} \lim_{N\to\infty} F_{\kappa,r}(\|p - q\|_1, N) = 0. \tag{8.14}$$

This last result is not trivial, since the thermodynamical limit introduces nonanalytical behaviors that could produce finite entropy differences between probability distributions infinitesimally close. We conclude this section by noting that Lesche stability of the $(\kappa, r)$ family of entropies follows also from the general proof given in [31].

## IX. CONCLUSIONS

In order to unify several entropic forms, the canonical MaxEnt principle has been applied to a generic trace-form entropy obtaining the differential-functional equation (2.8) for the corresponding generalized logarithm, when the ensuing distribution function is required to be expressed in terms of the generalized exponential through the natural relation (2.6).

The solution of this equation yields the biparametric family of logarithms

$$\ln_{\{\kappa,r\}}(x) = x^r \frac{x^{\kappa} - x^{-\kappa}}{2\ \kappa}; \tag{9.1}$$

the corresponding entropy [34–36] is

$$S_{\kappa,r}(p) = -\sum_{i=1}^{N} p_i^{1+r} \frac{p_i^{\kappa} - p_i^{-\kappa}}{2\ \kappa}. \tag{9.2}$$

This entropy is a mathematically and physically sound entropy when the parameters $\kappa$ and $r$ belong to the region shown in Fig. 1 and, therefore, the $(\kappa,r)$ logarithm satisfies the set of properties (4.1)–(4.6). In particular these entropies satisfy the Lesche stability condition.

Distribution functions obtained by extremizing the entropy (9.2) have power-law asymptotic behaviors: such behaviors could be relevant for describing anomalous systems; a comparison between several one-parameter distribution functions is shown in Fig. 3.

In addition, we have shown that several important one-parameter generalized entropies (Tsallis entropy, Abe entropy, and $\kappa$ entropy) are specific cases of this family; when the deformation parameters vanish, the family collapses to the Shannon entropy.

Our approach yielded also new one-parameter logarithms belonging to this family, whose corresponding exponentials can be explicitly given by algebraic methods.

There remains the question of the relevance of each mathematically sound entropy to specific physical situations. In fact a wide class of deformed logarithms satisfy a set of reasonable mathematical properties and physical constraints, in particular concavity, related to thermodynamic stability, and the Lesche inequality, related to the experimental robustness.

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