

Work extraction in the spin-boson model

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We show that work can be extracted from a two-level system (spin) coupled to a bosonic thermal bath. This is possible due to different initial temperatures of the spin and the bath, both positive (no spin population inversion), and is realized by means of a suitable sequence of sharp pulses applied to the spin. The extracted work can be of the order of the response energy of the bath, therefore much larger than the energy of the spin. Moreover, the efficiency of extraction can be very close to its maximum, given by the Carnot bound, at the same time the overall amount of the extracted work is maximal. Therefore, we get a finite power at efficiency close to the Carnot bound. The effect comes from the backreaction of the spin on the bath, and it survives for a strongly disordered (inhomogeneously broadened) ensemble of spins. It is connected with generation of coherences during the work-extraction process, and we deduced it in an exactly solvable model. All the necessary general thermodynamical relations are deduced from the first principles of quantum mechanics and connections are made with processes of lasing without inversion and with quantum heat engines.

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I. INTRODUCTION

A known feature of technological progress is the increase of human ability to control and design the microscopic world. Recent efforts in manipulating simple quantum systems, e.g., in the context of quantum computing or quantum chemistry, is one aspect of this general trend. Another aspect is the field of quantum thermodynamics, whose main objective is in designing and studying new thermodynamic processes in the domain where quantum features of matter are relevant. In particular, this activity aims to improve our understanding of standard thermodynamics [1–3] by addressing its concepts from the first principles of quantum mechanics [4–6]. The current activity in quantum thermodynamics includes quantum engines [7–12], general aspects of work extraction from quantum systems [13], thermodynamic aspects of quantum-information theory [14,15,64], and limits of thermodynamic concepts such as the second law [4,16,5] and the temperature [17]. There were also much earlier applications concerning, in particular, thermodynamic aspects of lasers and masers [18].

Our present purpose is to study work extraction from a two-temperature system on the basis of the known spin-boson model [19–22]: a two-level system coupled to a bosonic thermal bath. The motivation to use a two-level system is obvious: it is almost everywhere, and it is the minimal model having nontrivial quantum features. The necessity of the bath has to be stressed separately since, in the usual practice of quantum systems manipulation, the bath is a serious hindrance. As follows, the process of work extraction really needs external thermal baths: The second law in Thomson's formulation—which is derived as a theorem in quantum mechanics [23–28]—forbids work extraction from an equilibrium system by means of cyclic processes generated by external fields. The easiest way to employ an equilibrium system in work extraction is to attach it to a thermal bath having a different temperature, thus forming a local-equilibrium state. The overall system is then out of equilib-

rium, and work extraction from cycles is not forbidden, at least in principle. This was shown explicitly in [4,16,5].

This general restriction determined how standard quantum work-extraction (also known as amplification or lasing/masing) processes are designed [18]. The most traditional lasers and masers operate by extracting work from an ensemble of two-level systems having a negative temperature, in other words, population inversion, which is a strongly nonequilibrium state. More recent schemes of lasing without inversion employ nonequilibrium states of three- (four-, multi) level systems without population inversion of energy levels, but with initially sizable nondiagonal terms of the corresponding density matrix in the energy representation, usually called coherences [29,30]. These schemes attracted attention due to both their conceptual novelty and the fact that nonzero nondiagonal elements represent a weaker form of nonequilibrium than population inversion, and thus their preparation can be an easier task [29,30].

The mechanism of work extraction proposed in the present paper differs from the standard ones in several aspects.

(i) Work extraction (amplification, lasing) can be achieved in two-level systems without population difference and without using an initially coherent state. A setup consisting of a positive temperature spin interacting with a thermal bath at some higher or lower temperature suffices to extract work and thereby amplify pulsed fields acting on the spin. Moreover, the extracted work can be of the order of the bath's response energy, which is larger than the energy of the spin. Thus, when viewed as lasing without inversion, the presented mechanism offers definite advantages over the existing schemes.

(ii) The effect survives for a disordered ensemble of spins, where the spins have a random energy with a large dispersion. The reason for the survival is the possibility of combining the work-extraction process with the spin-echo phenomena [31,32]. As a consequence, we have a phenomenon even more amazing than the original spin-echo: a high-

temperature, completely disordered ensemble of spins can serve as a medium of work extraction.

(iii) The efficiency of work extraction can approach its maximally possible value given by Carnot bound. Moreover, the efficiency is maximized simultaneously with the overall amount of the extracted work. In addition, the power of work (i.e., the work divided over the total duration of the work-extraction process) is finite. Thus, in marked contrast to the original Carnot cycle [1,3] and some of its realizations in quantum engines [12], the three basic objectives of a good work-extraction process are met together: a large amount of extracted work, high efficiency, and finite power.

The origin of the presented mechanism is that, besides well-known effects of dissipation and decoherence induced by a thermal bath on a spin interacting with it, there is another effect, the presence of which is frequently not acknowledged. This is the backreaction of the spin to the bath, which in combination with external fields influences the spin's dynamics. The effect exists even for relatively small—but generic—bath-spin coupling constants, but is typically neglected from standard weak-coupling theories [33]. Our present treatment of the bath-spin interaction is exact and allows us to study the full influence of backreaction and memory effects.

This paper is organized as follows. In Sec. II, we recall a version of the spin-boson model we work with. It has become one of the most popular models in the theory of open quantum systems [19–21,34]. First in this section, we recall how to solve this model, following similar steps to Ref. [21], giving details on the factorized initial conditions and the Ohmic spectrum of the bath. Further, we describe a characteristic of this model, which to our knowledge was never commented on, namely the fact that correlated initial conditions are equivalent (at least for one time quantities) to factorized initial conditions for sufficiently long times. In Sec. III, we introduce the action of external fields on our system. The definition of work is recalled also in this section. The next section presents experimental realizations of our setup in various situations of two-level system(s) interacting with a thermal bath. In Sec. V, we obtain general limits on work extraction from a two-temperature system. The next two sections describe our basic results on the work extraction, efficiency, and the power of work. The last section offers our main conclusions and compares our results with the ones existing in the literature. Several technical questions are considered in the Appendixes. We have tried to make this paper reasonably self-contained. This especially concerns the concepts and relations of standard thermodynamics, which are not accepted uncritically, but, in many situations, are derived from the first principles of quantum mechanics.

II. THE MODEL

As is common when dealing with open systems, the Hamiltonian \hat{H} is composed by three parts,

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_I. \quad (1)$$

\hat{H}_S stands for the Hamiltonian of a two-level system (spin $\frac{1}{2}$),

$$\hat{H}_S = \frac{\varepsilon}{2} \hat{\sigma}_z, \quad \varepsilon \equiv \hbar\Omega, \quad (2)$$

where $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ are Pauli's matrices, and where the energy levels are $\pm\varepsilon/2$.

The spin interacts with a thermal bath which is a set of harmonic oscillators. In some cases, this may be taken in the literal sense, when harmonic oscillators represent phonons or photons. It is also known that rather general classes of thermal baths can be effectively represented via harmonic oscillators [35,36]. Thus for the Hamiltonian of the bath we take

$$\hat{H}_B = \sum_k \hbar\omega_k \hat{a}_k^\dagger \hat{a}_k, \quad [\hat{a}_l, \hat{a}_k^\dagger] = \delta_{kl}, \quad (3)$$

where \hat{a}_k^\dagger and \hat{a}_k are creation and annihilation operators of the bath oscillator with the index k . The thermodynamic limit for the bath will be taken later on.

The next important point is to specify the interaction between the spin and the bath. Recall that any reasonable model of a thermal bath is expected to drive a nonstationary state of the spin toward a stationary state. In this respect, for two-level systems, one distinguishes two types of relaxation processes and the corresponding times scales [3,18,37–39].

(i) \mathcal{T}_2 -time scale related to the relaxation of the average transversal components $\langle \hat{\sigma}_x \rangle$ and $\langle \hat{\sigma}_y \rangle$ of the spin (decoherence). Note that the very notion of the transversal components is defined by the form (2) of the spin Hamiltonian.

(ii) \mathcal{T}_1 -time scale related to the relaxation of $\langle \hat{\sigma}_z \rangle$. It is customary to have situations where

$$\mathcal{T}_2 \ll \mathcal{T}_1, \quad (4)$$

the main physical reason being that the transversal components are not directly related to the energy of the spin.

Our basic assumption on the relaxation times is (4) [40]. Moreover, to facilitate the solution of the model, we will disregard \mathcal{T}_1 time as being very large, thereby restricting the times of our interest to those much shorter than \mathcal{T}_1 . The interaction Hamiltonian is thus chosen such that it induces only transversal relaxation,

$$\hat{H}_I = \frac{\hbar}{2} \hat{X} \hat{\sigma}_z, \quad \hat{X} \equiv \sum_k g_k (\hat{a}_k^\dagger + \hat{a}_k), \quad (5)$$

where g_k are the coupling constants to be specified later, and where \hat{X} is the collective coordinate operator of the bath.

The last ingredient of our model is external fields which are acting on the spin. However, before discussing them in the next section, we shall recall how the model with Hamiltonian Eq. (1) is solved without external fields [21].

A. Heisenberg equations and their exact solution

Heisenberg equations for operators $\hat{\sigma}_z(t)$ and $\hat{a}_k(t)$ read from Eqs. (2), (3), (5), and (1)

$$\dot{\hat{\sigma}}_z = 0, \quad \dot{\hat{\sigma}}_z(t) = \hat{\sigma}_z(0), \quad (6)$$

$$\dot{\hat{a}}_k = \frac{i}{\hbar}[H, \hat{a}_k] = -i\omega_k \hat{a}_k - \frac{i}{2}g_k \hat{\sigma}_z. \quad (7)$$

Equations (6) and (7) are solved as

$$\hat{a}_k(t) = e^{-i\omega_k t} \hat{a}_k(0) + \frac{g_k \hat{\sigma}_z}{2\omega_k} (e^{-i\omega_k t} - 1), \quad (8)$$

and then

$$\hat{X}(t) = \hat{\eta}(t) - \hat{\sigma}_z G(t), \quad (9)$$

where

$$G(t) \equiv \sum_k \frac{g_k^2}{\omega_k} (1 - \cos \omega_k t) \quad (10)$$

quantifies the reaction of the spin on the collective operator of the bath, and where we denoted

$$\hat{\eta}(t) = \sum_k g_k [\hat{a}_k^\dagger(0) e^{i\omega_k t} + \hat{a}_k(0) e^{-i\omega_k t}] \quad (11)$$

for the quantum noise operator [41]. Recalling the standard relations

$$\hat{\sigma}_\pm = \hat{\sigma}_x \pm i\hat{\sigma}_y, \quad [\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm, \quad \hat{\sigma}_z \hat{\sigma}_\pm = \pm \hat{\sigma}_\pm, \quad (12)$$

and using Eq. (9) and $[\hat{X}(t), \hat{\sigma}_\pm(t)] = 0$ —since they belong to different Hilbert spaces—one derives

$$\dot{\hat{\sigma}}_\pm = \frac{i}{\hbar}[H, \hat{\sigma}_\pm] = \pm i(\Omega + \hat{X})\hat{\sigma}_\pm = i[\pm\Omega \pm \hat{\eta}(t) - G(t)]\hat{\sigma}_\pm. \quad (13)$$

These equations are solved as

$$\hat{\sigma}_\pm(t) = \exp[\pm i\Omega t - iF(t)] \hat{\Pi}_\pm(0, t) \hat{\sigma}_\pm(0), \quad (14)$$

$$\hat{\Pi}_\pm(t_0, t_1) \equiv \mathcal{T} \exp \left[\pm i \int_{t_0}^{t_1} ds \hat{\eta}(s) \right], \quad (15)$$

$$F(t) \equiv \int_0^t ds G(s) = \sum_k \frac{g_k^2}{\omega_k} \left(t - \frac{\sin \omega_k t}{\omega_k} \right), \quad (16)$$

where \mathcal{T} stands for the time-ordering operator. It is seen again from Eq. (14) that there are two effects generated by the bath-spin interaction: besides random influences entering with the quantum noise $\hat{\eta}(t)$, there is a deterministic influence generated by the backreaction term $F(t)$, somewhat similar to damping (friction) in the problem of quantum Brownian motion.

B. Factorized initial conditions

Let us assume that initially, at the moment $t=0$, the bath and the spin are in the following factorized state:

$$\rho(0) = \rho_S(0) \otimes \rho_B(0) = \rho_S(0) \otimes \frac{e^{-\beta \hat{H}_B}}{\text{tr} e^{-\beta \hat{H}_B}}, \quad (17)$$

where $\rho_S(0)$ is the initial density matrix of the spin, and where the bath is initially at equilibrium with temperature $T=1/\beta$.

Factorized initial conditions are adequate when the spin is prepared independently from the equilibrium bath and then is brought in contact to it at the initial time [42], for example, injection of an electronic spin into a quantum dot, or creation of an exciton by external radiation. Yet another situation where factorized initial conditions can be adopted is a (strong) selective measurement of $\hat{\sigma}_z$ by an external apparatus. In this case, $\rho_S(0)$ is an eigenstate of $\hat{\sigma}_z$ upon which the selection was done. Nonfactorized initial states are commented upon below, in Sec. II D.

The equilibrium relations

$$\langle \hat{a}_k^\dagger(0) \rangle = \langle \hat{a}_k(0) \rangle = 0, \quad (18)$$

$$\langle \hat{a}_k^\dagger(0) \hat{a}_k(0) + \hat{a}_k(0) \hat{a}_k^\dagger(0) \rangle = \coth \left(\frac{\beta \hbar \omega_k}{2} \right) \quad (19)$$

derived from Eq. (17) imply that the quantum noise is a stationary Gaussian operator with

$$\langle \hat{\eta}(t) \rangle = 0, \quad (20)$$

and having the time-ordered correlation function,

$$\begin{aligned} K_{\mathcal{T}}(t-t') &= \langle \mathcal{T}[\hat{\eta}(t) \hat{\eta}(t')] \rangle_{\hat{\eta}} \\ &= \sum_k g_k^2 \left[\coth \left(\frac{\beta \hbar \omega_k}{2} \right) \cos \omega_k(t-t') \right. \\ &\quad \left. - i \text{sgn}(t-t') \sin \omega_k(t-t') \right], \end{aligned} \quad (21)$$

where the average $\langle \cdots \rangle$ is taken over the initial state (17). It can be written as

$$K_{\mathcal{T}}(t) = K(t) - i\dot{G}(t), \quad (22)$$

where

$$\begin{aligned} K(t-t') &= \text{Re}[K_{\mathcal{T}}(t-t')] = \frac{1}{2} \langle \hat{\eta}(t) \hat{\eta}(t') + \hat{\eta}(t') \hat{\eta}(t) \rangle \\ &= \sum_k g_k^2 \coth \left(\frac{\beta \hbar \omega_k}{2} \right) \cos \omega_k(t-t') \end{aligned} \quad (23)$$

is the symmetrized correlation function.

