Propagation in a two-dimensional weighted local small-world network

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Propagation properties in a two-dimensional network with local small-world effects corresponding to the influence zone of each active site are studied. Two different weights based on characteristic times are introduced. The propagation of the front (here a forest fire front) in such a network exhibits two thresholds: the first one is geometrical corresponding to the percolation threshold and the second one is dynamical and results from the weighting procedure. The geometrical threshold is found to be a second-order phase transition as for regular networks. Further results are provided on the fractal dimension of the area covered during the propagation below the percolation threshold.

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I. INTRODUCTION

At the end of the last decade, Watts and Strogatz [1] proposed a network model called the small-world network (SWN) having the main properties of a social network, i.e., the clustering and possible connections between any two sites in the network within a finite number of steps. This model seems to fit very well many social behaviors like disease propagation. It is built from a regular network, but in addition to the number k of nearest neighbors, ϕ long-range connections are randomly introduced. If we allow only a certain proportion p of the network sites to participate in the propagation process, there is a percolation threshold p_c corresponding to the appearance of a large cluster whose size behaves as a power law with p [2]. This threshold was extensively investigated for one-dimensional systems and its dependence on k and ϕ satisfies the following equation [3,4]:

$$\phi = \frac{(1-p_c)^{k/2}}{p_c}. \tag{1}$$

The exponent $k/2$ refers to the number of neighbors in the propagation direction. In the case of epidemics, this threshold corresponds to the smallest concentration of susceptible individuals leading to a disease outbreak [3]. Similar systems like scale-free networks were used to model problems with hot nodes like computer viruses [5]. In such networks, the distribution of connections is power-law decaying. Furthermore, systems like airport networks require both long-range connections and the introduction of weights in either their nodes or links [6]. Long-range connections and weights are also necessary to correctly model fire propagation through a network of vegetation elements (forest fire front) which is an ever-topical problem [7]. Fire simulations on regular networks seem to not fit the experimental data because they are limited to nearest neighbors while radiation and firebrand impacts are dominant propagation mechanisms.

Fire spread is mainly dependent on characteristic lengths, namely, the mean free path of radiation from the flame and the spotting distance due to firebrands. This leads to a definition of impact parameters beyond the nearest neighbors. Furthermore, weights are given at each heated or burning site as a result of thermal degradation and combustion processes. It is then necessary to build a model including such features to improve physical insight. On the other hand, it is important to investigate the propagation behavior and thresholds in a fire network.

This is the aim of the present paper where a local small-world network (LSWN) is proposed. In real systems, the impact parameter depends on fire and fuel conditions (wind, terrain slope, fuel moisture content, fuel type and loading, etc.). It was found that the LSWN is in excellent agreement with known experimental data [8]. It is then used to study the geometrical and dynamical propagation thresholds and their dependence on impact parameter. For the geometrical threshold, a fractal investigation is included.

II. MODEL DESCRIPTION

The present model is based on the usual SWN initially proposed by Watts and Strogatz [1]. This network is described from the number of nearest neighbors k and averaged number of long-range connections ϕ (shortcuts) randomly generated throughout the whole network. The concept of connections $(k+\phi)$ in the SWN is, in the present model, replaced by that of impact parameters defining an influence zone. This means that for a given node, its influence zone is characterized by one (isotropic propagation) or two impact parameters in such a way that all active sites present in this zone are connected to this node. In the present work, the action of firebrands is neglected. As a consequence, the impact parameters can be either identical to the characteristic lengths l_x and l_y directly related to the radiative impact (deterministic case) or generated following a Poisson-like distribution based on these lengths (random case). As the present SWN is restricted to the influence zone, it is called a local small world network. It can then be considered as a regular network for the whole system, while it is regarded as

