Generalized mean-field theory for Ising spins in small world networks

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A generalization of mean-field theory for random systems is described. The results of that analytic model could be reconciled with the results of numerical calculations of the Curie temperature for a system of Ising spins in small world (SW) networks by introducing the effective interaction energy associated with long-range links which exceeds the real energy of spin interaction. Such a model describes qualitatively well the increasing Curie temperature T_C with the growth of the long-range links fraction p in the two-dimensional SW system with fixed coordination number. On the basis of simple physical considerations, concentration dependences $T_C(p)$ are found for SW systems of different dimensions.

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I. INTRODUCTION

The aim of the present paper is to generalize mean-field theory for a system of Ising spins in nonregular networks presenting the small world (SW) network—the graph with peculiar properties [1,2]. Ordinary networks (regular and nonregular) refer to the lattices the sites of which are connected with their neighbors only. By contrast, in SW networks there are random connections both between near and far (in geometrical sense) sites. It is precisely these far, or long-range, links (shortcuts) that are responsible for the special features of SW networks.

The Ising problem in SW networks arises in a similar way as for ordinary lattices—in network sites Ising spins are placed which interact (the interaction energy J) with their nearest neighbors only—that is, with those ones which are directly connected with a given spin. However, in SW networks some geometrically removed spins turn out to be the nearest neighbors that, naturally, are favorable to magnetic ordering. The existence or absence of the ordered magnetic state is governed by the fraction of long-range links and the dependence of the interaction energy J_{SW} ascribed to those links on their geometrical length. Numerical calculations for the power dependence $J_{SW}=J^+r_{ij}^{\alpha}$ (r_{ij} is the geometrical distance between sites i,j) show that a phase transition at finite temperature is possible even in the one-dimensional system [3–7], though it occurs at $\alpha=0$ only, when the interaction energy along those links is independent of the distance and, certainly, at a sufficiently high fraction of those links. At $\alpha>0$, the phase transition is absent [8].

Most conclusions about the properties of the Ising model in SW networks have been obtained by numerical Monte Carlo calculations. Particularly, it has been establushed that the type of the proceeding magnetic phase transition corresponds to the mean-field model[1] [6]. It allows one to hope that a properly generalized mean-field theory could provide a correct description of that transition. The necessity of generalizing the traditional mean-field theory is dictated by the following considerations. In that theory the transition temperature is defined by the formula $T_C=\langle z\rangle J$, where $\langle z\rangle$ is the coordination number averaged over lattice sites. Imagine that the generation of SW networks from the regular lattice with the coordination number z is handled by means of randomly replacing short links with long ones (see below), but in such a way that the average link number for a site remains fixed ($\langle z\rangle=z$). As this takes place, the transition temperature has to be constant, as well. That conclusion contradicts the results of numerical calculations [3–8] and is connected to the fact that in traditional mean-field theory the local fields are assumed to be the same in all sites, while, in fact, those fields vary from one site to another in a *random* fashion. The generalized theory has to take that into account.

In the present paper, we shall describe such a theory and use it to study the magnetic state of an Ising spin system in SW networks. Data obtained with the help of that theory could be reconciled with the numerical results of investigating similar systems by introducing the effective interaction energy through the long-range links which exceeds the real energy of spin interaction. Such a model explains the rising of the Curie temperature $T_C(p)$ with p increasing in the SW system with fixed coordination number.

In Sec. II, the generalized mean-field theory is described based on the distribution function of local magnetic fields. In the cases when the exact solution is known, the Curie temperature calculated with this theory turns out to be significantly closer to the exact value than that calculated with traditional mean-field theory. In Sec. III, the approach formulated is used for describing the properties of the Ising spin system in SW networks. Distribution functions of local magnetic fields are calculated for two-dimensional SW networks with varying and fixed coordination number, and the dependences of the Curie temperature on the fraction of sites with long links (in the former case) or the fraction of those links (in the latter case) are found. A comparison is made with the results of numerical calculations, and a conception of the effective interaction energy through long links is introduced. The simple considerations presented in Sec. IV lead to establishing the concentration dependences $T_C(p)$ of the Curie temperature in SW systems of different dimensions. At last, Sec. V is devoted to concluding remarks.

[1]Namely, the magnetization j close to the Curie temperature T_C depends on the temperature T according to the law $j\propto(T_C-T)^{\beta}$, where $\beta=1/2$.