Since $\hat{\eta}(t)$ is a Gaussian random operator, one can use Wick's theorem for decomposing higher-order products [43]. Due to the factorized structure (17) of the initial state, the common averages of $\hat{\eta}$ and various spin operators can be taken independently. For example, averaging Eq. (14) and using Wick's theorem together with the arithmetic relation $k!2^k(2k-1)!! = (2k)!$, one gets

$$\langle \hat{\sigma}_{\pm}(t) \rangle = e^{\pm i\Omega t - if(t)} \langle \hat{\Pi}_{\pm}(0, t) \rangle \langle \hat{\sigma}_{\pm}(0) \rangle = e^{\pm i\Omega t - \xi(t)} \langle \hat{\sigma}_{\pm}(0) \rangle, \quad (24)$$

where for $t_2 \geq t_1$

$$\begin{aligned} \langle \hat{\Pi}_{\pm}(t_1, t_2) \rangle &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \int_{t_1}^{t_2} \cdots \int_{t_1}^{t_2} ds_1 \cdots ds_{2k} \\ &\quad \times \langle \mathcal{T}[\hat{\gamma}(s_1) \cdots \hat{\gamma}(s_{2k})] \rangle \\ &= \exp \left[-\frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} ds_1 ds_2 K_{\mathcal{T}}(s_1 - s_2) \right] \\ &= \exp[-\xi(t_2 - t_1) + iF(t_2 - t_1)], \end{aligned} \quad (25)$$

and where

$$\xi(t) = \frac{1}{2} \int_0^t \int_0^t ds_1 ds_2 K(s_1 - s_2) = \int_0^t ds_1 \int_0^{s_1} ds_2 K(s_2). \quad (26)$$

As seen from Eq. (24), $\xi(t)$ characterizes the decay of $\langle \hat{\sigma}_{\pm} \rangle$ due to the interaction with the bath.

C. Ohmic spectrum of the bath

The coupling with the bath can be parametrized via the spectral density function $J(\omega)$,

$$J(\omega) = \sum_k g_k^2 \delta(\omega - \omega_k). \quad (27)$$

In the thermodynamic limit, the number of bath oscillators goes to infinity, and $J(\omega)$ becomes a smooth function, whose form is determined by the underlying physics of the system-bath interaction.

We shall be mainly working with the Ohmic spectrum,

$$J(\omega) = \gamma \omega e^{-\omega/\Gamma}, \quad (28)$$

where γ is the dimensionless coupling constant, and where Γ is the maximal characteristic frequency of the bath's response. This spectrum and its relevance for describing quantum open systems were widely discussed in the literature; see, e.g., [20].

1. Quantum noise correlation function and decay times

The correlation function of the quantum noise in the Ohmic case, using Eqs. (23), (27), and (28), is given by

$$K(t) = \int_0^{\infty} d\omega J(\omega) \coth \left[\frac{\hbar\omega}{2T} \right] \cos \omega t \quad (29)$$

$$= \gamma \int_0^{\infty} d\omega \omega \coth \left[\frac{\hbar\omega}{2T} \right] e^{-\omega/\Gamma} \cos \omega t. \quad (30)$$

Recall that the decay factor $\xi(t)$ is related to $K(t)$ via Eq. (26): $\ddot{\xi}(t) = K(t)$. Properties of these functions are worked out in Appendix A. In particular, for $\xi(t)$ one gets from Eqs. (A13) and (A14) the following exact expression:

$$\xi(t) = \gamma \ln \left[\frac{\Gamma^2 \left(1 + \frac{T}{\hbar\Gamma} \right) \sqrt{1 + \Gamma^2 t^2}}{\Gamma \left(1 + \frac{T}{\hbar\Gamma} - i \frac{Tt}{\hbar} \right) \Gamma \left(1 + \frac{T}{\hbar\Gamma} + i \frac{Tt}{\hbar} \right)} \right], \quad (31)$$

where Γ is Euler's gamma function. It is seen that the temperature is controlled by the dimensionless parameter $T/(\hbar\Gamma)$.

Let us now determine the behavior of this quantity for low and high temperatures. Using Eq. (A7), one obtains for $\hbar\Gamma/T \gg 1$ (low temperatures)

$$\xi(t) = \gamma \ln \left[\frac{\hbar\beta}{\pi t} \sinh \left(\frac{\pi t}{\hbar\beta} \right) \right] + \frac{\gamma}{2} \ln[1 + \Gamma^2 t^2]. \quad (32)$$

This implies two regimes of decay: power-law and exponential,

$$t \ll \hbar\beta: e^{-\xi(t)} = (1 + \Gamma^2 t^2)^{-\gamma/2}, \quad (33)$$

$$t \geq \hbar\beta: e^{-\xi(t)} = e^{-t/T_2}, \quad T_2 = \frac{\hbar}{\gamma T \pi}. \quad (34)$$

For $\hbar\Gamma/T \ll 1$ (high temperatures), one uses Eq. (A12) to get

$$\xi(t) = \frac{2\gamma T}{\hbar\Gamma} \left[\Gamma t \arctan(\Gamma t) - \frac{1}{2} \ln(1 + \Gamma^2 t^2) \right]. \quad (35)$$

This time the possible regimes of decay can be approximated as Gaussian and exponential,

$$t \lesssim 1/\Gamma: e^{-\xi(t)} \simeq e^{-t^2/T_2^2}, \quad T_2 = \sqrt{\frac{\hbar^2}{\gamma T \Gamma}}, \quad (36)$$

$$t \gg 1/\Gamma: e^{-\xi(t)} = e^{-t/T_2}, \quad T_2 = \frac{\hbar}{2\gamma T}. \quad (37)$$

In this latter case, as seen from Eq. (A12), $K(t)$ behaves as an approximate δ function: $K(t) \simeq 2\gamma T \Gamma / [\hbar(1 + t^2 \Gamma^2)]$ with the strength $2\gamma T/\hbar$ determined by parameters γ and T . Note that in all the above cases, the characteristic times of decay become shorter upon increasing the temperature T or coupling constant γ , as is expected. The Gaussian regime of decay was also numerously observed in NMR experiments (see [44] and references therein). This regime is the basis of the quantum Zeno effect [19] and was recently predicted to govern the reduction process in quantum measurements [45].

2. The G factor

Finally, we will indicate the form of the backreaction functions $G(t)$ and $F(t)$ in the Ohmic case [see Eq. (10)]. As will be seen below, these functions are rather important for our purposes,

$$G(t) = \gamma \Gamma \left(1 - \frac{1}{1 + \Gamma^2 t^2} \right), \quad (38)$$

$$F(t) = \gamma [\Gamma t - \arctan(\Gamma t)]. \quad (39)$$

Since $G(t)$ becomes equal to a constant on the characteristic time $1/\Gamma$, it is justified to call the latter the response time of the bath.

D. Correlated initial conditions

Most papers on the system bath models assume factorized initial conditions. However, in many situations the use of such a condition is difficult to justify *a priori*, since it implies a possibility of switching the system-bath interaction. Nonfactorized initial conditions can be Gibbians that are modified at the initial time, as considered in [6] for the Caldeira-Leggett model and in [5] for the spin-boson model.

For our present purposes it is sensible to use the following correlated initial conditions for the spin and the bath:

$$\rho(0) = \frac{1}{Z} \exp[-\beta_S \hat{H}_S - \beta(\hat{H}_I + \hat{H}_B)],$$

$$Z = \text{tr} e^{-\beta_S \hat{H}_S - \beta(\hat{H}_I + \hat{H}_B)}, \quad (40)$$

where β is the inverse temperature of the bath and β_S is that of the spin.

The initial condition (40) with $\beta_S \neq \beta$ can be generated from the equilibrium equal-temperature state of the overall system via cooling or heating the bath by means of some superbath. During this process $\hat{\sigma}_z$ is conserved, and the bath relaxes to its new temperature under an ‘‘external field’’ $\pm \frac{1}{2} \hat{X}$ generated by the interaction Hamiltonian \hat{H}_I with $\hat{\sigma}_z = \pm 1$. More details of this procedure are given in Appendix B.

In the thermodynamic limit for the bath, the correlated initial condition (40) is equivalent to the factorized condition (17) with

$$\rho_S(0) = \frac{1}{\text{tr} e^{-\beta_S \hat{H}_S}} e^{-\beta_S \hat{H}_S}, \quad (41)$$

that is, when starting from the factorized initial condition (17) and (41), the dynamics of the overall system builds up a correlated state which at times t much longer than the response time of the bath, $t \gg 1/\Gamma$ (ergodic limit) [46], is equivalent to Eq. (40). By saying ‘‘equivalent,’’ we mean that the initial conditions (17), (41), and (40) produce the same values for spin’s observables and for collective observables of the bath (i.e., the ones involving summation over all bath oscillators). This equivalence is further discussed in Appendix C.

As for the initial state of the spin, it can be deduced from Eq. (40) or from Eq. (41),

$$\langle \hat{\sigma}_z \rangle = -\tanh \left[\frac{\beta_S \varepsilon}{2} \right], \quad \langle \hat{\sigma}_x \rangle = \langle \hat{\sigma}_y \rangle = 0. \quad (42)$$

In the following, we will use the factorized initial condition (17) since it is technically simpler. The time limit $t \rightarrow \infty$ will be taken before any perturbation acts on the system to ensure the equivalence with the correlated initial condition (40).

III. PULSED DYNAMICS

The external fields acting on the spin are described by a time-dependent Hamiltonian

$$\hat{H}_F(t) = \frac{1}{2} \sum_{k=x,y,z} h_k(t) \hat{\sigma}_k, \quad (43)$$

with magnitudes $h_k(t)$, which is to be added to \hat{H} defined in Eq. (1) such that the overall Hamiltonian is time-dependent,

$$\hat{H}(t) = \hat{H} + \hat{H}_F(t). \quad (44)$$

Equation (43) represents the most general external field acting on the spin. We shall concentrate on the *pulsed* regime of external fields which is well known in NMR and ESR physics [18,31,32,37–39,47]. For example, it was used to describe spin-echo phenomena [31,32] or processes that switch off undesired interactions, such as those causing decoherence [22,34,37,38].

A pulse of duration δ is defined by a sudden switching on of the external fields at some time $t > 0$, and then suddenly switching them off at time $t + \delta$. It is well known that during a sudden switching, the density matrix does not change [1], while the Hamiltonian gets a finite change. Let us for the moment keep arbitrary the concrete form of external fields in the interval $(t, t + \delta)$. The Schrödinger evolution operator of the spin+bath from time zero until some time $t + \tau$, $\tau > \delta$ reads

$$\mathcal{T} \exp \left[-\frac{i}{\hbar} \int_0^{t+\tau} ds \hat{H}(s) \right] = e^{-i(t+\tau-t)\hat{H}/\hbar} \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] e^{-i\hat{H}t/\hbar} \quad (45)$$

$$= e^{-i\tau\hat{H}/\hbar} \hat{U}_P(t) e^{-i\hat{H}t/\hbar}. \quad (46)$$

The left-hand side (LHS) of Eq. (45) contains the full time-dependent Schrödinger-representation Hamiltonian $\hat{H}(s)$, while on the right-hand side (RHS) of this equation we took into account that the actual time dependence is present only between t and $t + \delta$. The terms $e^{-i\hat{H}t/\hbar}$ and $e^{-i(t+\tau-t)\hat{H}/\hbar}$ stand for the free (unpulsed) evolution in time intervals $(0, t)$ and $(t + \delta, t + \tau)$, respectively. In Eq. (46), we denoted

$$\hat{U}_P(t) \equiv e^{i\delta\hat{H}/\hbar} \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] \quad (47)$$

$$= \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_t^{t+\delta} ds e^{i(s-t)\hat{H}/\hbar} \hat{H}_F(s) e^{i(t-s)\hat{H}/\hbar} \right]$$

$$= \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_0^\delta ds e^{is\hat{H}/\hbar} \hat{H}_F(s+t) e^{-is\hat{H}/\hbar} \right] \quad (48)$$

for the pulse evolution operator. The transition from Eq. (47) to Eq. (48) can be made by recalling that $\hat{H}(t) = \hat{H} + \hat{H}_F(t)$ and then by noting that the expressions in these equations satisfy the same first-order differential equation in δ with the same boundary condition at $\delta=0$ [48].

We focus on pulses so short that the influence of the spin Hamiltonian $\hbar\Omega\hat{\sigma}_z/2$ and the interaction Hamiltonian \hat{H}_I can be neglected during the interval δ . This means that one can take the first term in the Taylor expansion ($0 < s < \delta$),

$$\begin{aligned} e^{is\hat{H}/\hbar}\hat{H}_F(s+t)e^{-is\hat{H}/\hbar} &= \hat{H}_F(s+t) + \frac{is}{\hbar}[\hat{H},\hat{H}_F(s+t)] + \dots \\ &= \hat{H}_F(s+t) + \frac{is}{\hbar}\left[\frac{\hbar\Omega}{2}\hat{\sigma}_z + \hat{H}_I, \hat{H}_F(s+t)\right] \\ &+ \dots \approx \hat{H}_F(s+t). \end{aligned} \quad (49)$$

Thus, for the pulse evolution operator one gets

$$\hat{U}_P(t) = \mathcal{T} \exp\left[-\frac{i}{\hbar} \int_0^\delta ds \hat{H}_F(s+t)\right]. \quad (50)$$

The generalization of the evolution operator (46) to an arbitrary number of short pulses is straightforward.