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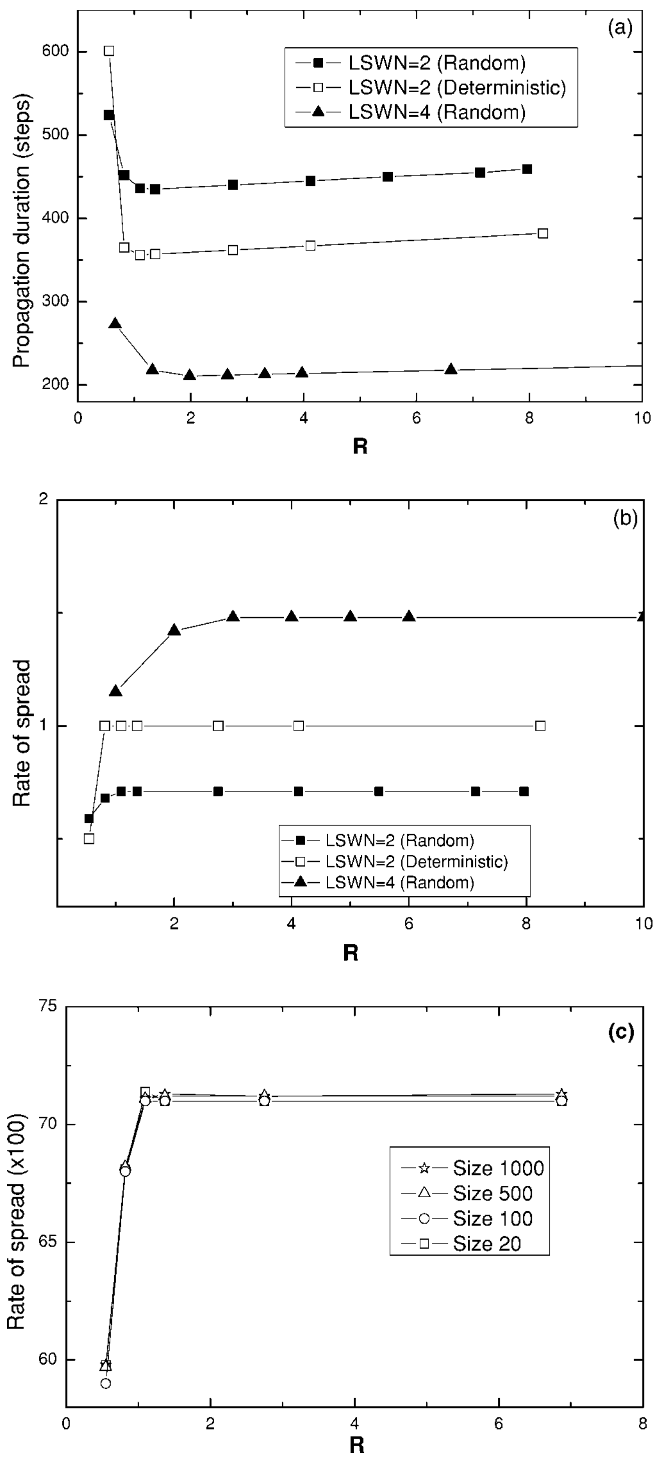

FIG. 1. (a) Fire propagation duration and (b) rate of spread vs R for a system size of 300 and different impact parameters. (c) Rate of spread vs R for different system sizes and $l=2$.

a SWN at the local scale. It is worthy of note that the SWN effect will be strengthened by accounting for firebrand impact.

This LSWN model uses a double weighting procedure on sites based on the knowledge of two characteristic times, namely, the time for a site to achieve a complete combustion (t_c) and that of thermal degradation before ignition (t_{TD}). In other words, a given site needs n_c time steps to burn after ignition, while a fire-exposed site located at the limit of the influence zone needs n_{TD} time steps to reach ignition. Moreover, the ignition time of any exposed site exponentially decays with its distance to the burning site. Fuel type and moisture content do not affect the weighting procedure while the influence zone should be modified.

Let us now consider the dynamic aspect of the LSWN. For each time step, a burning site will increase by one unit the thermal degradation level of the sites connected to it before they reach ignition. Once ignited, during t_c, each burning site contributes to the thermal degradation and ignition of the sites located in its influence zone. The present model predicts the fire area, whose thickness is defined as the distance covered by fire during the combustion time t_c.