II. GENERALIZED MEAN-FIELD THEORY FOR ISING SPINS IN A REGULAR SQUARE LATTICE

In accordance with the exact solution of the Ising model [9,10], the Curie temperature for a two-dimensional square lattice of spins with ferromagnetic interaction equals $T_C = 2.27J$, where J is the interaction energy of the nearest neighbors. The mean-field result $T_C = 4J$ ($z=4$) differs significantly from the exact one. Besides the known defect of the mean-field theory, which does not take into account the spin correlation, another disadvantage of that model is suggesting an *equivalence* of all lattice sites. It results in assuming that the "mean field" is the same in all sites though that field is *randomly* varied from one site to another. Therefore, it is natural to incorporate this randomness in the scheme and to see how much higher would be the accuracy of the solution obtained with that generalized mean-field theory. For the first time, such an approach has been used in Ref. [11] for considering the system of randomly positioned magnetic dipoles. Later, it was analyzed in detail in a series of papers [12], where the starting point is the defined *ad hoc* distribution function of pair interaction energies. The essence of that model is in replacing the standard mean-field equation

$$j = \tanh\left(\frac{\lambda j}{T}\right), \tag{1}$$

where j is the reduced system magnetization λ and is the mean-field constant, by its generalized analog

$$j = \int_{-\infty}^{\infty} \tanh\left(\frac{h}{T}\right) F(j;h)\,dh, \tag{2}$$

in which $F(j;h)$ is the distribution function of local magnetic fields h generated by all lattice spins (in the Ising model, by nearest neighbors only) in a given site of the lattice. Notice that the mean-field equation (1) is equivalent to the equation

$$j = \tanh\left[\frac{\overline{h}(j)}{T}\right], \tag{3}$$

where

$$\overline{h}(j) = \int_{-\infty}^{\infty} hF(j;h)\,dh.$$

In the framework of such a generalization, the problem is reduced to determining the distribution function $F(j;h)$, which allows one to investigate the magnetic properties of the relevant system with the help of Eq. (3).[2]

To estimate how much better is the result of the mean-field theory in its generalized form (3), let us consider the square lattice in which site spins with orientations $\uparrow,\downarrow$ are placed. The probabilities of the corresponding orientations are equal to $(1+j)/2$ and $(1-j)/2$. The effective field h of the interaction of the central spin with its nearest neighborhood is the sum of terms of the same absolute value J (J is the interaction energy for spins of the same direction) whose sign is defined by the relative orientation of the central and neighbor spins. Thus, the field h depends on the configuration of Ising spins in the first coordination sphere. For the square lattice, there are $2^4=16$ such configurations, in all. Their distribution over the effective field h, along with probabilities of those configurations, is displayed in Table I.

TABLE I. Configurations of spins of nearest neighbors in the square lattice and their probabilities.

Configuration of moments	Effective magnetic field (h/J)	Number of configurations	Configuration probability for magnetization (j)
↑↑↑↑	+4	$C_4^0=1$	$\left(\frac{1+j}{2}\right)^4$
↑↑↑↓	+2	$C_4^1=4$	$4\left(\frac{1+j}{2}\right)^3(1-j/2)$
↑↑↓↓	0	$C_4^2=6$	$6\left(\frac{1+j}{2}\right)^2(1-j/2)^2$
↑↓↓↓	−2	$C_4^3=4$	$4\left(\frac{1+j}{2}\right)(1-j/2)^3$
↓↓↓↓	−4	$C_4^4=1$	$\left(\frac{1-j}{2}\right)^4$

The relevant distribution function reads

$$F_0(j;h) = \frac{1}{16}[(1+j)^4\delta(h-4J) + 4(1+j)^3(1-j)\delta(h-2J) + 6(1+j)^2(1-j)^2\delta(h) + 4(1+j)(1-j)^3\delta(h+2J) + (1-j)^4\delta(h+4J)]. \tag{4}$$

Substituting it into Eq. (2) one finds

$$j^2 = \frac{2\tanh(2K) + \tanh(4K) - 2}{2\tanh(2K) - \tanh(4K)}, \quad K = \frac{J}{T}. \tag{5}$$

This relationship determines the temperature dependence of the system magnetization and Curie temperature T_C. The latter is defined by the condition $j=0$ which is reduced to the equation $2\tanh(2K)+\tanh(4K)=2$ whose solution gives $K = K_C^{(0)} \approx 0.323$. From here on, it follows $T_C \approx 3.10J$, which is much closer to the exact result than the traditional mean-field estimate. Also notice that the mean field defined by the distribution function (4) equals $\overline{h}=4jJ$, which, in accordance with Eq. (3), leads to the ordinary mean-field Curie temperature value.