Note that in obtaining Eq. (50), we do not require that the bath Hamiltonian \hat{H}_B during the pulse is neglected. Since the external fields are acting on the spin only, the influence of the bath Hamiltonian disappears by itself from $e^{is\hat{H}/\hbar}\hat{H}_F(s+t)e^{-is\hat{H}/\hbar}$, and is perfectly kept in the general evolution operator (45) and (46), once the interaction Hamiltonian \hat{H}_I was neglected.

Recalling the orders of magnitude $\hbar\Omega$ and $\hbar\gamma\Gamma$ of the spin energy and the interaction energy, respectively—in particular, recall Eq. (9), $\hbar G = \hbar \int_0^\infty d\omega J(\omega)/\omega$, and Eq. (28)—one gets the following qualitative criterion for the validity of the short pulsing regime:

$$\delta \ll \min(\Omega^{-1}, [\gamma\Gamma]^{-1}). \quad (51)$$

As it should be, for very small γ and a fixed Γ , the second restriction on δ is weaker than the first one. More quantitative conditions for the validity of the pulsed regime were studied recently in the context of decoherence suppression by external pulses [34].

To deal with the pulsed dynamics in the Heisenberg representation, one introduces the following superoperators:

$$\mathcal{E}_t \hat{A} \equiv e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar}, \quad (52)$$

$$\mathcal{P}_t \hat{A} \equiv \hat{U}_P^\dagger(t) \hat{A} \hat{U}_P(t). \quad (53)$$

Then the Heisenberg evolution of an operator \hat{A} corresponding to Eqs. (46) and (50) reads

$$\hat{A}(t+\tau) = \mathcal{E}_t \mathcal{P}_t \mathcal{E}_\tau \hat{A} = e^{i\hat{H}(t+\tau)/\hbar} \hat{U}_P^\dagger(t) e^{i\hat{H}\tau/\hbar} \hat{A} e^{-i\hat{H}\tau/\hbar} \hat{U}_P(t) e^{-i\hat{H}t/\hbar}. \quad (54)$$

A. Definition of work

The action of external fields on the system is connected with flow of work. The work done in the time interval $(0, t)$ is standardly defined as the increase of the average overall energy of the spin and bath defined by the time-dependent Hamiltonian $\hat{H}(t)$ [1–3],

$$W(0, t) = \text{tr}[\rho(t)\hat{H}(t)] - \text{tr}[\rho(0)\hat{H}(0)]. \quad (55)$$

Due to the conservation of energy of the entire system (spin+bath+work source), work is equal to the energy given by the corresponding work source (source of external fields).

Since the external fields are acting only on the spin, there is a differential formula for the work which uses only quantities referring to the local state of the spin and which thus illustrates that the work sources exchange energy only through the spin,

$$\frac{dW}{dt} = \text{tr}\left(\rho_S(t) \frac{\partial \hat{H}_F(t)}{\partial t}\right), \quad (56)$$

where $\hat{H}_F(t)$ as defined by Eq. (43) is the contribution of the external fields into the spin's Hamiltonian, and where $\rho_S(t)$ is the density matrix of the spin. Equations (55) and (56) relate with each other by the von Neumann equations of motion

$$\dot{\rho} = \frac{i}{\hbar}[\hat{H}(t), \rho(t)] = \frac{i}{\hbar}[\hat{H} + \hat{H}_F(t), \rho(t)] \quad (57)$$

for the common density matrix $\rho(t)$ of the spin and the bath, where \hat{H} is the Hamiltonian without external fields [49].

More specifically, we are interested in the work due to a pulse. For the above example of a single pulse at time t , this quantity reads from Eqs. (46), (54), and (55)

$$\begin{aligned} W(0, t+\delta) &= W(t, t+\delta) \\ &= \text{tr}\{[\rho(t+\delta) - \rho(t)]\hat{H}\} \\ &= \text{tr}\{\rho(t)[\mathcal{P}_t \hat{H} - \hat{H}]\}. \end{aligned} \quad (58)$$

This expression is directly generalized to several successive pulses: assume that the pulse \mathcal{P}_t at time t was followed by another pulse $\mathcal{P}_{t+\tau}$ at time $t+\tau$ with $\tau > 0$. The work done during the first pulse is given by Eq. (58), while the work done during the second pulse reads

$$\begin{aligned} W(t+\tau, t+\tau+\delta) &= \text{tr}\{(\rho(t+\tau+\delta) - \rho(t+\tau))\hat{H}\} \\ &= \text{tr}\{\rho(t+\tau)[\mathcal{P}_{t+\tau} \hat{H} - \hat{H}]\} \\ &= \text{tr}\{\rho(0)\mathcal{E}_t \mathcal{P}_t \mathcal{E}_\tau [\mathcal{P}_{t+\tau} \hat{H} - \hat{H}]\}. \end{aligned} \quad (59)$$

Summing this up with $W(0, t+\delta)$, one gets for the complete work for the two-pulse situation

$$W(0, t+\tau+\delta) = \text{tr}\{\rho(0)[\mathcal{E}_t \mathcal{P}_t \mathcal{E}_\tau \mathcal{P}_{t+\tau} \hat{H} - \hat{H}]\}, \quad (60)$$

as should be.

B. Parametrization of pulses

As seen from Eqs. (43) and (53), and taking into account the condition (51) which, to all effects, can be taken as $\delta \rightarrow 0$, any pulse corresponds to the most general unitary operation in the Hilbert space of the spin (this would correspond to a rotation in the classical language). It is convenient to parametrize pulses by coefficients $c_{a,b}$ as

$$\mathcal{P} \hat{\sigma}_a \equiv \hat{U}_P^\dagger(t) \hat{\sigma}_a \hat{U}_P(t) = \sum_{b=\pm, z} c_{a,b} \hat{\sigma}_b, \quad a = \pm, z. \quad (61)$$

For more detailed applications, we will need the explicit form of $\hat{U}_p^\dagger(t)$ [see Eq. (50)] as a 2×2 unitary matrix whose determinant can be taken to be unity without loss of generality,

$$\hat{U}_p^\dagger(t) = \begin{pmatrix} e^{-i\phi} \cos \vartheta & -e^{-i\psi} \sin \vartheta \\ e^{i\psi} \sin \vartheta & e^{i\phi} \cos \vartheta \end{pmatrix}, \quad (62)$$

where

$$0 \leq \phi, \quad \psi \leq 2\pi, \quad 0 \leq \vartheta \leq \frac{\pi}{2}. \quad (63)$$

Parametrizations similar to Eq. (62) are frequently applied in NMR and ESR experiments [18,31,32,37–39,47] where the spin is rotated certain degrees over a well-defined axis by tuning the parameters of the laser (microwave) pulse applied.

IV. REALIZATIONS OF THE MODEL

Once the model with all its ingredients has been defined, we discuss some of its realizations and provide some numbers. A two-level system coupled to a thermal bath is a standard model for practically all fields where quantum systems are studied: NMR, ESR, quantum optics, superionics, Joseph junctions, etc. Two particular conditions are, however, necessary to apply the model we study: the condition $T_1 \gg T_2$ on the characteristic relaxation times and the availability of sufficiently strong pulses. On the other hand, we can allow for rather short times T_2^* , since as we will see this time scale can be overcome with the spin-echo technique.

There are experimentally realized examples of two-level systems which have sufficiently long T_2 times, satisfy in $T_1 \gg T_2$, e.g., T_1 exceeds T_2 by several orders of magnitude, and admit strong pulses of external fields. For atoms in optical traps, where $T_2 \sim 1$ s, $1/\Gamma \sim 10^{-8}$ s, there are efficient methods for creating nonequilibrium initial states and for manipulating atoms by external laser pulses [50]. For an electronic spin injected or optically excited in a semiconductor, $T_2 \sim 1$ μ s [51], and for an exciton created in a quantum dot $T_2 \sim 10^{-9}$ s [52]; in both situations $1/\Gamma \sim 10^{-9} - 10^{-13}$ s, and femtosecond (10^{-15} s) laser pulses are available. In the case of NMR physics, $T_2 \sim 10^{-6} - 10^3$ s, $1/\Gamma \sim 1$ μ s, and the duration of pulses can vary between 1 ps and 1 μ s [37,38,53].

In all the above examples, the response time $1/\Gamma$ of the bath is much shorter than the internal time $1/\Omega$ of the spin. Sometimes it is argued that such a separation is related to the large size of the bath and is something generic by itself. This is clearly incorrect, since as seen from the derivation in Sec. II, the dimensionless parameter Ω/Γ has to do with the form of the bath-spin interaction, rather than with the size of the bath. Moreover, several examples of bath-spin interaction are known and were analyzed both experimentally and theoretically, where $\Omega/\Gamma \sim 1$. For example, Ref. [54] focuses on relaxation of nuclear spins with hyperfine frequencies $\Omega \approx 700$ MHz, $1/T_2 < 90$ MHz, and the ratio Ω/Γ may vary between 10 and 0.1.

Another important parameter that characterizes our setup is the initial polarization $|\langle \hat{\sigma}_z \rangle|$ of the spin. It is known in NMR and ESR physics that the response of magnetic atoms

(nucleus) to external dc magnetic field is best characterized by the frequency/field ratio [18], which is, for example, equal to 42 MHz/T for a proton. For an electron, this ratio is 10^3 times larger due to the difference between atomic and nuclear Bohr magnetons, and for ^{15}N it is 10 times smaller. Thus at temperature $T=1$ K and magnetic field $B=1$ T, the equilibrium polarization of a proton is only $|\langle \hat{\sigma}_z \rangle| = \tanh(\hbar\mu_B/2k_B T) = 10^{-3}$, while for an electron it is ~ 1 .

Exact solution versus various approximations

The model as stated above—that is, with the Hamiltonian (1) and (44)—is exactly solvable for all temperatures and all bath-spin coupling constants. It is useful at this point to remind the reader what are the specific reasons to insist on this feature. The model with Hamiltonian (1) is a particular case of a more general spin-boson model, where the influence of T_1 time is retained either via an additional term $\propto \hat{\sigma}_x$ in the Hamiltonian of the spin, or via an additional coupling in the interaction Hamiltonian. This model is in general not solvable, and what is worse there are no reliable approximate methods which apply for a fixed (maybe weak) coupling to the bath and for all temperatures including the very low ones. The standard weak-coupling theories—both Markovian, leading to well-known Bloch equations, and non-Markovian ones—are satisfactory only for sufficiently high temperatures, while at low temperatures weak-coupling series are singular, and different methods of their resummation produce different results. In this context, compare, e.g., convolutionless master equations extensively discussed in [19] with a convolutional one worked out in [55].

This situation becomes even more problematic under driving by external fields. The objects studied by us—such as work and energy of the spin—can be rather fragile to various not very well-controlled approximations, since there are general limitations governing their behavior: Thomson's formulation of the second law and restrictions on work extraction from a two-temperature system (discussed below). These limitations are derived from the first principles of quantum mechanics [23–28] and have to be respected in any particular model.

V. GENERAL RESTRICTIONS ON WORK EXTRACTION

The setup of two systems having initially different temperatures and interacting with a source of work allows us to draw a number of general relations on work extraction. We start from the following general assumptions.

(i) *Out of equilibrium initial conditions.* The initial conditions at the moment $t=0$ are given by Eq. (40), where the bath and the spin have initially different temperatures T and T_S , respectively. Recall from the discussion in Sec. II D that after a small lapse, this initial condition is equivalent to the factorized one (17) and (41). We use the former one since it is more convenient when dealing with the general restrictions on the work extraction.

(ii) *Cyclic external fields.* For the following derivation, the Hamiltonian $\hat{H}_F(t)$ of external fields acting on the spin is completely arbitrary. In particular, it need not be composed

by pulses, where it would vanish outside of the pulses. The only general assumption made on $\hat{H}_F(t)$ is that its action is cyclic at some final time t_f ,

$$\hat{H}_F(0) = \hat{H}_F(t_f) = 0. \quad (64)$$

We can find the following two relations (derived explicitly in Appendix D):

$$W \geq \left(1 - \frac{T}{T_S}\right) \Delta H_S, \quad (65)$$

$$W \geq \left(1 - \frac{T_S}{T}\right) (\Delta H_I + \Delta H_B), \quad (66)$$

where

$$\Delta H_k = \text{tr}\{\hat{H}_k[\rho(t_f) - \rho(0)]\}, \quad k = S, I, B \quad (67)$$

are the changes of the corresponding average energies of the spin, bath, and interaction, with $\rho(t_f)$ being the complete density matrix of the spin and bath at time t_f , and where the total work reads

$$W = \Delta H_S + \Delta H_I + \Delta H_B. \quad (68)$$

Here are the implications of Eqs. (65) and (66).

(i) If $T_S > T$ and work is extracted, $W < 0$, Eq. (65) implies

$$\Delta H_S < 0, \quad \Delta H_I + \Delta H_B > 0. \quad (69)$$

The system loses energy, while the bath gains it and the amount of the extracted work $|W|$ is then bounded from above by $|\Delta H_S|$.

(ii) If $T = T_S$, both Eqs. (65) and (66) produce $W \geq 0$, which is, in fact, the statement of the second law in Thomson's formulation: no work can be extracted from an equilibrium system by means of cyclic perturbations.

(iii) If $T_S < T$, inequalities in Eq. (69) are reversed: now work extraction implies that

$$\Delta H_S > 0, \quad \Delta H_I + \Delta H_B < 0, \quad (70)$$

and $|W|$ is then bounded from above by $|\Delta H_I + \Delta H_B|$.

These conclusions are close to what one could have expected from the standard (phenomenological) thermodynamical reasoning [1]. However, it should be emphasized that in contrast to typical textbook derivations, Eqs. (65) and (66) were derived starting from first principles (see Appendix D), and, moreover, their derivation is by no means restricted to a weak bath-spin coupling, a condition which need not be satisfied in practice.

Efficiency and heat

Another useful notion is the efficiency η of the work extraction, which shows how economically nonequilibrium, two-temperature resource is employed in work extraction [1–3]. The special importance of efficiency is related to the fact that in standard thermodynamics, it is bounded from above by Carnot's value, which is a system-independent quantity.