Defining t_{NN} as the time required for the nearest neighbor to reach ignition, directly related to t_{TD}, a dynamic threshold propagation for a single burning site can be defined by the ratio

$$R = \frac{t_c}{t_{\mathrm{NN}}}. \tag{2}$$

In the case of a single burning site, fire spread occurs for $R \geqslant 1$; the unit value of R corresponds to the propagation threshold. If $R < 1$, the burning site will be completely consumed before the nearest neighbors (and consequently the whole influence zone) reach ignition. In the case of multiple burning sites (characteristic of real fire fronts), this threshold is obviously smaller than 1. This behavior will be illustrated in the present paper.

Like regular networks and the SWN, the LSWN is characterized by a geometric propagation threshold (percolation) corresponding to the minimum concentration of occupied (active) sites (p_c) allowing propagation through a large part of the system. In the case of regular networks, this propagation reaches the opposite end of the network and the largest cluster size diverges, while for the SWN, the largest cluster ξ starts increasing with p as a power law [$\xi = (p - p_c)^x$, x being real positive] above this threshold [2]. In the present model, the largest cluster should diverge since the whole system can be viewed as a regular network while the SWN effect appears only locally. Therefore, we should obtain the same critical properties as for the regular network (with connections beyond the nearest neighbor) while the fire pattern exhibits a SWN behavior. Indeed, it was found (see Fig. 5 of [8]) that the SWN effect seems to increase as p decreases.

III. RESULTS AND DISCUSSION

The statistical averaging process is carried out by generating a network of 90 000(300 × 300) sites. The propagation process is initiated by igniting the first line of sites (300 sites) instead of a single site. This process provides the average time and rate of spread as well as the average "mass" m (proportion) of burned sites during fire propagation. We are not interested here in the propagation probability distribution and limit ourselves to an isotropic propagation process (i.e., $l_x = l_y = l$).

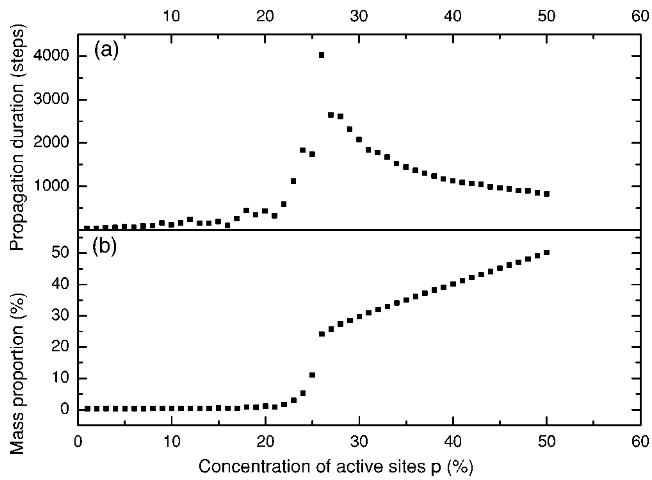

FIG. 2. (a) Fire propagation duration and (b) burned mass proportion vs concentration p for a system size of 300. $l=2$ and $t_c=30$ time steps.

A. Dynamical threshold

Let us consider now the fire propagation through a homogeneous system (all sites are active) as an illustrative example. Unlike point-ignition conditions where the change in propagation regime is known to be abrupt for $R=1$ as previously discussed, the line-ignition threshold is smaller than unity. For the latter conditions, a site can be a nearest neighbor of more than one burning site, which in turn leads to an increase in its rate of thermal degradation. This is clearly seen in Fig. 1, where the rate of spread and propagation duration are plotted versus R. The dynamic transition occurs for a threshold value $R \approx 0.5$, whatever the impact parameter.

A randomly generated parameter yields a smoother variation of the propagation duration and rate of spread versus R. This effect increases with the impact parameter. For a given impact parameter, increase in R leads to an increase in t_c which allows more burning sites to contribute to the propagation process. However, beyond a certain value of R or t_c, the whole of the influence zone contributes to this process. The maximum rate of spread, given by l/t_c, is then reached. For any homogeneous system larger than the influence zone, the rate of spread remains nearly constant as shown in Fig. 1(c).

In the remaining part of the paper, it is assumed that the dynamical criterion $R>1$ is satisfied to ensure propagation.