The temperature dependences $j(T)$ of the magnetization corresponding to the traditional mean-field theory [Eq. (1)], its generalized form (2), and the exact solution

$$j = [1 - \mathrm{sh}^{-4}(2K)]^{1/8} \tag{6}$$

of the Ising problem for the square lattice [10] are presented in Fig. 1. It is seen that the result of the generalization considered in the present paper is significantly closer to the exact result.

[2]Different approaches to the problem considered and a comprehensive bibliography can be found in the reviews in [13–15].

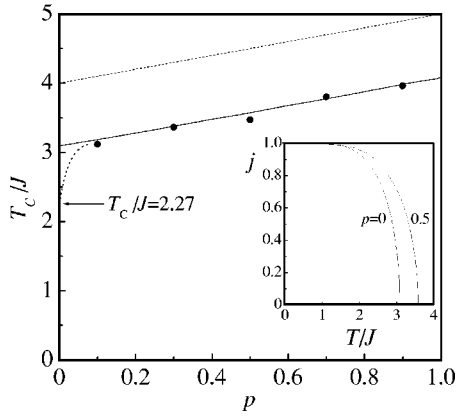

FIG. 1. Dependences $T_C(p)$ of the Curie temperature for two-dimensional SW systems with nonfixed average coordination number on the fraction p of sites with long links. Points: numerical calculations [16] for the square lattice 20×20 (T_C has been determined at the level $j=0.1$). The solid line: analytical calculation with the help of Eq. (9) of generalized mean-field theory. Dashed line: the result of the traditional mean-field theory. Inset: the temperature dependence of magnetization calculated by means of Eq. (8).

Analogously, different two-dimensional lattices (hexagonal and triangle) for which exact solution are known [9] could be considered. The relevant results are collected in Table II.

It is seen that the results of the generalized mean-field theory are very close to the results of the Bethe approach. In any case, that generalization improves significantly the traditional mean-field result and, hence, is more preferable for describing the properties of complex (in particular, nonregular) Ising systems.

III. ISING MODEL FOR SW NETWORKS

A. SW networks with nonfixed coordination number

One could generate SW networks from a regular network by different procedures. For instance, in [16] additional connections between randomly chosen sites of the original square lattice which are not nearest neighbors have been added (with the condition that each site may have no more than one such extra link). The fraction p of the sites with those random links has been changed from 0 to 1, so the average coordination number z does not remain constant but increases from 4 to 5 ($\Delta z=1$).

TABLE II. Curie temperatures for two-dimensional Ising lattices.

	T_C/J			
Lattice type (z)	MFT[a]	Generalized MFT[b]	Bethe[c]	Exact result [9]
Hexagonal (3)	3	2.11	1.82	1.52
Square (4)	4	3.10	2.89	2.27
Triangle (6)	6	5.08	4.93	3.64

[a]MFT, mean-field theory.
[b]Present paper.
[c]Bethe approach [10].

For SW networks with magnetization j and fraction p of sites possessing long links, the probabilities of different spin configurations "near by" a given site are listed in Table III, which are necessary for calculating the distribution function of local fields.

The relevant distribution function reads

$$F(j,h) = (1-p)F_0(j,h) + \frac{1}{32}p[(1+j)^5\delta(h-5J) + 5(1+j)^4(1-j)\delta(h-3J) + 10(1+j)^3(1-j)^2\delta(h-J) + 10(1+j)^2(1-j)^3\delta(h+J) + 5(1+j)(1-j)^4\delta(h+3J) + (1-j)^5\delta(h+5J)]. \quad (7)$$

Substituting it into Eq. (2) one gets the equation

$$(1-p)[8(1+j^2)\tanh(4K) + 16(1-j^2)\tanh(2K)] + p[(5+10j^2+j^4)\tanh(5K) + 5(3+j^2)(1-j^2)\tanh(3K) + 10(1-j^2)^2\tanh(K)] = 16, \quad (8)$$

which determines the system magnetization $j(T)$ and Curie temperature. The latter could be defined setting $j=0$ in Eq. (8). Then the equation determining T_C reads

$$2\tanh(2K) + \tanh(4K) = 2 - \frac{1}{8}p[5\tanh(5K) + 15\tanh(3K) + 10\tanh(K) - 8\tanh(4K) - 16(\tanh)(2K)]. \quad (9)$$

At $p=0$ it transforms into the previous equation (5), and at $p>0$ it has the solution $K_C<K_C^{(0)}=0.323$ which corresponds to the enhanced Curie temperature when compared to the regular lattice with $p=0$.