Though our system starts out of equilibrium due to differ-

ent *initial* temperatures of the spin and the bath, the notion of efficiency should be studied for it anew, since it does not automatically fall into the class of heat-engine models, as studied in textbooks of thermodynamics and statistical physics [1–3].

(i) There is no working body which operates cyclically between two thermal baths. With us, cyclic processes are defined with respect to the work source.

(ii) The interaction between the systems having different temperatures—in the case discussed here, the spin and the bath—need not be weak.

(iii) We do not require that our systems always stay very close to equilibrium. In contrast, both during and immediately after the work-extraction process, the spin is in a non-equilibrium state, which in general cannot be described in terms of a time-dependent temperature.

However, in spite of all these differences, we can define the notion of efficiency, and this will be an equally useful characterization of the work-extraction process [1–3].

Recall that external fields are acting exclusively on the spin variables and not on those of the bath. This implies that when during work extraction the source of work receives energy $|W|$, this energy consists of a contribution coming directly from the spin and of a part which comes to the work source from the bath but through the spin. In this context, one can write the change of energy of the spin as

$$\begin{aligned} \frac{d}{dt} \text{tr}(\rho_S(t) \hat{H}_S(t)) &= \text{tr} \left[\left(\frac{d}{dt} \rho_S(t) \right) \hat{H}_S(t) \right] + \text{tr} \left[\rho_S(t) \left(\frac{\partial}{\partial t} \hat{H}_S(t) \right) \right] \\ &= \frac{d}{dt} Q + \frac{d}{dt} W, \end{aligned} \quad (71)$$

where in our case the Hamiltonian of the spin reads from Eqs. (2) and (43) [note the analogy with Eq. (44)]

$$\hat{H}_S(t) = \frac{\varepsilon}{2} \hat{\sigma}_z + \frac{1}{2} \sum_{k=x,y,z} h_k(t) \hat{\sigma}_k. \quad (72)$$

The partial time derivative in Eq. (71) stresses that we are in a Schrödinger representation. When deriving Eq. (71), we have used $\partial_t \hat{H}_S(t) = \partial_t \hat{H}_F(t)$ and Eq. (56). The last equality in Eq. (71) serves as a definition of heat (dQ) [56].

Integrating this from 0 to t_f and using Eqs. (64) and (68), we obtain

$$\Delta Q = -(\Delta H_I + \Delta H_B). \quad (73)$$

Note that in the above definition of heat, the average interaction energy is attributed to the heat received from the bath, although by itself it depends also on the variables of the spin; see Eq. (5). The reason for this asymmetry is clearly contained in the very initial statement of the problem, where we—quite in accordance with the usual practice of statistical physics—restricted the work source to act only on the spin.

All this being said, one can now proceed for $W < 0$ (work extraction) with the usual definition of efficiency as the ratio of the useful energy $|W|$ to the maximal energy involved in the work extraction,

$$\eta \equiv \frac{|W|}{\max(|\Delta H_S|, |\Delta H_I + \Delta H_B|)}. \quad (74)$$

For $T_S > T$, Eqs. (68) and (69) and $W < 0$ imply $|W| = |\Delta H_S| - |\Delta H_I + \Delta H_B|$, and then Eq. (74) results in

$$\eta = \frac{|W|}{|\Delta H_S|}. \quad (75)$$

Analogously, for $T_S < T$ we have

$$\eta = \frac{|W|}{|\Delta H_I + \Delta H_B|} = \frac{|W|}{|W| + |\Delta H_S|} \quad (76)$$

from $|W| = |\Delta H_I + \Delta H_B| - |\Delta H_S|$.

It is now seen from Eqs. (65), (66), (69), and (70) that the efficiency is always bounded by the Carnot value,

$$\eta \leq 1 - \frac{\min(T, T_S)}{\max(T, T_S)}. \quad (77)$$

VI. WORK EXTRACTION VIA TWO PULSES

A. Setup of pulsing

Let us now show in detail the setup of work extraction. The spin and the bath are prepared in the state (40) with different temperatures T_S and T for the spin and the bath, respectively. Thus, the initial average population difference $\langle \hat{\sigma}_z \rangle$ is given by Eq. (42).

Alternatively, we can prepare the spin+bath in the state (17) and (41). In this case, one waits for a time $t \gg 1/\Gamma$ for ensuring the robustness of the results. Then the setup does not depend on details of the initial preparation, because the initial conditions (17), (41), and (40) have become equivalent.

The final ingredient of the setup are pulses

$$\mathcal{P}_t = \mathcal{P}_1, \quad \mathcal{P}_{t+\tau} = \mathcal{P}_2, \quad (78)$$

applied at times t and $t+\tau$, respectively.

B. Formulas for work

The work done for the first pulse reads [as defined in Eqs. (58)]

$$W_1 = \frac{\varepsilon}{2} \langle \mathcal{P}_1 \hat{\sigma}_z - \hat{\sigma}_z \rangle_t + \frac{\hbar}{2} \langle (\mathcal{P}_1 \hat{\sigma}_z - \hat{\sigma}_z) \hat{X} \rangle_t, \quad (79)$$

where for any operator \hat{A} the average

$$\langle \hat{A} \rangle_t = \text{tr}[\hat{A} \rho(t)] \quad (80)$$

refers to the time t just before the application of the pulse. The value of W_1 is worked out by recalling the parametrization (62), the evolution of the collective bath coordinate \hat{X} as given by Eq. (9), and finally the initial condition (17) and (41). The final result reads

$$W_1 = (1 - c_{z,z}^{(1)}) \left[\frac{\hbar}{2} G - \frac{\varepsilon}{2} \langle \hat{\sigma}_z \rangle \right], \quad (81)$$

where $G = \gamma \Gamma$ is the limit of $G(t)$ for $\Gamma \gg 1$, and where $c_{z,z}^{(1)}$ is the corresponding parametrization coefficient of the first pulse as defined by Eq. (61).

As follows from $T_S > 0$ and $\langle \hat{\sigma}_z \rangle < 0$ [see Eq. (42)], the work W_1 is always positive. This is in agreement with the thermodynamic wisdom of local equilibrium: the second term on the RHS of Eq. (81) is the contribution from the spin energy and it is positive, since the spin was in equilibrium before the application of the first pulse. Another positive term $\frac{1}{2}(1 - c_{z,z}^{(1)})\hbar G$ on the RHS of Eq. (81) comes from the interaction Hamiltonian (the bath operators, and thus the bath Hamiltonian, are not influenced by this first pulse). Again, it is intuitively expected that the interaction Hamiltonian should make the average energy costs higher.

The work done for the second pulse reads analogously to Eq. (79),

$$W_2 = \frac{\varepsilon}{2} \langle \mathcal{P}_2 \hat{\sigma}_z - \hat{\sigma}_z \rangle_{t+\tau} + \frac{\hbar}{2} \langle (\mathcal{P}_2 \hat{\sigma}_z - \hat{\sigma}_z) \hat{X} \rangle_{t+\tau}, \quad (82)$$

where the averages $\langle \cdots \rangle_{t+\tau}$ refer to the time just before the application of the second pulse.

Equation (82) is worked out in Appendix G with the result for the total work $W = W_1 + W_2$ being

$$W = -\frac{\varepsilon}{2} (1 - c_{z,z}^{(2)} c_{z,z}^{(1)}) \langle \hat{\sigma}_z \rangle + \varepsilon e^{-\xi(\tau)} \text{Re} \{ c_{+,z}^{(1)} c_{z,+}^{(2)} e^{i\Omega\tau} \langle e^{i\chi_2 \hat{\sigma}_z} \hat{\sigma}_z \rangle \} \quad (83)$$

$$+ \frac{\hbar G}{2} (1 - c_{z,z}^{(1)}) + \frac{\hbar}{2} (1 - c_{z,z}^{(2)}) [G(\tau) + g_2(\tau) c_{z,z}^{(1)}] \quad (84)$$

$$+ e^{-\xi(\tau)} \text{Re} \{ c_{+,z}^{(1)} c_{z,+}^{(2)} e^{i\Omega\tau} [i\hbar \dot{\xi}(\tau) \langle e^{i\chi_2 \hat{\sigma}_z} \hat{\sigma}_z \rangle + \hbar g_2(\tau) \langle e^{i\chi_2 \hat{\sigma}_z} \rangle] \}. \quad (85)$$

The detailed explanation of various terms in this expression and of their physical meaning comes as follows.

The first term on the RHS of Eq. (83) is the contribution from the initial spin energy. The second term comes from the transversal degrees of freedom excited by the first pulse. The factor $e^{-\xi(\tau)}$ accounts for the reduction of these terms in the time interval τ between pulses. Recall that the parametrization coefficients $c_{+,z}^{(1,2)}$ and $c_{z,+}^{(1,2)}$ for the first and the second pulse are defined in Eq. (61).

The terms in Eqs. (84) and (85) are the joint contribution from the bath Hamiltonian (3) and from the interaction Hamiltonian (5). The last of them couples to the transversal degrees of the spin, as reflected by the presence of $e^{-\xi(\tau)}$. Recall that the averages $\langle \cdots \rangle$ in Eqs. (83) and (85) refer to the initial state Eqs. (17) and (41). Finally, the factors

$$g_2(\tau) \equiv G - G(\tau) = \frac{\gamma \Gamma}{1 + \tau^2 \Gamma^2}, \quad (86)$$

$$\chi_2(\tau) = -\gamma \arctan(\tau\Gamma) \quad (87)$$

(the lower index 2 refers to the two-pulse situation) come from the backreaction of the spin to the bath.

Next we note that the behavior of $W=W_1+W_2$ is controlled by five dimensionless parameters (see Appendix G), which for the Ohmic case reads

$$W = W_1 + W_2 = \frac{\hbar\gamma\Gamma}{2} w\left(\frac{T}{\hbar\Gamma}, \gamma, \frac{\varepsilon}{\hbar\Gamma}, \langle\hat{\sigma}_z\rangle, \tau\Gamma\right). \quad (88)$$

Note that the spin temperature T_S enters only through the initial (at $t=0$) $\langle\hat{\sigma}_z\rangle$ as given by Eq. (42).

There are two situations within the present setup, where work extraction is not prohibited: $T>T_S$ and $T<T_S$. We deal with them separately, since for these cases the work-extraction effect exists in different ranges of the parameters.

C. Work extraction for $T>T_S$

It was seen above that the first pulse always costs work, since it is applied on the spin whose state is (initially) in local equilibrium at temperature T_S . However, the first pulse can do more than simply wasting work. Consider, for example, a $\pi/2$ pulse in the y direction [57],

$$\mathcal{P}_1 = \mathcal{P}\left(\frac{\pi}{2}; y\right), \quad (89)$$

where

$$\mathcal{P}(\varphi; y)\hat{\sigma}_z \equiv e^{i\varphi\hat{\sigma}_y/2}\hat{\sigma}_z e^{-i\varphi\hat{\sigma}_y/2} = \hat{\sigma}_z \cos \varphi - \hat{\sigma}_x \sin \varphi, \quad (90)$$

$$\mathcal{P}(\varphi; y)\hat{\sigma}_x \equiv e^{i\varphi\hat{\sigma}_y/2}\hat{\sigma}_x e^{-i\varphi\hat{\sigma}_y/2} = \hat{\sigma}_z \sin \varphi + \hat{\sigma}_x \cos \varphi, \quad (91)$$

$$\mathcal{P}(\varphi; y)\hat{\sigma}_y \equiv \hat{\sigma}_y. \quad (92)$$

This pulse excites the transversal component $\langle\hat{\sigma}_x\rangle$ which starts to decay under action of the bath, and thus correlations between the spin and the bath are established. The proper second pulse is then applied at time τ , for instance $+\pi/2$ in the x direction,

$$\mathcal{P}_2 = \mathcal{P}\left(\frac{\pi}{2}; x\right), \quad (93)$$

where

$$\mathcal{P}(\varphi; x)\hat{\sigma}_z \equiv e^{i\varphi\hat{\sigma}_x/2}\hat{\sigma}_z e^{-i\varphi\hat{\sigma}_x/2} = \hat{\sigma}_z \cos \varphi + \hat{\sigma}_y \sin \varphi, \quad (94)$$

$$\mathcal{P}(\varphi; x)\hat{\sigma}_y \equiv e^{i\varphi\hat{\sigma}_x/2}\hat{\sigma}_y e^{-i\varphi\hat{\sigma}_x/2} = -\hat{\sigma}_z \sin \varphi + \hat{\sigma}_y \cos \varphi, \quad (95)$$

$$\mathcal{P}(\varphi; x)\hat{\sigma}_x \equiv \hat{\sigma}_x. \quad (96)$$

Note that our choice of pulses corresponds to

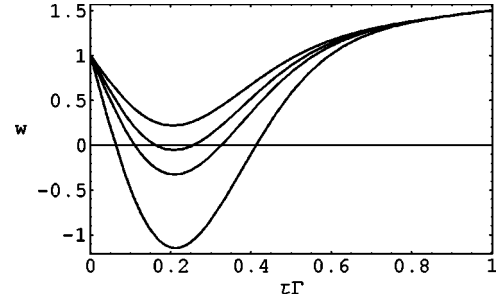


FIG. 1. Dimensionless total work w [see Eq. (88) in the text, $W=\hbar\gamma\Gamma w/2$] vs dimensionless time $\tau\Gamma$ (the waiting time between the two pulses) in the regime $T>T_S$. We compare the extracted work for different values of the initial polarization (or, equivalently, of the initial temperature) of the spin. $T/\hbar\Gamma=10$; $\gamma=1$; $\varepsilon/\hbar\Gamma=0.01$; $\langle\hat{\sigma}_z\rangle=-0.8, -0.5, -0.4, -0.3$ (from bottom to top). The two pulses are given by Eqs. (89), (93), and (97). Work extraction disappears for larger $\langle\hat{\sigma}_z\rangle$, that is, for closer (initial) temperatures of the spin and the bath.

$$c_{+,z}^{(1)} = 1, \quad c_{z,+}^{(2)} = \frac{1}{2i}, \quad c_{z,z}^{(1)} = 0, \quad c_{z,z}^{(2)} = 0. \quad (97)$$

It appears that not only is some work extracted by the second pulse, but the overall work by the two pulses can be negative for properly chosen time τ ,

$$W = W_1 + W_2 < 0, \quad (98)$$

as seen in Figs. 1 and 2. This is one of the central results of this paper.