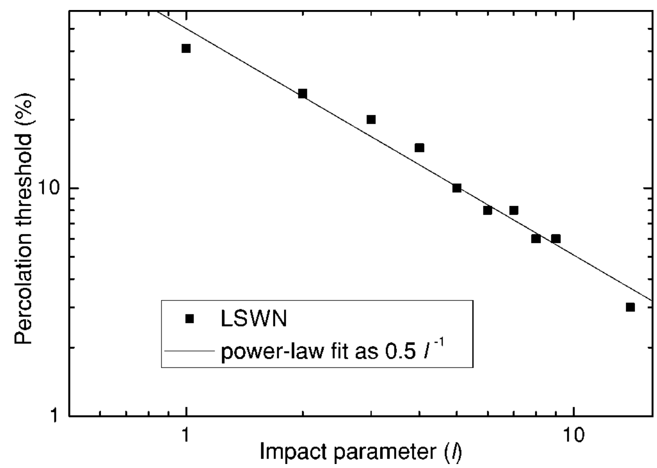

FIG. 4. Percolation threshold p_c vs impact parameter l for a system size of 300. The solid line is a power-law fit of LSWN values. $t_c=30$ time steps.

B. Percolation threshold

Propagation process through heterogeneous media is now studied by introducing a proportion p of active sites. The p dependence of the average burned mass and fire propagation duration are shown in Fig. 2 for $l=2$. This figure shows the usual second-order phase transition observed in regular networks, namely, the cluster divergence at p_c (in terms of propagation duration) and the sharp change of mass behavior. The mass m corresponds to the probability of belonging

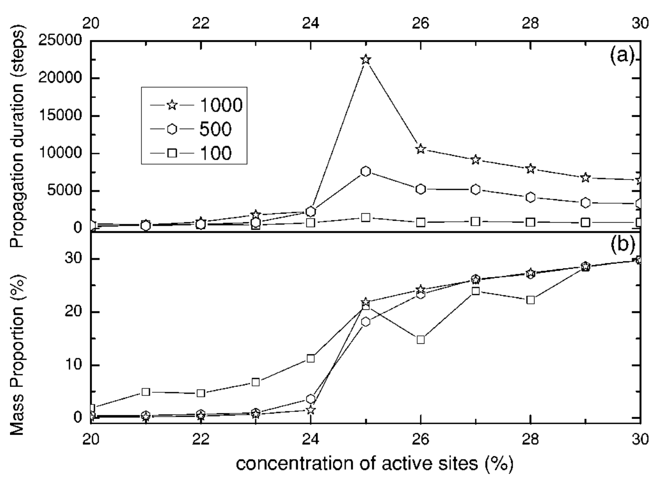

FIG. 3. Finite size effect: (a) fire propagation duration and (b) burned mass proportion vs concentration p for different system sizes. $l=2$ and $t_c=30$ time steps.

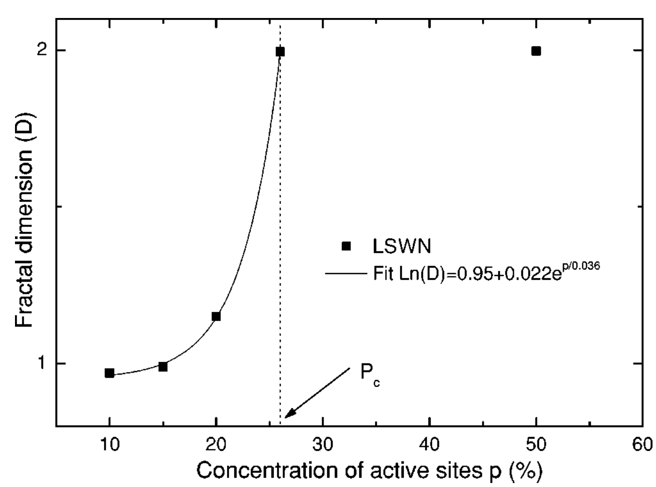

FIG. 5. Fractal dimension of the area covered by the fire vs concentration p. $l=2$ and $t_c=30$ time steps. The dotted line corresponds to the percolation threshold p_c. The solid line is a fit of LSWN values.