The results of numerical Monte Carlo calculations [16] for the considered system with different fractions p of sites with long links are shown in Fig. 1. In the range $p\gtrsim0.1$, the Curie temperature $T_C(p)$ increases linearly with p. Also in Fig. 1, the dependence of $T_C(p)$ calculated with the help of Eq. (9) and the dependence of $T_C(p)=J(4+p)$ corresponding to the traditional mean-field theory are presented. Though the results of our analytical calculation agree well with the numerical calculations [16], they do not reflect the principal thing. In fact, the exact value $T_C(p=0)=2.27$ signals the fast growth of T_C in the range $0<p\lesssim0.1$ (shown by the dashed curve[3]) not reflected in the results of [16]. Therefore, it is not excluded that in the case all phenomena associated with special features of SW networks take place for the most part in the narrow range $0\leqslant p\lesssim0.1$. That makes the above-mentioned procedure of generating SW networks to be non-

[3]There are no Monte Carlo calculations for the case considered that could more definitely determine the form of this curve. The more stepwise is the curve, the better is our approximation of the correct behavior.

TABLE III. Configurations of nearest neighbors (spins) in SW networks with the probability p of long-range links (symbols $\Uparrow$, $\Downarrow$ correspond to spin states at the long-link end).

Configuration of spins	Effective magnetic field, h/J	Number of configurations	Configuration probability at magnetization j
$\uparrow\uparrow\uparrow\uparrow$	+4	$C_4^0=1$	$\left(\frac{1+j}{2}\right)^4(1-p)$
$\uparrow\uparrow\uparrow\uparrow+\Uparrow$	+5	$C_4^0=1$	$\left(\frac{1+j}{2}\right)^4 p\left(\frac{1+j}{2}\right)$
$\uparrow\uparrow\uparrow\uparrow+\Downarrow$	+3	$C_4^0=1$	$\left(\frac{1+j}{2}\right)^4 p\left(\frac{1-j}{2}\right)$
$\uparrow\uparrow\uparrow\downarrow$	+2	$C_4^1=4$	$4\left(\frac{1+j}{2}\right)^3\left(\frac{1-j}{2}\right)(1-p)$
$\uparrow\uparrow\uparrow\downarrow+\Uparrow$	+3	$C_4^1=4$	$4\left(\frac{1+j}{2}\right)^3\left(\frac{1-j}{2}\right)p\left(\frac{1+j}{2}\right)$
$\uparrow\uparrow\uparrow\downarrow+\Downarrow$	+1	$C_4^1=4$	$4\left(\frac{1+j}{2}\right)^3\left(\frac{1-j}{2}\right)p\left(\frac{1-j}{2}\right)$
$\uparrow\uparrow\downarrow\downarrow$	0	$C_4^2=6$	$6\left(\frac{1+j}{2}\right)^2\left(\frac{1-j}{2}\right)^2(1-p)$
$\uparrow\uparrow\downarrow\downarrow+\Uparrow$	+1	$C_4^2=6$	$6\left(\frac{1+j}{2}\right)^2\left(\frac{1-j}{2}\right)^2 p\left(\frac{1+j}{2}\right)$
$\uparrow\uparrow\downarrow\downarrow+\Downarrow$	−1	$C_4^2=6$	$6\left(\frac{1+j}{2}\right)^2\left(\frac{1-j}{2}\right)^2 p\left(\frac{1-j}{2}\right)$
$\uparrow\downarrow\downarrow\downarrow$	−2	$C_4^3=4$	$4\left(\frac{1+j}{2}\right)\left(\frac{1-j}{2}\right)^3(1-p)$
$\uparrow\downarrow\downarrow\downarrow+\Uparrow$	−1	$C_4^3=4$	$4\left(\frac{1+j}{2}\right)\left(\frac{1-j}{2}\right)^3 p\left(\frac{1+j}{2}\right)$
$\uparrow\downarrow\downarrow\downarrow+\Downarrow$	−3	$C_4^3=4$	$4\left(\frac{1+j}{2}\right)\left(\frac{1-j}{2}\right)^3 p\left(\frac{1-j}{2}\right)$
$\downarrow\downarrow\downarrow\downarrow$	−4	$C_4^4=1$	$\left(\frac{1-j}{2}\right)^4(1-p)$
$\downarrow\downarrow\downarrow\downarrow+\Uparrow$	−3	$C_4^4=1$	$\left(\frac{1-j}{2}\right)^4 p\left(\frac{1+j}{2}\right)$
$\downarrow\downarrow\downarrow\downarrow+\Downarrow$	−5	$C_4^4=1$	$\left(\frac{1-j}{2}\right)^4 p\left(\frac{1-j}{2}\right)$