The time τ needed for work extraction should be neither too short (otherwise the two pulses will effectively sum into one, and we know that no work extraction is achieved by a single pulse) nor too long (otherwise the transversal degree of freedom excited by the first pulse will decay, and we will have two isolated single pulses). This is seen in Figs. 1 and 2. Note that the choice of pulses is obviously important for

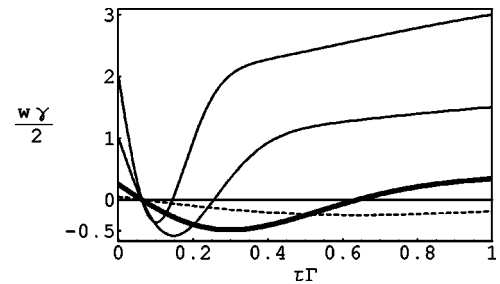


FIG. 2. The ratio $W/\hbar\Gamma = w\gamma/2$ [see Eq. (88) in the text] vs the dimensionless time $\tau\Gamma$ for two pulses in the regime $T>T_S$. We compare the extracted work for different values of the dimensionless bath-spin coupling constant γ . $T/\hbar\Gamma=10$; $\varepsilon/\hbar\Gamma=0.01$; $\langle\hat{\sigma}_z\rangle=-0.8$; and $\gamma=4$ (upper solid curve), $\gamma=2$ (lower solid curve), $\gamma=0.5$ (bold curve), and $\gamma=0.1$ (dotted curve). The two pulses are given by Eqs. (89), (93), and (97). It is seen that the maximal extracted work is a nonmonotonous function of the dimensionless coupling constant γ .

having work extraction. Equations (89) and (93) represent only one particular example leading to work extraction in the regime $T > T_S$.

As for the magnitude of the extracted work, one notes from Eq. (88) and Fig. 1 that it is of order of $\hbar\Gamma/2$, which is basically the response energy of the bath. This is not occasional, since as seen from Eq. (70), the work in this regime $T > T_S$ is coming from the bath.

Noting the ratio $\varepsilon/(\hbar\Gamma)=0.01$ in Fig. 1—this and even smaller ratios are usual for the realizations of the model as we discussed in Sec. IV—we conclude that the extracted work can be several orders of magnitude larger than the energy of the spin. On the other hand, the extracted work is limited by $\sim T$, which is the characteristic thermal energy available in the bath. Indeed, as seen in Fig. 1, the extracted work can be of order $\hbar\Gamma/2$, while the bath temperature is nearly 20 times larger: $T=10\hbar\Gamma$. Not unexpectedly, work extraction disappears when the temperatures T and T_S are close to each other; see Fig. 1.

Let us return once again to the optimal time interval τ . As Figs. 1 and 2 show, the value of τ at which the extracted work is maximal is roughly of the same order of magnitude as $1/\Gamma$. However, the optimal τ can be much larger (e.g., $\sim 10^3/\Gamma$) for smaller coupling constants γ , that is, one can increase the waiting time between the pulses at the expense of reducing the magnitude $\propto \hbar\gamma\Gamma$ of the extracted work.

D. Work extraction for $T < T_S$

Let us now turn to scenarios of work extraction in the regime $T_S > T$. As seen from Eq. (69), if there is work extraction at all in this regime, the work should come from the average energy difference of the spin, while $\Delta H_I + \Delta H_B$ is then necessarily positive. Since the latter quantity is of order of $\gamma\Gamma$ (response energy of the bath), and the spin's energy difference is obviously of order ε , there are two ways to try to achieve work extraction, that is, to get $W = |\Delta H_I + \Delta H_B| - |\Delta H_S| < 0$: One should either take $\varepsilon/(\hbar\Gamma) \sim 1$ or take the dimensionless coupling constant γ very small. The second way did not lead to work extraction, since the required coupling constants are so small that the spin effectively decouples from the bath. In contrast, the first case with $\varepsilon/(\hbar\Gamma) \sim 1$ led to a sizable work extraction, as seen in Fig. 3. Recall in this context that systems with $\varepsilon/(\hbar\Gamma) = \Omega/\Gamma \sim 1$ are well known; see Sec. IV for details.

As compared to the previous regime, here the choice of pulses has to be different for the work extraction to be possible. For example,

$$\mathcal{P}_1 = \mathcal{P}\left(-\frac{\pi}{2}; x\right), \quad \mathcal{P}_2 = \mathcal{P}\left(-\frac{\pi}{2}; y\right), \quad (99)$$

for the first and the second pulses, respectively; see Eqs. (92) and (96) for the definitions of pulses. We see from Eq. (62) that this choice amounts to substituting

$$c_{+,z}^{(1)} = i, \quad c_{z,+}^{(2)} = \frac{1}{2}, \quad c_{z,z}^{(1)} = 0, \quad c_{z,z}^{(2)} = 0, \quad (100)$$

into Eqs. (82), (84), and (85).

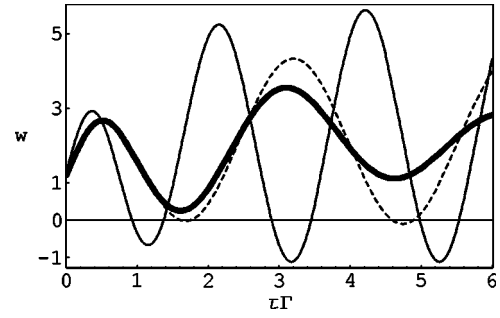


FIG. 3. Dimensionless work w [see Eq. (88)] vs dimensionless time $\tau\Gamma$ for two pulses in the regime $T < T_S$ in the case of $\gamma=0.1$, $\langle \hat{\sigma}_z \rangle = -0.01$. The choice of pulses is given by Eqs. (99) and (100). Full line: $T/\hbar\Gamma=0.1$, $\varepsilon/\hbar\Gamma=3$; dashed line: $T/\hbar\Gamma=0.1$, $\varepsilon/\hbar\Gamma=2$; bold line: $T/\hbar\Gamma=1$, $\varepsilon/\hbar\Gamma=3$.

E. Efficiency of work extraction

We shall now discuss the efficiency of work extraction as defined by Eqs. (74), (75), and (76). To calculate it, one needs to know the total work given by Eqs. (81), (83), and (85), and the contribution ΔH_S to the work W coming from the average energy of the spin t , which is given by the RHS of Eq. (83).

The efficiency as a function of $\tau\Gamma$ is presented by Figs. 4 and 5 for $T > T_S$ and $T < T_S$, respectively. There are several important things to note.

(i) For $T > T_S$, the efficiency can be very close to unity, if the temperatures T and T_S are sufficiently separated from each other, which is the case in Fig. 4. It is, however, always limited by Carnot's value, as given by Eq. (77). For $T < T_S$, the efficiency is sizable, but is rather below the corresponding Carnot value.

(ii) The work and efficiency are maximized over $\tau\Gamma$ *simultaneously*.

(iii) Recall in this context that in standard thermodynamics, efficiencies close to the optimal value are connected to very small work per unit of time (zero power of work), since they are achievable for very slow processes. This is not the case with the presented setup. As seen from Figs. 1, 2, and 3, the work is extracted on times which are of order of $1/\Gamma$

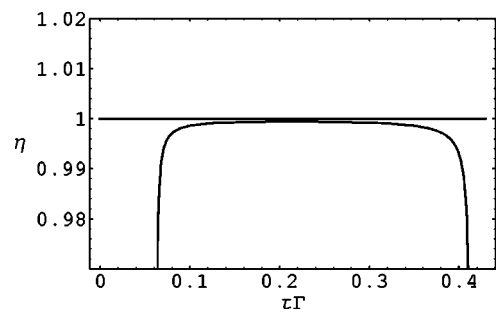


FIG. 4. Efficiency η vs dimensionless time $\tau\Gamma$ for two pulses in the regime $T > T_S$, $T/\hbar\Gamma=10$, $\gamma=1$, $\varepsilon/\hbar\Gamma=0.01$, $\langle \hat{\sigma}_z \rangle = -0.8$. The two pulses are given by Eqs. (89), (93), and (97). The efficiency is slightly below the corresponding Carnot's value and is maximized over $\tau\Gamma$ almost simultaneously with the dimensionless work w ; see Fig. 1.

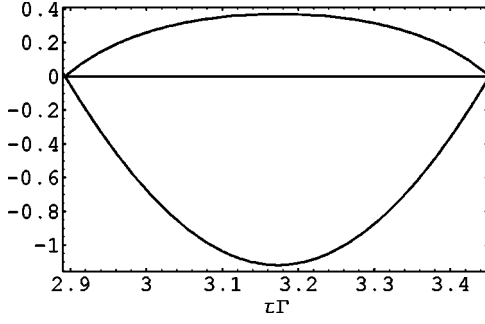


FIG. 5. Efficiency η (upper curve) and dimensionless work w (lower curve) vs dimensionless time $\tau\Gamma$ for two pulses in the regime $T < T_S$, $T/\hbar\Gamma=0.1$, $\gamma=0.1$, $\varepsilon/\hbar\Gamma=3$, $\langle\hat{\sigma}_z\rangle=-0.01$. The two pulses are given by Eqs. (99) and (100). The efficiency is below the corresponding Carnot's value 0.99 and is maximized over $\tau\Gamma$ almost simultaneously with the dimensionless work w .

(response time of the bath), which is typically much smaller than the internal characteristic time $1/\Omega$ of the spin. Thus, in Fig. 4 we have nearly optimal efficiencies together with the maximal work and a finite power of work.

VII. WORK EXTRACTION VIA SPIN-ECHO PULSES

So far we assumed that we deal either with a single spin coupled to the bath, or, equivalently, with an ensemble of identical noninteracting spins each coupled with its own bath [58]. However, many experiments—especially in NMR physics—are done on ensembles of noninteracting spins which are not in identical environments. The difference lies in the different energies ε . This can be caused by inhomogeneous fields contributing into energy ε , or by action of environment, e.g., chemical shifts for nuclear spins [37–39] or effective g factors for electronic spins in a quantum dot. It is customary to regard these energies as random quantities, so that the collective outcomes from such ensembles are obtained by averaging over $\varepsilon=\hbar\Omega$ the corresponding expressions for a single spin. We shall assume that the distribution of Ω is Gaussian with average Ω_0 and dispersion d [59],

$$P(\Omega) = \frac{1}{\sqrt{2\pi d}} e^{-(\Omega - \Omega_0)^2/(2d)}. \quad (101)$$

It is now clear that by averaging over $P(\Omega)$, the oscillating terms $e^{i\Omega\tau}$ will produce $\sim e^{-d\tau^2/2}$, that is, a strong decay on characteristic times

$$T_2^* \propto 1/\sqrt{d}. \quad (102)$$

For $\tau/T_2^* \gg 1$, all the terms containing $e^{i\Omega\tau}$ will be zero after averaging, and the corresponding averaged work for two pulses will always be positive, as seen from Eqs. (81), (83), (84), and (85). Indeed, all possible negative values of the full work W were related to transversal degrees of freedom excited by the first pulse. These terms come with the factor $e^{i\Omega\tau}$, which is connected to the free evolution in the time interval τ between the two pulses. Due to the decay of these terms after $\tau/T_2^* \gg 1$, it is impossible to extract work from this ensemble via two pulses.

However, we can extract work even in the strongly disordered situation with T_2^* being short, if we combine our work-extraction setup with the spin-echo phenomenon [31,32]. For our present purposes, this amounts to applying a π pulse, for instance in the x direction,

$$\mathcal{P}_\pi \hat{\sigma}_z = -\hat{\sigma}_z, \quad \mathcal{P}_\pi \hat{\sigma}_y = -\hat{\sigma}_y, \quad \mathcal{P}_\pi \hat{\sigma}_x = \hat{\sigma}_x, \quad (103)$$

right in the middle of two pulses \mathcal{P}_1 and \mathcal{P}_2 (to be tuned later on) applied at times t and $t+2\tau$, respectively. The work done by the first pulse reads from Eq. (81) after averaging over $P(\Omega)$ given by Eq. (101),

$$W_1 = (1 - c_{z,z}^{(1)}) \left[\frac{\hbar}{2} G - E \right], \quad (104)$$

where

$$E = -\frac{\hbar}{2} \int d\Omega P(\Omega) \Omega \tanh \frac{\beta_S \hbar \Omega}{2} < 0 \quad (105)$$

is the average initial energy of the ensemble of spins. The work done by the π pulse at time $t+\tau$ is found from Eq. (82) by substituting there the parameters $c_{z,z}^{(2)}=-1$ and $c_{z,+}^{(2)}=0$ of this pulse,

$$W_\pi = \hbar G(\tau) + \hbar g_2(\tau) c_{z,z}^{(1)} - 2E c_{z,z}^{(1)}, \quad (106)$$

where $g_2(\tau)$ is defined in Eqs. (86). It is seen that $W_\pi > 0$, because the π pulse does not couple properly with the transversal degrees of freedom excited by the first pulse. Thus, both pulses \mathcal{P}_1 and \mathcal{P}_π waste work.