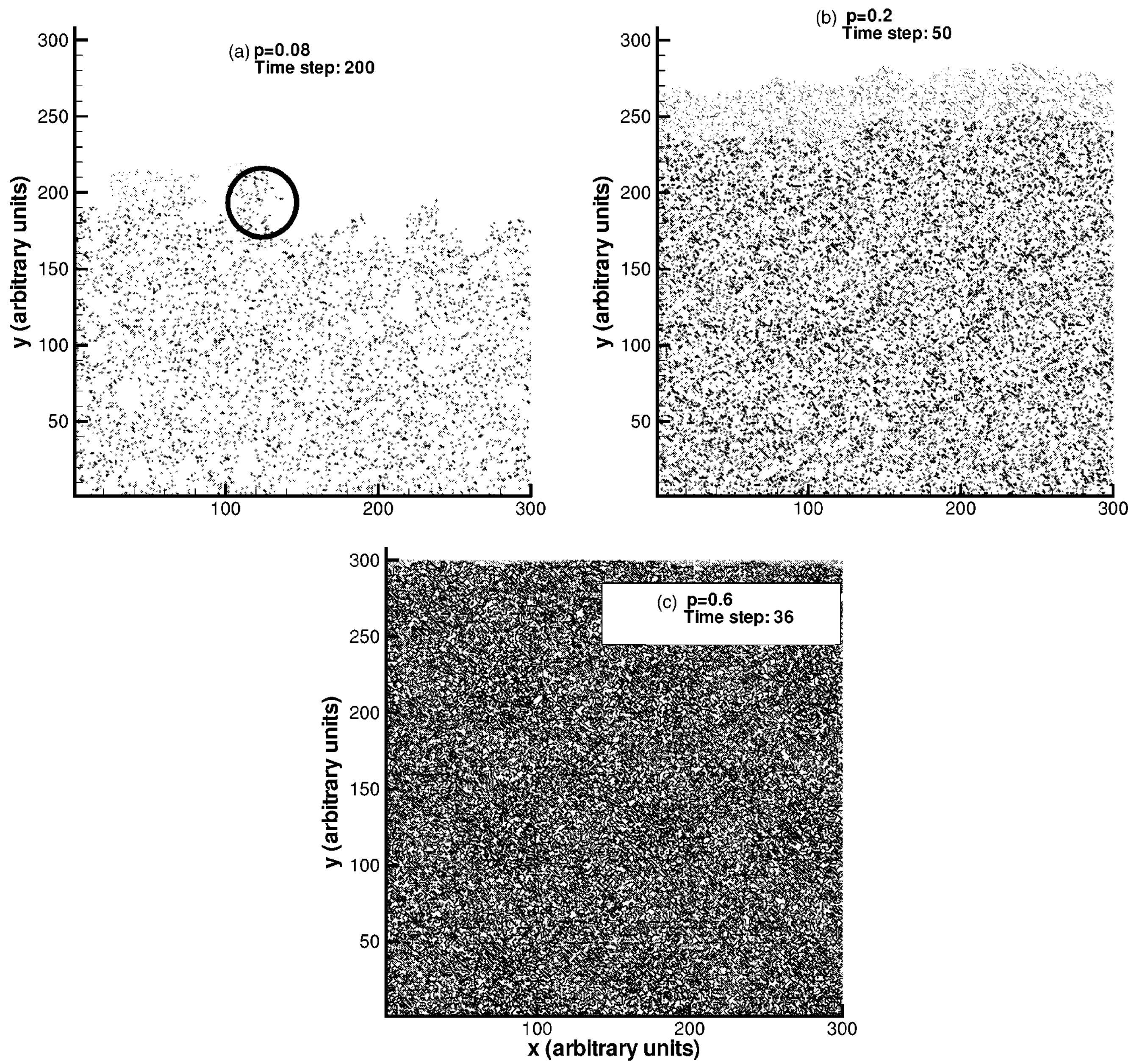

FIG. 6. Propagation patterns for p=(a) 0.08, (b) 0.2, and (c) 0.6. $l=6$ and $t_c=30$ time steps.

to the largest cluster. This second-order phase transition is observed for any impact parameter value, which reveals the regular network properties of the system.