appropriate to analyze the properties of the considered systems in the whole range $0 \leqslant p \leqslant 1$. We believe that it is more proper to analyze the feasibility of the generalized mean-field theory for the Ising problem in SW networks by considering systems the average coordination number of which remains fixed for every p value.

B. SW networks with fixed coordination number

To fix the coordination number one needs to use another procedure (see, for example, [6]) which converts the regular square lattice $N \times N$ with the total links number of $2N^2$ (at $N \gg 1$) into an SW network through three stages.

(i) Choose randomly some fraction p of the original lattice links (the number of such links equals $M=2pN^2$) and disconnect their ends from their own sites.

(ii) Reconnect one of two ends of every link to a randomly chosen lattice site, leaving free the second end.

(iii) Connect those free ends of every link to randomly chosen lattice sites excluding the formation of loops (connections with both ends switching on a single site) and connection doubling.

The described procedure does not change the total number of links and, hence, the average coordination number $\langle z\rangle = \Sigma_{i,j} z_{ij}/N^2$ (z_{ij} is the coordination number of the site ij) remains fixed: $\langle z\rangle = z$. After performing that procedure, the regular lattice converts into an SW network and every site is characterized by two random numbers s and k, the former determining the number of short links (remaining after the first procedure stage) and the latter defining the number of long links (formed during the second and third procedure stages). Before performing the above-described procedure of link randomization, $s=4$, $k=0$ for every site, while $0 \leqslant s \leqslant 4$, $k \geqslant 0$ after its completion.

After the first stage (i.e., after removing some fraction of links), the probability $w_1(s)$ that a given site conserves s links ($s=0,1,2,3,4$) equals

$$w_1(s) = C_4^s (1-p)^s p^{4-s}. \tag{10}$$

After the second stage, some sites acquire new connections. The probability $w_2(k_2)$ that a given site gets k_2 additional links is equal to $w_2(k_2) = C_M^{k_2} (1/N^2)^{k_2} (1-1/N^2)^{M-k_2}$ where $M=2pN^2$ is the total number of switchings made in the course of the stage.

From the "point of view" of any site, the third stage of the procedure is no different from the second stage, except for the restrictions connected with generating loops and doubling. However, for the network with a great number of sites those restrictions are not essential because the portion of connections falling into this category is on the order of $1/N$. Thus, we shall not take that exclusion rule into account and assume that after completing the third stage the probability $w_3(k_3)$ that a given site acquires k_3 additional links is expressed by the same formulas as $w_2(k_2)$—that is, $w_3(k_3) = C_M^{k_3} (1/N^2)^{k_3} (1-1/N^2)^{M-k_3}$.

Hence, the probability $w_{2,3}(k)$ that after the two last stages a given lattice site obtains k links equals

$$\begin{aligned} w_{2,3}(k) &= \sum_{k_2+k_3=k} w_2(k_2) w(k_3) \\ &= \left(\frac{1}{N^2}\right)^k \left(1-\frac{1}{N^2}\right)^{2M-k} \sum_{n=0}^{k} C_M^n C_M^{k-n} \\ &= C_{2M}^k \left(\frac{1}{N^2}\right)^k \left(1-\frac{1}{N^2}\right)^{2M-k}. \end{aligned} \tag{11}$$

This probability is independent of the total number of lattice sites if the latter is large enough. In fact, at $M\to\infty$ but $k \ll M$, Eq. (11) transforms into a Poisson distribution

$$w_{2,3}(k) = \frac{(4p)^k e^{-4p}}{k!} \tag{12}$$

with mean value $\langle k\rangle = 4p$ (at $p=1$, as one would expect for the square lattice, $\langle k\rangle = 4$).

According to the Ising model, all neighbors (i.e., sites directly connected with a given site) generate at a given site the effective field whose sign is defined by the mutual orientation of spins in those two sites.