Ultimately, the total work $W=W_1+W_\pi+W_2$ done by the three pulses together is derived in Appendix H to be

$$\begin{aligned} W = & \frac{\hbar G}{2} (1 + c_{z,z}^{(2)} c_{z,z}^{(1)}) + \hbar G(\tau) (2 - c_{z,z}^{(2)} - c_{z,z}^{(1)}) - \frac{\hbar G(2\tau)}{2} \\ & (1 + c_{z,z}^{(2)} c_{z,z}^{(1)} - c_{z,z}^{(1)} - c_{z,z}^{(2)}) \\ & + e^{-4\xi(\tau)+\xi(2\tau)} \text{Re}\{c_{z,z}^{(1)} c_{z,z}^{(2)} [2\hbar \dot{\xi}(\tau) - \hbar \dot{\xi}(2\tau)] \\ & \times [\sin \chi_3 + im \cos \chi_3] - \hbar g_3 [\cos \chi_3 - im \sin \chi_3]\} \end{aligned} \quad (107)$$

$$\begin{aligned} - E(1 + c_{z,z}^{(2)} c_{z,z}^{(1)}) + e^{-4\xi(\tau)+\xi(2\tau)} \text{Re}\{c_{z,z}^{(1)} c_{z,z}^{(2)} (2E \cos \chi_3 \\ - i\hbar \Omega_0 \sin \chi_3)\}, \end{aligned} \quad (109)$$

where

$$g_3(\tau) = G - G(2\tau) = \frac{\gamma\Gamma}{1 + 4\tau^2\Gamma^2}, \quad (110)$$

$$\chi_3(\tau) = 2F(\tau) - F(2\tau) = \gamma [\arctan(2\tau\Gamma) - 2 \arctan(\tau\Gamma)], \quad (111)$$

are the backreaction factors for the considered setup of pulses, and where

$$m = - \int d\Omega P(\Omega) \tanh \frac{\beta_S \hbar \Omega}{2} < 0 \quad (112)$$

is the average magnetization of the ensemble.

As compared to Eqs. (83)–(85), which present the work for two pulses, Eqs. (107)–(109) are different in several aspects.

(i) There are no oscillating factors $e^{i\Omega\tau}$ which after averaging over the distribution $P(\Omega)$ would produce damping on times T_2^* . This is due to the π pulse (103) in the middle of two pulses (spin-echo setup). A simple explanation on why the terms $\propto e^{i\Omega\tau}$ are absent is as follows. Assume that the interaction with the bath is absent and the spin moves under dynamics generated by the free Hamiltonian $\hat{H}_S = (\hbar\Omega/2)\hat{\sigma}_z$. Denote by $\mathcal{E}_t^{(0)}$ the corresponding Heisenberg evolution operator: $\mathcal{E}_t^{(0)}\hat{A} = \exp[(it/\hbar)\hat{H}_S]\hat{A}\exp[-(it/\hbar)\hat{H}_S]$. It is now seen with help of Eq. (103) that the factor $e^{i\Omega\tau}$ drops out (as if the time had been reversed),

$$\mathcal{E}_\tau^{(0)}\mathcal{P}_2\mathcal{E}_\tau^{(0)}\hat{\sigma}_+ = e^{i\Omega\tau}\mathcal{E}_\tau^{(0)}\mathcal{P}_2\hat{\sigma}_+ = e^{i\Omega\tau}\mathcal{E}_\tau^{(0)}\hat{\sigma}_- = e^{i\Omega\tau}e^{-i\Omega\tau}\hat{\sigma}_- = \hat{\sigma}_-. \quad (113)$$

(ii) The decay (decoherence) factor $e^{-4\xi(\tau)+\xi(2\tau)}$ in Eqs. (108) and (109) is different from $e^{-\xi(2\tau)}$. The last decay factor is the one generated by the free (unpulsed) evolution during the time 2τ . Only in the exponential regime $\xi(t) \simeq t/T_2$ will we have $e^{-4\xi(\tau)+\xi(2\tau)} \simeq e^{-\xi(2\tau)}$. (Recall that the exponential regime is present for the Ohmic spectrum at long times, see Sec. II C.) For Gaussian decay $\xi(t) \simeq t^2/T_2^2$, $e^{-\xi(2\tau)}$ predicts sizable decay in contrast to $e^{-4\xi(\tau)+\xi(2\tau)} \simeq 1$. This partial inhibition of decay due to π pulse(s) is known in NMR physics [37] and has been recently reinterpreted as a quantum error-correction scheme [22,34].

(iii) Now there are two independent parameters which characterize the initial state of the ensemble of spins: E and m . The work W in Eqs. (107)–(109) can be expressed in the dimensionless form similar to Eq. (88),

$$W = \frac{\hbar\gamma\Gamma}{2} w \left(\frac{T}{\hbar\Gamma}, \gamma, \frac{\Omega_0}{\Gamma}, \frac{T_S}{\hbar\Gamma}, \frac{d}{\Gamma^2}, \pi\Gamma \right). \quad (114)$$

It is now more convenient to account for the temperature of the spin via $T_S/\hbar\Gamma$, and there is a new dimensionless parameter d/Γ^2 which quantifies the ratio of the response time $1/\Gamma$ to $T_2^* = 1/\sqrt{d}$. The average magnetization m is expressed via $T_S/\hbar\Gamma$ and d/Γ^2 .

Figure 6 describes a scenario of work extraction in the regime $T_S > T$ and for pulses

$$\mathcal{P}_1 = \mathcal{P}\left(\frac{\pi}{2}; x\right), \quad \mathcal{P}_2 = \mathcal{P}\left(-\frac{\pi}{2}; y\right). \quad (115)$$

This choice of pulses amounts to substituting

$$c_{-,z}^{(1)} = i, \quad c_{z,+}^{(2)} = \frac{1}{2}, \quad c_{z,z}^{(2)} = 0, \quad c_{z,z}^{(1)} = 0 \quad (116)$$

in Eqs. (107)–(109).

Recall that in the regime $T_S > T$, there is a positive contribution to the total work coming from the bath, and sizable average frequencies $\Omega_0/\Gamma \geq 5$ are needed to overcome this contribution, as seen from Fig. 6. This restriction on the (average) frequency is similar to the one present in the two-pulse work-extraction scenario for the nondisordered ensemble of spins in the regime $T_S > T$.

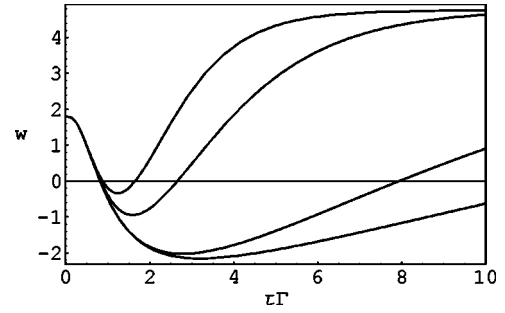


FIG. 6. Dimensionless work for three spin-echo pulses. $T/\hbar\Gamma = 10.5, 1, 0.5$ (from top to bottom); $T_S/\hbar\Gamma = 10^3$; $d/\Gamma^2 = 10^2$; $\gamma = 0.1$; $\Omega_0/\Gamma = 8$. Work extraction is poor or disappears for smaller d/Γ^2 or Ω_0/Γ , because there is too much random thermal energy in the ensemble.

It is seen from Fig. 6 that the initial high-temperature ensemble of spins is strongly disordered: $d/\Gamma^2 = 10^2 \gg 1$. This ratio cannot be much larger, since there will be too much random energy in the ensemble, that is, the positive term $-E$ in Eq. (109) will be too large and cannot be compensated by potential negative terms. Simultaneously, the average magnetization $|m|$ will be too small. For the same reasons, there are no interesting scenarios of work extraction for a strongly disordered ensemble in the regime $T_S < T$: the average magnetization $|m|$ is too small.

VIII. CONCLUSION

This paper describes several related scenarios of work extraction based on the spin-boson model: spin- $\frac{1}{2}$ interacting with external sources of work and coupled to a thermal bath of bosons. The work sources act only on the spin, since the bath is viewed as something out of any direct access. The model has two basic characteristic features. First, the transversal relaxation time T_2 is assumed to be much shorter than the longitudinal relaxation time T_1 . This condition allows the notion of local equilibrium, because once transversal components decay at time T_2 , the spin can be described via a temperature different from that of the bath. Second, the external fields are acting in the regime of short and strong pulses. This feature makes the analytical treatment feasible. Both these idealizations are well known in NMR/ESR physics and related fields, and were applied and discussed extensively in the literature [22,34,37–39,60]. It may be of interest to see in the future how precisely finite T_1 times and finite pulsing times influence the work-extraction effect.

The work is extracted from an initial local-equilibrium state of the spin at temperature T_S which is not equal to the temperature T of the equilibrium bath. As we recalled several times, Thomson's formulation of the second law prohibits work extraction via cyclic processes from an equilibrium state of the entire system: $T = T_S$ [23–28]. In this spirit, one would expect that work extraction is also absent when external fields are acting only on the spin in a local equilibrium

state [37,39]. We have shown, however, that this is not the case. It is possible to extract work in this latter setup due to the common action of the following factors: (i) backreaction of the spin to the thermal bath; (ii) generation of coherences (i.e., transversal components of the spin) during the work extraction process.

With help of the spin-echo phenomenon, it is possible to extract work from a disordered ensemble of spins having random frequencies. This ensemble can even be strongly disordered in the sense that the relaxation time \mathcal{T}_2^* induced by the disorder is much smaller than both the \mathcal{T}_2 time and the response time of the bath.

As to provide further perspectives on the obtained results, let us discuss them in two related contexts, those of lasing without inversion and quantum heat engines.

A. Comparing with lasing without inversion

As we discussed in the Introduction, besides the standard lasing effect, where work is extracted from a spin having population inversion (i.e., having a negative temperature), there are schemes of lasing which operate with a weaker form of nonequilibrium, since they employ three- or higher-level atoms which are initially in a state with nonzero coherences (i.e., nonzero off-diagonal elements of the density matrix in the energy representation). There are numerous works both theoretical and experimental, partially reviewed in [29,30], showing that in such systems one can have various scenarios of lasing without inversions in populations of atomic energy levels. In quantum optics, lasers without inversion are expected to have several advantages over those with inversion.

The effects described by us also qualify as lasing without inversion (or more precisely gain or work extraction without inversion). There are, however, several important differences as compared to the known mechanisms.

(i) We do not require coherences present in the initial state. Our mechanism operates starting from an initial local equilibrium state of the spin, which by itself is stable with respect to decoherence (i.e., to both \mathcal{T}_2 and \mathcal{T}_2^* time scales). It does employ coherences, however, but they are generated in the course of the work-extraction process, which, in particular, means that all the energy costs needed for their creation are included in the extracted work.

(ii) We do not need to have three-level systems: the effect is seen already for two-level ones.

(iii) In one of our scenarios, the extracted work comes from the bath if its temperature is higher than the initial temperature of the spin. Due to this fact, the extracted work can be much larger than the energy change of the spin. Thus the work extracted per cycle of operation can be much larger than for the standard lasing mechanism, where it is of order of the spin's energy.

(iv) The work is extracted due to an initial difference between the temperature of the spin and that of the bath. This difference can be created, e.g., by cooling or heating up the

bath on times shorter than the \mathcal{T}_1 time. Alternatively, one can cool or heat up the spin with the same restriction on the times. The latter preparation of an initially nonequilibrium state is similar to the analogous one in the standard lasing mechanism, except that no population inversion has to be created (i.e., no overcooling of the spin), and the spin's temperature can be increased or decreased.

B. Comparing with quantum heat engines

The standard thermodynamic model of a heat engine is a system (working body) operating cyclically between two thermal baths at different temperature and delivering work to an external source [1,3]. The work produced during a cycle, as well as the efficiency of the production, depend on the details of the operation. The upper bound on the efficiency is given by the Carnot expression, which is system-independent (universal). This efficiency is reached for the Carnot cycle during the very slow (slower than all the characteristic relaxation times) and therefore reversible mode of operation [1,3]. Though Carnot's cycle illustrates the best efficiency ever attainable, it is rather poor as a model for a real engine. This is explicitly caused by the very long duration of its cycle: the work produced in a unit of time is very small (zero power). This problem initiated the field of finite-time thermodynamics, which studies, in particular, how precisely the efficiency is to be sacrificed so as to reach a finite power of work [62].

In a similar spirit, a number of researchers transferred these ideas into quantum domain designing models for engines where the basic setup of the classic heat engine is retained, while the working body operating between the baths is quantum [8–10,12,13].

Our setup for work extraction can also be viewed as a model for a quantum engine. It is, however, of a nonstandard type since there is no working body operating between two different-temperature systems (in our case these are the bosonic thermal bath and the ensemble of spins). The two systems couple directly and the work source is acting on only one of them. In spite of this difference, the notion of efficiency can be defined along the standard lines, and it is equally useful as the standard one; in particular, it is always bound from above by the Carnot value. We have shown that the efficiency can approach this value at the same time as the extracted work approaches its maximum. This is a necessary condition for a large efficiency to be useful in practice. Moreover, the whole process of work extraction takes a finite time of order of the response time of the bosonic bath, which is actually much smaller than relaxation times of the spin. Thus, the three desired objectives can be achieved simultaneously: maximal work, maximal efficiency, and a large power of work.

ACKNOWLEDGMENT

We would like to thank L. G. Suttorp for fruitful discussions.