In order to investigate finite size effects on the percolation threshold, the p dependence of the burned mass and fire propagation duration for different system sizes and $l=2$ are plotted in Fig. 3. The peak of fire propagation duration becomes more pronounced and the mass variation sharper for larger systems. However, the transition threshold p_c seems to be not affected by the system size. For sizes smaller than 100, fluctuations appear. Although more statistics are needed to accurately determine p_c, there is evidence that the correlation length is smaller than 100 in this case.

Now let us examine the impact parameter dependence on this threshold to evaluate the SWN effect. This is shown in Fig. 4 where p_c appears power-law decreasing with the impact parameter l. The power-law fit exponent (-1) is that predicted for the SWN by Eq. (1) in the propagation direction where the number of nearest neighbors $k/2=1$ and $\phi=l-1$ is the impact parameter excluding the nearest neighbor.

For the limit $l=1$, corresponding to the nearest neighbors, the known bond percolation threshold for a square regular network $(p_c=1/2)$ is reproduced by the power-law fitting function.

C. Fractal dimension

When the propagation starts with a single site belonging to the largest cluster, it covers a complex zone whose dimension is less than 2 (i.e., the fractal dimension). The probability that this initial site belongs to the largest cluster depends on the proportion p of active sites. As previously done, the propagation is initiated along the first line of sites to ensure the largest propagation process. In this case, at least one of the initial sites belongs to the largest cluster. Moreover, other sites that belong to other clusters may participate in the propagation.

Above the percolation threshold, the propagation area covers the whole system, i.e., its fractal dimension is 2.

Below this threshold, the propagation covers only a part of the system characterized by a fractal dimension defined by its scaling power-law exponent. Using the mass m, $m \propto L^{D-2}$ where D is the fractal dimension and L the system size. Figure 5 shows the p dependence of the fractal dimension D. This behavior can be used to determine the numerical value of p_c. For an impact parameter $l=2$, $p_c \approx 0.26$. It is worthy of note that in the limit of $p=0$, the value of D tends to 1 which corresponds to the fractal dimension of the first line of sites. The transition shown in Fig. 5 between the p dependence of the fractal dimension below p_c and its nondependence above p_c has been previously observed [9] for agregates (diffusion-limited aggregation DLA).

At about p_c, the mass does not behave as a power-law function of the size. This means that the dimension becomes multifractal at the percolation threshold resulting from random generation of the impact parameter. A self-similar behavior has been obtained by Albinet *et al.* [10] considering only deterministic cases of fire front propagation. For such cases, we found an overestimation of the rate of spread [8].

For very small concentrations of active sites [Fig. 6(a)], most of the clusters are of finite size and the SWN effect during propagation becomes dominant as the cluster sizes are close to the impact parameter. The fragmented front structure appears as a ballistic propagation [e.g., the circled zone in Fig. 6(a)]. For high values of p, the propagation becomes mainly diffusive [Fig. 6(c)] with a predominance of the regular network effect. The crossover between ballistic and diffusive regimes, as observed by Sapoval *et al.* [9], appears in Fig. 6(b).

IV. CONCLUSION

The propagation of a front through a local SWN including a weighting procedure based on two characteristic times is presented. This model exhibits a geometrical and a dynamical propagation threshold. The former is a second-order phase transition (as for regular networks) whatever the impact parameter while the l dependence of p_c shows the SWN effect in the propagation direction.

The dynamical threshold results from the weighting procedure. Applied to fire propagation, this threshold occurs at $R=1$ for point-ignition conditions while it appears at $R \approx 0.5$ for line-ignition conditions. Beyond a certain value of R or t_c, a maximum rate of spread is reached as the whole influence zone contributes to the propagation process.

The sharp variation of the fractal dimension as a function of p at the percolation threshold can be used to determine the numerical value of p_c.

Finally, the concentration p appears to be a good measure of the SWN effect. The crossover between the ballistic regime where the SWN effect is dominant and the diffusive regime where the regular network effect is clearly identified.

An extensive study of statistical and scaling properties of clusters and thresholds is planned.

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