Long links appearing in the course of generating SW network and directly connecting the sites that before have been connected by a number of links distinguish such a network from the regular one [2]. In the framework of the mean-field theory, the special role of such links could be taken into account by ascribing the effective interaction energy J^+ (differing from the interaction energy J of the nearest neighbors) to the pair of sites connected by such a link. The distribution of numbers of short links is defined by the function (10) and the distribution of long links by the relation (12). The probability $w^+(s,k)$ that a site has s short and k long links is[4]

$$w^+(s,k) = w_1(s) w_{2,3}(k) = \frac{C_4^s}{k!} e^{-4p} (4p)^k p^{4-s} (1-p)^s. \tag{13}$$

For a ferromagnetic spin interaction ($J, J^+ > 0$), the total effective field h on the site which has s short links and k long links equals $h_{ss'kk'} = (s-2s')J + (k-2k')J^+ = (\beta_+ - \beta_-)(sJ + kJ^+)$ where $s' \leqslant s$, $k' \leqslant k$ are the numbers of corresponding links connecting a given site with spins which are antiparallel to the spin in the central site, and β_+, β_- are portions of parallel and antiparallel spins averaged over the system. Their probable values depend on the reduced system magnetization j: $\beta_\pm = (1\pm j)/2$.

The probabilities that s' among s short links and k' among k long links produce negative contributions to the field are, respectively, equal to

$$\begin{aligned} w_{ss'}(j) &= 2^{-s} C_s^{s'} (1+j)^{s-s'} (1-j)^{s'}, \\ w_{kk'}(j) &= 2^{-k} C_k^{k'} (1+j)^{k-k'} (1-j)^{k'}. \end{aligned} \tag{14}$$

Thus, the distribution function $F(j,h)$ of fields in sites of SW networks reads

$$F(j,h) = \sum_{s=0}^{4} \sum_{s' \leqslant s} \sum_{k=0}^{\infty} \sum_{k' \leqslant k} w^+(s,k) w_{ss'}(j) w_{kk'}(j) \delta(h - h_{ss'kk'}). \tag{15}$$

In the absence of macroscopic system magnetization ($j=0$) the distribution function (15), as one would expect, is symmetrical: $F(0,h) = F(0,-h)$. Notice also that the state of a site specified by the parameters s,k (total numbers of links of every kind) and s',k' (the number of antiferromagnetic neighbors) could be degenerated in the sense that the same effective field $h_{ss'kk'}$ could be associated with various sets of

[4] In Eq. (13) we neglect improbable generating new short links at the third stage of our procedure.

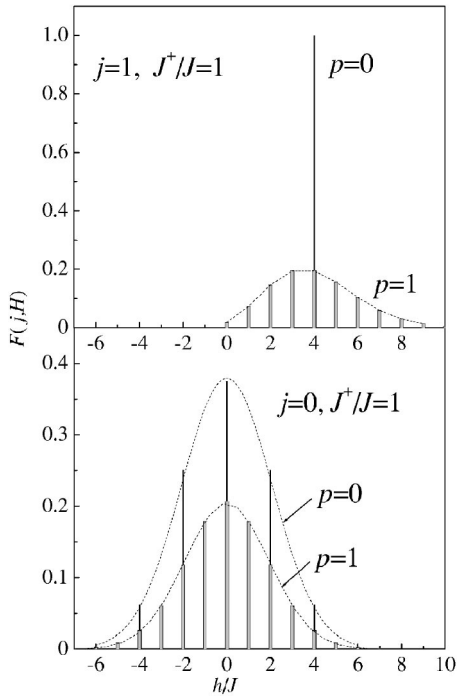

FIG. 2. The distribution function $F(j,h)$ of effective fields for two-dimensional SW systems with the fixed average coordination number at $J^+=J$ (interactions through short and long links are the same) for various magnetizations j and fractions p of long links. Dashed envelops: Gauss distribution ($j=0$) with zero mean value and Poisson distribution ($j=1$) with the mean value $\langle h\rangle=4J$. If $J^+ > J$, the distributions remain the same for $p=0$ and are shifted in the direction of higher h for $p=1$.

those parameters. The form of the distribution function (15) and its evolution with a variation of the magnetization are shown in Fig. 2.