APPENDIX A: QUANTUM NOISE GENERATED BY AN OHMIC BATH

Here we discuss properties of the function,

$$K(t) = \gamma \int_0^\infty d\omega \omega \coth(\hbar\omega\beta/2) e^{-\omega\Gamma} \cos \omega t. \quad (\text{A1})$$

In the given integration domain, one can use

$$\coth(\hbar\omega\beta/2) = 1 + 2 \frac{1}{e^{\hbar\omega\beta} - 1} = 1 + 2 \sum_{n=1}^{\infty} e^{-\hbar\omega\beta n} \quad (\text{A2})$$

and get from Eq. (A1),

$$K(t) = \gamma \Gamma^2 \frac{1 - \Gamma^2 t^2}{(1 + \Gamma^2 t^2)^2} + 2\gamma \sum_{n=1}^{\infty} \frac{(\Gamma^{-1} + \hbar\beta n)^2 - t^2}{[(\Gamma^{-1} + \hbar\beta n)^2 + t^2]^2}. \quad (\text{A3})$$

With the help of a standard relation,

$$\sum_{n=1}^{\infty} \frac{1}{t^2 + y^2(n + \kappa)^2} = \frac{i}{2ty} \left[\psi\left(1 + \kappa - i\frac{t}{y}\right) - \psi\left(1 + \kappa + i\frac{t}{y}\right) \right], \quad (\text{A4})$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ with Γ being the Euler Gamma function,

$$\sum_{n=1}^{\infty} \frac{(\kappa + n)^2 - t^2}{[(\kappa + n)^2 + t^2]^2} = \frac{1}{2} [\psi'(1 + \kappa - it) + \psi'(1 + \kappa + it)]. \quad (\text{A5})$$

Combining Eq. (A5) with Eq. (A3) and $\kappa = 1/(\hbar\beta\Gamma)$, one ends up with the following formula:

$$K(t) = \gamma \Gamma^2 \frac{1 - \Gamma^2 t^2}{(1 + \Gamma^2 t^2)^2} + \frac{\gamma \Gamma^2}{\hbar^2} \left[\psi'\left(1 + \frac{1}{\hbar\Gamma\beta} - i\frac{t}{\hbar\beta}\right) + \psi'\left(1 + \frac{1}{\hbar\Gamma\beta} + i\frac{t}{\hbar\beta}\right) \right]. \quad (\text{A6})$$

Let us now consider separately the cases of low and high temperatures. For $\hbar\Gamma\beta \gg 1$, one uses the known relation

$$\Gamma\left(1 - i\frac{t}{\hbar\beta}\right) \Gamma\left(1 + i\frac{t}{\hbar\beta}\right) = \frac{\pi t}{\hbar\beta \sinh[\pi t/(\hbar\beta)]}, \quad (\text{A7})$$

and obtains from Eq. (A6),

$$K(t) = \gamma \Gamma^2 \frac{1 - \Gamma^2 t^2}{(1 + \Gamma^2 t^2)^2} + \frac{\gamma}{t^2} - \frac{\gamma T^2 \pi^2}{\hbar^2} \frac{1}{\sinh^2[\pi t/(\hbar\beta)]}. \quad (\text{A8})$$

For small t ($t \ll 1/\Gamma$), $K(t)$ is positive, as it should be,

$$K(t) = \gamma \Gamma^2 + \frac{\gamma T^2 \pi^2}{3\hbar^2}. \quad (\text{A9})$$

In contrast, for $t \sim \hbar\beta \gg 1/\Gamma$ it becomes negative, namely, the noise is anticorrelated,

$$K(t) = 3\gamma \frac{1}{\Gamma^2 t^4} - \frac{\gamma T^2 \pi^2}{\hbar^2} \frac{1}{\sinh^2[\pi t/(\hbar\beta)]}. \quad (\text{A10})$$

At the end it is again correlated in the limit of very large times $t \gg \hbar\beta$ where the first term on the RHS of Eq. (A10) dominates (this domain is shrunk for low temperatures).

In the high-temperature limit $\hbar\beta\Gamma \ll 1$, one can use in Eq. (A6) the Stirling formula,

$$\psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + \dots, \quad z \gg 1 \quad (\text{A11})$$

and then the quasiclassical limit for the quantum noise reads (after some more simplifications)

$$K(t) = \gamma \Gamma^2 \frac{1 - \Gamma^2 t^2}{(1 + \Gamma^2 t^2)^2} + \frac{2\gamma T \Gamma}{\hbar} \frac{1}{1 + t^2 \Gamma^2}. \quad (\text{A12})$$

In the purely classical limit, the first term on the RHS can be neglected and we return (for $t\Gamma \gg 1$) to the classical white noise with the strength $2\pi\gamma T$.

Finally, in the context of Eq. (A6) we notice the following useful relations:

$$\begin{aligned} \dot{\xi}(t) &= \int_0^t dt' K(t') \\ &= \gamma \Gamma \left\{ \frac{t\Gamma}{1 + \Gamma^2 t^2} + \frac{iT}{\hbar\Gamma} \left[\psi\left(1 + \frac{1}{\hbar\Gamma\beta} - i\frac{t}{\hbar\beta}\right) - \psi\left(1 + \frac{1}{\hbar\Gamma\beta} + i\frac{t}{\hbar\beta}\right) \right] \right\}, \end{aligned} \quad (\text{A13})$$

$$\begin{aligned} \xi(t) &= \int_0^t dt' \int_0^{t'} dt'' K(t'') \\ &= \gamma \ln \left[\frac{\Gamma^2 \left(1 + \frac{1}{\hbar\Gamma\beta}\right) \sqrt{1 + t^2 \Gamma^2}}{\Gamma\left(1 + \frac{1}{\hbar\Gamma\beta} - i\frac{t}{\hbar\beta}\right) \Gamma\left(1 + \frac{1}{\hbar\Gamma\beta} + i\frac{t}{\hbar\beta}\right)} \right], \end{aligned} \quad (\text{A14})$$

which are used in the main text.

APPENDIX B

Here we shortly outline how the two-temperature state (40) can be prepared starting from the overall equilibrium state,

$$\begin{aligned} \rho(0) &= \frac{1}{Z} \exp[-\beta_S \hat{H}_S - \beta_B (\hat{H}_1 + \hat{H}_B)], \\ Z &= \text{tr} e^{-\beta_S \hat{H}_S - \beta_B (\hat{H}_1 + \hat{H}_B)}, \end{aligned} \quad (\text{B1})$$

which has equal temperatures of the spin and the bath.

Assume that the bath was subjected to another much larger thermal bath (superbath) at temperature T different from T_S , so that the total Hamiltonian of the spin, bath, and superbath reads

$$\hat{H}_{\text{total}} = \hat{H} + \hat{H}_{\text{sup}}, \quad (\text{B2})$$

where the latter operator \hat{H}_{sup} characterizes the *weak* interaction of the bath with the superbath and contains also the self-Hamiltonian of the superbath. Thus

$$[\hat{H}_S, \hat{H}_{\text{sup}}] = 0. \quad (\text{B3})$$

Now the statement of this appendix is that under the action of the superbath at temperature T , the common state of the spin and the bath will relax to the state Eq. (B11) or Eq. (40) with different temperatures for the spin and the bath. The reason is that due to Eq. (B3), $\hat{\sigma}_z$ is conserved during the whole evolution generated by the superbath, so that $\hat{\sigma}_z$ does not relax and keeps its value given by Eq. (B1) [63]. In contrast, the variables of the bath—including \hat{X} —do not have such a protection, so they relax under the influence of the superbath. Let us now substantiate this statement.

Because $[\hat{H}_S, \hat{H}] = 0$, the initial equilibrium state $\rho(0)$ of the spin and the bath can be represented as

$$\rho(0) = \sum_{j=\pm 1} p_{jj} \rho_{jj}(0) |j\rangle\langle j|, \quad (\text{B4})$$

where

$$p_{jj} = \frac{e^{-j\beta_S \varepsilon/2}}{2 \cosh(\beta_S \varepsilon/2)}, \quad j = \pm 1 \quad (\text{B5})$$

are probabilities for the spin to be up or down, respectively, $|j\rangle$ is the eigenstate of $\hat{H}_S = (\varepsilon/2)\hat{\sigma}_z$ with eigenvalue $j = \pm 1$, and where

$$\rho_{jj}(0) = \frac{1}{Z_j} \exp\left[-\beta_S \left(\frac{j}{2}\hat{X} + \hat{H}_B\right)\right],$$

$$Z_j = \text{tr}_B e^{-\beta_S(j/2\hat{X} + \hat{H}_B)}, \quad j = \pm 1 \quad (\text{B6})$$

are conditional states of the bath.

The total initial state of the spin, bath, and superbath thus reads

$$\rho_{\text{total}}(0) = \sum_{j=\pm 1} p_{jj} \rho_{\text{sup}}(0) \otimes \rho_{jj}(0) |j\rangle\langle j|, \quad (\text{B7})$$

where $\rho_{\text{sup}}(0)$ is the initial equilibrium state of the superbath. Note that due to weak coupling between the bath and superbath, their initial states can be assumed to be factorized.

As follows from Eqs. (B3) and (B4), the time-dependent state of the total system consisting of spin, bath, and superbath can be presented as

$$\rho_{\text{total}}(t) = \sum_{j=\pm 1} p_{jj} \Omega_{jj}(t) |j\rangle\langle j|, \quad (\text{B8})$$

where $\Omega_{jj}(t)$ —the conditional joint state of the bath and superbath—satisfies the von Neumann equation

$$i\hbar \dot{\Omega}_{jj} = \left[\frac{j}{2}\hat{X} + \hat{H}_B + \hat{H}_{\text{sup}}, \Omega_{jj} \right] \quad (\text{B9})$$

with the initial condition

$$\Omega_{jj}(0) = \rho_{\text{sup}}(0) \otimes \rho_{jj}(0). \quad (\text{B10})$$

Thus, Ω_{jj} moves according to the Hamiltonian $(j/2)\hat{X} + \hat{H}_B + \hat{H}_{\text{sup}}$. It is now clear that in the weak-coupling limit of the bath-superbath interaction, the marginal conditional state $\text{tr}_{\text{sup}} \Omega_{jj}(t)$ will—for sufficiently long times t —relax to Gibbs distribution at temperature T (equal to that of the superbath) and with Hamiltonian $(j/2)\hat{X} + \hat{H}_B$. Thus the (unconditional) marginal state of the spin and the bath will indeed relax to

$$\rho \propto \exp[-\beta_S \hat{H}_S - \beta(\hat{H}_I + \hat{H}_B)]. \quad (\text{B11})$$

APPENDIX C

Here we explain in detail why the initial conditions (17), (41), and (40) are equivalent.

One can write the full Hamiltonian \hat{H} defined in Eq. (1) as

$$\hat{H} = \sum_k \hbar \omega_k \left(\hat{a}_k^\dagger + \frac{g_k \hat{\sigma}_z}{2\omega_k} \right) \left(\hat{a}_k + \frac{g_k \hat{\sigma}_z}{2\omega_k} \right) + \frac{\varepsilon}{2} \hat{\sigma}_z - \sum_k \frac{\hbar g_k^2}{4\omega_k}, \quad (\text{C1})$$

and diagonalize it via a unitary operator,

$$\hat{U} = \exp\left[\sum_k \frac{g_k \hat{\sigma}_z}{2\omega_k} (\hat{a}_k^\dagger - \hat{a}_k) \right],$$

$$\hat{U} \hat{a}_k \hat{U}^\dagger = \hat{a}_k - \frac{g_k \hat{\sigma}_z}{2\omega_k}, \quad \hat{U} \hat{\sigma}_z \hat{U}^\dagger = \hat{\sigma}_z. \quad (\text{C2})$$

Let us define the operators

$$\hat{b}_k = \hat{a}_k + \frac{g_k \hat{\sigma}_z}{2\omega_k}, \quad [\hat{b}_k, \hat{b}_l^\dagger] = \delta_{kl}. \quad (\text{C3})$$

As follows from Eqs. (C1) and (40), various operator functions

$$f(\hat{b}_k, \hat{b}_l^\dagger, \dots) \hat{s}, \quad \hat{s} = \hat{1}, \hat{\sigma}_z, \hat{\sigma}_+, \hat{\sigma}_-, \quad (\text{C4})$$

where f is an arbitrary function of arbitrary combinations of \hat{b}_k 's and \hat{b}_l^\dagger 's, has on the correlated state (40) exactly the same statistics (e.g., the same average) as the corresponding operator function $f(\hat{a}_k, \hat{a}_l^\dagger, \dots) \hat{s}$ on the factorized state (17) and (41). One should notice here that Eq. (C4) represents every function of the spin and the bath operators since the Pauli matrix always forms linear functions due to the fact that $\sigma^2 = 1$. Indeed, if \hat{s} is equal to $\hat{1}$ or to $\hat{\sigma}_z$, one has using Eqs. (C1) and (C2)

$$\begin{aligned} \frac{1}{\text{tr} e^{-\beta\hat{H}}} \text{tr}[f(\hat{b}_k, \hat{b}_l^\dagger, \dots) \hat{s} e^{-\beta\hat{H}}] &= \frac{1}{\text{tr} e^{-\beta\hat{H}}} \text{tr}[\hat{U} f(\hat{b}_k, \hat{b}_l^\dagger, \dots) \hat{s} e^{-\beta\hat{H}} \hat{U}^\dagger] = \frac{1}{\text{tr} e^{-\beta\hat{H}}} \text{tr}[f(\hat{U}\hat{b}_k\hat{U}^\dagger, \hat{U}\hat{b}_l^\dagger\hat{U}^\dagger, \dots) \hat{U}\hat{s}\hat{U}^\dagger e^{-\beta\hat{H}}] \\ &= \frac{\text{tr}[f(\hat{a}_k, \hat{a}_l^\dagger, \dots) \hat{s} e^{-\beta\sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k} e^{-\beta(\epsilon/2)\hat{\sigma}_z}]}{\text{tr} e^{-\beta\sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k} \text{tr} e^{-\beta(\epsilon/2)\hat{\sigma}_z}} = \langle f(\hat{a}_k, \hat{a}_l^\dagger, \dots) \rangle \langle \hat{s} \rangle, \end{aligned} \quad (\text{C5})$$

where the averages $\langle \dots \rangle$ are taken over the factorized state (17) and (41).

Likewise, for $\hat{s} = \hat{\sigma}_\pm$,

$$\frac{1}{\text{tr} e^{-\beta\hat{H}}} \text{tr}[f(\hat{b}_k, \hat{b}_l^\dagger, \dots) \hat{\sigma}_\pm e^{-\beta\hat{H}}] = \frac{1}{\text{tr} e^{-\beta\hat{H}}} \text{tr}[f(\hat{U}\hat{b}_k\hat{U}^\dagger, \hat{U}\hat{b}_l^\dagger\hat{U}^\dagger, \dots) \hat{U}\hat{\sigma}_\pm\hat{U}^\dagger e^{-\beta\hat{H}}] \quad (\text{C6})$$

$$= \frac{\text{tr}[f(\hat{a}_k, \hat{a}_l^\dagger, \dots) e^{\pm\sum_k (g_k/2\omega_k)(\hat{a}_k^\dagger - \hat{a}_k)} \hat{\sigma}_\pm e^{-\beta\sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k} e^{-\beta(\epsilon/2)\hat{\sigma}_z}]}{\text{tr} e^{-\beta\sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k} \text{tr} e^{-\beta(\epsilon/2)\hat{\sigma}_z}} \quad (\text{C7})$$

$$= \langle f(\hat{a}_k, \hat{a}_l^\dagger, \dots) e^{\pm\sum_k (g_k/2\omega_k)(\hat{a}_k^\dagger - \hat{a}_k)} \rangle \langle \hat{\sigma}_\pm \rangle \quad (\text{C8})$$

$$= 0, \quad (\text{C9})$$

where when going from Eq. (C6) to Eq. (C7) we used Eq. (C2), and where Eq. (C9) follows from $\langle \hat{\sigma}_\pm \rangle = 0$.