To determine the Curie temperature for a nonregular network one substitutes the distribution function (15) in Eq. (2) of the mean-field theory. The T_C value is defined by the boundary of the nonzero solution for the resulting the equation

$$j = \sum_{s=0}^{4} \sum_{s' \leqslant s} \sum_{k=0}^{\infty} \sum_{k' \leqslant k} w^+(s,k) f_{ss'kk'}(j) \tanh(h_{ss'kk'}), \quad (16)$$

where

$$f_{ss'kk'}(j) = 2^{-(s+k)} C_s^{s'} C_k^{k'} (1+j)^{(s+k)-(s'+k')} (1-j)^{(s'+k')}. \quad (17)$$

The dependence $T_C(p)$ found by this way is shown in Fig. 3 and at $J=J^+$ demonstrates nonsignificant (less than 10% only) lowering of $T_C(p)$ with increasing p. That conclusion is contradictory to the results of numerical calculations [6], which show a nearly 50% variation of the Curie temperature with increasing p from 0 to 1: the fast growth in the range of $0<p\lesssim 0.1$ and rather slow growth at $p\gtrsim 0.1$. Within the first range, the Ising system quickly converts into the mean-field system and the discrepancy between our mean-field result $T_C(0)=3.09$ and the exact result $T_C(0)=2.27$ is substantially

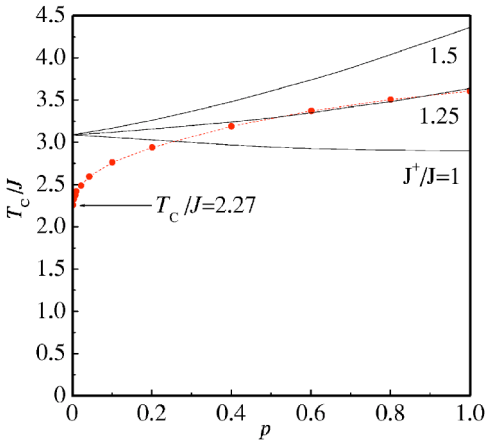

FIG. 3. Dependences $T_C(p)$ of the Curie temperature for two-dimensional SW systems with fixed average coordination number on the fraction p of long links. Points: numerical calculations [6] for the square lattice 200×200. Solid curves: analytical calculations with the help of Eq. (16) of generalized mean-field theory for different values of the J^+/J ratio.

canceled. Therefore, in the second range one could expect agreement between numerical Monte Carlo calculations and our analytical mean-field calculations. However, that agreement is accomplished by assigning some higher effective interaction energy $J^+\approx 1.25J$ to the long links only.

Introducing the effective energy J^+ differing from the real interaction energy J is, in fact, the way to rectify the mean-field theory not considering spin correlation. Actually, every site transmits information about its own state (i.e., about the spin orientation) to its neighbors by means of its links. The short link transmits that information to the nearest neighbor and through it to three next sites else (in the square lattice)—that is, in four sites, totally (one might not take more removed sites into account). The long link transmits information to the remoted site and, further, to four sites else—i.e., in five sites, totally. Hence, the long link turns out to be more effective by $5/4=1.25$ times. Though these considerations are not rigorous, they explain the enhanced effectiveness of long links qualitatively and provide a reasonable estimation of that enhancement.

IV. CONCENTRATION DEPENDENCE OF THE CURIE TEMPERATURE

To gain some insight into the physical reason for the concentration dependence $T_C(p)$ (i.e., enhancing T_C with increasing the concentration p of long links in SW networks) let us consider arguments explaining the absence of the ordered ferromagnetic state in the one-dimensional system without long links and the possibility of that state appearing in a SW network. In the first case, splitting the spin chain into two domains with opposite magnetization results in increasing the system energy by $2J$. However, the boundary between those domains could be placed at any of N chain sites that corresponds to raising the entropy by $k\ln N$. Thus, the variation of the free energy equal to $\Delta F=2J-kT\ln N$ is always negative at high enough number N; i.e., domain for-

mation is efficient. In the case of SW networks, the possibility to choose the position of the domain boundary (without additional increasing energy due to long links) is significantly limited: the number of sites suitable for that boundary lowers by about $pN \gg 1$ times comparing with the original site number N and equals $\sim N/pN = 1/p$. Now, the variation of the free energy is $\Delta F \sim J - kT \ln(1/p)$. It is positive (i.e., domain formation is profitable) if $T < T_C^{(1D)}$ where

$$kT_C^{(1D)} \sim J/|\ln p|. \tag{18}$$

Another matter is the two-dimensional lattice where the creation of the domain with perimeter length of L (in units equal to the lattice constant) leads to appearing L pairs of spins of opposite directions at the domain's boundary that result in increasing the system energy by $2LJ$. To calculate the entropy associated with that boundary, one needs to estimate the number of ways to draw a closed boundary of length L. As in every site the boundary could choose one of three directions; the number of those ways is about $G = 3^L$ (as the boundary is closed, that number is somewhat overestimated but for large L the error is insignificant [17]). Thus, the variation of the free energy equals $\Delta F \approx 2LJ - kT \ln G = L(2J - kT \ln 3)$. The ordered state is stable when the variation is positive—i.e., at $T < T_C^{(2D)}$ where[5] $kT_C^{(2D)} = 2J/\ln 3 = 1.82J$. The presence of long links lowers the number of ways to draw the boundary: it could not travel through the sites possessing those links because the system energy would be higher. The number of those forbidden sites is on the order of $(pL^2)^{1/2} = p^{1/2}L$, and near those sites the boundary could choose not three but only two directions. Hence, the number of possible boundaries of the length L reduces to $G_p \sim 3^{L-p^{1/2}L} 2^{p^{1/2}L}$, and the variation of the free energy equal to $\Delta F \approx 2LJ - kT \ln G_p = L\{2J - kT[\ln 3 - p^{1/2} \ln(3/2)]\}$ is positive at $T < T_C^{(2D)} + \Delta T_C^{(2D)}$ where

$$\Delta T_C^{(2D)} \approx p^{1/2} T_C^{(2D)} \left(1 - \frac{\ln 2}{\ln 3}\right) \propto p^{1/2}. \tag{19}$$

Analogous arguments also allow one to estimate the dependence $T_C(p)$ for the three-dimensional case where creation of a three-dimensional domain with surface area S leads to the formation of S pairs with opposite spin directions, resulting in increasing the energy by $2SJ$. If the number of variants to extend the surface in every site is $g \sim 1$, then the number of ways to create the domain with the surface area S is about $G = g^S$. Thus, the variation of the free energy equals $\Delta F \approx 2SJ - kT \ln G = S(2J - kT \ln g)$. The ordered state is stable at $T < T_C^{(3D)}$ where[6] $kT_C^{(3D)} = 2J/\ln g$. The presence of long links diminishes the number of ways to extend the surface: it could not pass through the sites with those links as the system energy would be higher. The number of forbidden sites is $\sim p^{2/3}S$, and near those the number of surface extension ways equals, not g, but $(g - \Delta g)$. Hence, the number of possible domains with surface S lowers down to $G_p \sim g^{S - p^{2/3}S}(g-1)^{p^{2/3}S}$, and the variation of free energy equal to $\Delta F \approx 2SJ - kT \ln G_p = S\{2J - kT[\ln g + p^{3/2} \ln(1 - \Delta g/g)]\}$ is positive at $T < T_C^{(3D)} + \Delta T_C^{(3D)}$ where

$$\Delta T_C^{(3D)} \approx p^{2/3} T_C^{(3D)} \left(1 - \frac{\ln(g-1)}{\ln g}\right) \propto p^{2/3}. \tag{20}$$

The dependences (18) and (19) are supported by numerical calculations [4,6] to estimate the applicability of Eq. (19)

As for the three-dimensional SW network, the dependence $\Delta T_C^{(3D)}$ is, according to [6], close to the linear one; however, the author of that result is doubtful of its accuracy.

V. CONCLUSIONS

We have shown that the results of analytical mean-field theory could be qualitatively reconciled with the results of numerical calculations of the Curie temperature for the system of Ising spins in SW networks by introducing the effective energy of interaction through long links that is somewhat higher than the real interaction energy. That allows us to explain the growth of the Curie temperature $T_C(p)$ with increasing p for a system with fixed coordination number.

The physical reason for the growth T_C with p is a lowering system entropy associated with the long links.

In conclusion, of course, the generalized mean-field theory does not lead to a quantitatively accurate description of the magnetic state of the Ising SW system, but allows one to obtain results which are qualitatively correct. Their advantage (as compared with more accurate but numerical methods) is that the results are obtained by a simple analytic method.

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[5] The Onsager exact solution $kT_C^{(2D)} = 2J/\ln(1+\sqrt{2}) = 2.27J$ changes this result unessentially.

[6] Agreement with the almost exact solution $kT_C^{(3D)} \approx 4.5J$ [10] is obtained at $g \approx 1.5$.

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