Now note that for the initial condition (40), $\hat{\sigma}_z(0)$ and the quantum noise operator $\hat{\eta}(t)$ are in general not independent variables, in contrast to the case of the factorized initial condition (17) and (41). However, for $t \gg 1/\Gamma$ they do become independent,

$$\hat{\eta}(t) = \hat{\eta}_b(t) + \hat{\sigma}_z[G(t) - G],$$

$$\hat{\eta}_b(t) \equiv \sum_k g_k [\hat{b}_k^\dagger(0) e^{i\omega_k t} + \hat{b}_k(0) e^{-i\omega_k t}], \quad (\text{C10})$$

where

$$G \equiv \sum_k \frac{g_k^2}{\omega_k} \quad (\text{C11})$$

is the limit of $G(t)$ for $t \gg 1/\Gamma$. Taking the latter limit in Eq. (C10), one gets that $\hat{\eta}(t)$ is equal to $\hat{\eta}_b(t)$ and is thus independent of $\hat{\sigma}_z$. Recalling that $\hat{\eta}_b(t)$ has on the state (40) the same statistics as $\hat{\eta}(t)$ on the factorized state (17) finishes the argument: the equivalence holds for times larger than the bath response time ($1/\Gamma$ for the Ohmic situation).

Note that the thermodynamic limit for the bath is essential for this conclusion. Otherwise, $G(t)$ would be a finite sum of cosines, and would not converge to G .

APPENDIX D: DERIVATION OF EQS. (65) AND (66)

Assume that the initial state of the spin and bath is

$$\rho(0) = \frac{1}{Z} \exp[-\beta_S \hat{H}_S - \beta(\hat{H}_I + \hat{H}_B)],$$

$$Z = \text{tr} e^{-\beta_S \hat{H}_S - \beta(\hat{H}_I + \hat{H}_B)}, \quad (\text{D1})$$

with different temperatures for the spin and the bath.

An external field $\hat{V}(t)$ is acting on the system,

$$\hat{H}(t) = \hat{H} + \hat{V}(t) \quad (\text{D2})$$

such that it is zero both initially and at the moment $t = \tau$,

$$\hat{V}(\tau) = \hat{V}(0) = 0. \quad (\text{D3})$$

This condition defines *cyclic process*. The total work which was done on this system reads

$$W = \Delta H_S + \Delta H_I + \Delta H_B, \quad (\text{D4})$$

where

$$\Delta H_k = \text{tr}\{\hat{H}_k[\rho(\tau) - \rho(0)]\}, \quad k = S, I, B \quad (\text{D5})$$

are the changes of the corresponding energies, and where $\rho(\tau)$ is the overall density matrix at time τ .

Recall that the relative entropy (see, e.g., [19])

$$S[\rho \parallel \sigma] \equiv \text{tr}(\rho \ln \rho - \rho \ln \sigma) \geq 0 \quad (\text{D6})$$

is non-negative for any density matrices ρ and σ . One now uses

$$\begin{aligned} S[\rho(\tau) \parallel \rho(0)] &= \text{tr}[\rho(\tau) \ln \rho(\tau) - \rho(\tau) \ln \rho(0)] \\ &= \text{tr}[\rho(0) \ln \rho(0) - \rho(\tau) \ln \rho(0)] \\ &= \beta_S \Delta H_S + \beta(\Delta H_I + \Delta H_B) \geq 0, \end{aligned} \quad (\text{D7})$$

where we used Eq. (D1) and $\text{tr}(\rho(\tau) \ln \rho(\tau)) = \text{tr}(\rho(0) \ln \rho(0))$ is due to the unitarity of the overall dynamics generated by the time-dependent Hamiltonian $\hat{H}(t)$.

Combining Eq. (D7) with Eq. (D4), one gets Eqs. (65) and (66),

$$W \geq \left(1 - \frac{T}{T_S}\right) \Delta H_S, \quad W \geq \left(1 - \frac{T_S}{T}\right) (\Delta H_I + \Delta H_B). \quad (\text{D8})$$

Finally, note that if we used the initial conditions

$$\rho(0) = \rho_S(0) \otimes \rho_B(0) = \frac{1}{\text{tr} e^{-\beta \hat{H}_S}} e^{-\beta \hat{H}_S} \otimes \frac{1}{\text{tr} e^{-\beta \hat{H}_B}} e^{-\beta \hat{H}_B} \quad (\text{D9})$$

we would not be able to conclude from the above derivation that the efficiency is limited by the Carnot value. Indeed, instead of Eqs. (D8) one has, respectively,

$$\beta_S \Delta H_S + \beta \Delta H_B \geq 0, \quad W \geq \Delta H_I + \left(1 - \frac{T}{T_S}\right) \Delta H_S. \quad (\text{D10})$$

The latter inequality is not informative with respect to Carnot's bound, since it cannot and should not in general be excluded that ΔH_I is sizeable.

However, for the model studied in the present paper, the initial conditions (D1) and (D9) are equivalent for sufficient long times as shown in Appendix C.

Let us emphasize the main points by which the present derivation differs from the standard textbook one.

(i) No postulates were used: the whole derivation is based on the quantum-mechanical equations of motion and certain assumptions on the initial conditions.

(ii) It was not assumed that the interaction between the system and the bath is small, a restrictive assumption which need not be satisfied in reality.

(iii) The fact of using the initial conditions in Eq. (D1) is important in the present derivation, though presumably Carnot's bound is valid in certain more general cases, such as, in our case, factorized initial conditions from Eq. (D9).

APPENDIX E: SOME CORRELATION FUNCTIONS

In this appendix and in the following ones, we study various dynamical aspects of the model defined by Eqs. (1)–(3) and (5). The initial conditions are given by Eqs. (17) and (41). Hereafter, $\langle \cdots \rangle$ means averaging over this initial condition. Let us define some correlation functions.

(a) For

$$t_3 \geq t_2 \geq t_1, \quad (\text{E1})$$

and recalling definitions (11) and (15), one derives using Wick's theorem in the same way as when deriving Eq. (25),

$$\begin{aligned} \langle \hat{\eta}(t_3) \hat{\Pi}_{\pm}(t_1, t_2) \rangle &= \pm \sum_{k=0}^{\infty} \frac{i(-1)^k}{(2k+1)!} \int_{t_1}^{t_2} \cdots \int_{t_1}^{t_2} ds_1 \cdots ds_{2k+1} \\ &\quad \times \langle \mathcal{T}[\hat{\eta}(t_3) \hat{\eta}(s_1) \cdots \hat{\eta}(s_{2k+1})] \rangle \\ &= \pm i \int_{t_1}^{t_2} ds K_{\mathcal{T}}(t_3 - s) \exp \left[-\frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} \right. \\ &\quad \left. \times ds_1 ds_2 K_{\mathcal{T}}(s_1 - s_2) \right] \quad (\text{E2}) \end{aligned}$$

$$\begin{aligned} &= \pm [i\dot{\xi}(t_3 - t_1) - i\dot{\xi}(t_3 - t_2) + G(t_3 - t_1) \\ &\quad - G(t_3 - t_2)] e^{-\xi(t_2 - t_1) + iF(t_2 - t_1)}, \quad (\text{E3}) \end{aligned}$$

where for deriving the last line we used the definition of $K_{\mathcal{T}}(t)$,

$$K_{\mathcal{T}}(t) = K(t) - i\dot{G}(t) = \ddot{\xi}(t) - i\ddot{F}(t). \quad (\text{E4})$$

Note that for $t_3 = t_2$, we can derive Eq. (E3) in a simpler way by employing Eq. (25) and

$$\langle \hat{\eta}(t_2) \hat{\Pi}_{\pm}(t_1, t_2) \rangle = \mp i \partial_{t_2} \langle \hat{\Pi}_{\pm}(t_1, t_2) \rangle. \quad (\text{E5})$$

(b) A correlation function $\langle \hat{\Pi}_{\pm}(t_1, t_2) \hat{\eta}(t_3) \rangle$ under the same condition (E1) is studied similarly to Eq. (E3), the only difference being that the time-ordered correlation function $K_{\mathcal{T}}(t_3 - s)$ in Eq. (E2) is substituted by the analogous time-antiorordered one [time-antiorordering comes due to Eq. (E1)],

$$K_{\mathcal{A}}(t_3 - s) = K_{\mathcal{T}}^*(t_3 - s). \quad (\text{E6})$$

These two functions are related by complex conjugation, as seen from Eq. (21). Thus,

$$\begin{aligned} \langle \hat{\Pi}_{\pm}(t_1, t_2) \hat{\eta}(t_3) \rangle &= \pm [i\dot{\xi}(t_3 - t_1) - i\dot{\xi}(t_3 - t_2) - G(t_3 - t_1) \\ &\quad + G(t_3 - t_2)] e^{-\xi(t_2 - t_1) + iF(t_2 - t_1)}. \quad (\text{E7}) \end{aligned}$$

As compared to Eq. (E3), the sign of G factors is seen to change.

(c) A correlation function between two $\hat{\Pi}$ factors for

$$t_4 \geq t_3 \geq t_2 \geq t_1 \quad (\text{E8})$$

is worked out as follows. First one notes

$$\langle \hat{\Pi}_{\pm}(t_3, t_4) \hat{\Pi}_{\mp}(t_1, t_2) \rangle = \left\langle \mathcal{T} \exp \left[\pm i \int_{t_1}^{t_4} ds \phi(s) \hat{\eta}(s) \right] \right\rangle, \quad (\text{E9})$$

where

$$\begin{aligned} \phi(s) &= -1, \quad t_1 \leq s \leq t_2, \\ &= 0, \quad t_2 \leq s \leq t_3, \\ &= 1, \quad t_3 \leq s \leq t_4. \quad (\text{E10}) \end{aligned}$$

One gets

$$\begin{aligned}
 \langle \hat{\Pi}_{\pm}(t_3, t_4) \hat{\Pi}_{\mp}(t_1, t_2) \rangle &= \exp \left[-\frac{1}{2} \int_{t_1}^{t_2} \int_{t_1}^{t_2} ds_1 ds_2 K_{\mathcal{T}}(s_1 - s_2) - \frac{1}{2} \int_{t_3}^{t_4} \int_{t_3}^{t_4} ds_1 ds_2 K_{\mathcal{T}}(s_1 - s_2) + \int_{t_1}^{t_2} \int_{t_3}^{t_4} ds_1 ds_2 K_{\mathcal{T}}(s_1 - s_2) \right] \\
 &= \exp[-\xi(t_2 - t_1) - \xi(t_4 - t_3) - \xi(t_4 - t_2) + \xi(t_4 - t_1) + \xi(t_3 - t_2) - \xi(t_3 - t_1)] \\
 &\quad \times \exp[iF(t_2 - t_1) + iF(t_4 - t_3) + iF(t_4 - t_2) - iF(t_4 - t_1) - iF(t_3 - t_2) + iF(t_3 - t_1)]. \quad (\text{E11})
 \end{aligned}$$

APPENDIX F: EVOLUTION OF THE QUANTUM NOISE AND $\hat{\Pi}$ FACTORS UNDER HEISENBERG DYNAMICS

Note from Eqs. (8) and (11) how the quantum noise and $\hat{\Pi}$ factors evolve under Heisenberg dynamics,

$$\mathcal{E}_t \hat{\eta}(\tau) \equiv e^{i\hat{H}t/\hbar} \hat{\eta}(\tau) e^{-i\hat{H}t/\hbar} = \hat{\eta}(t + \tau) + \hat{\sigma}_z [G(\tau) - G(t + \tau)], \quad (\text{F1})$$

$$\begin{aligned}
 \mathcal{E}_t \hat{\Pi}_{\pm}(t_1, t_2) &\equiv e^{i\hat{H}t/\hbar} \hat{\Pi}_{\pm}(t_1, t_2) e^{-i\hat{H}t/\hbar} \\
 &= \mathcal{T} \exp \left[\pm i \int_{t_1}^{t_2} ds \sum_k g_k [\hat{a}_k(t) e^{-i\omega_k s} + \hat{a}_k^\dagger(t) e^{i\omega_k s}] \right] \\
 &= \hat{\Pi}_{\pm}(t + t_1, t + t_2) \exp[\pm i \hat{\sigma}_z [F(t_2) - F(t_1) \\
 &\quad + F(t + t_1) - F(t + t_2)]]. \quad (\text{F2})
 \end{aligned}$$

When deriving these equations, we used $[\hat{\eta}(\tau + t), \hat{\sigma}_z] = 0$. Recall that $\hat{\sigma}_z$ is conserved under evolution generated by \hat{H} : $\mathcal{E}_t \hat{\sigma}_z = \hat{\sigma}_z$.

APPENDIX G: DERIVATIONS FOR TWO PULSES

The work done by the second pulse is defined as

$$\begin{aligned}
 \frac{1}{\hbar} W_2 &= \frac{1}{\hbar} \langle \mathcal{P}_2 (\hat{H}_S + \hat{H}_1) - (\hat{H}_S + \hat{H}_1) \rangle_{t+\tau} \\
 &= \frac{\Omega}{2} \langle \mathcal{P}_2 \hat{\sigma}_z - \hat{\sigma}_z \rangle_{t+\tau} + \frac{1}{2} \langle (\mathcal{P}_2 \hat{\sigma}_z - \hat{\sigma}_z) \hat{X} \rangle_{t+\tau} \quad (\text{G1}) \\
 &= \frac{\Omega}{2} (c_{z,z}^{(2)} - 1) \langle \hat{\sigma}_z \rangle_{t+\tau} + \frac{1}{2} (c_{z,z}^{(2)} - 1) \langle \hat{\sigma}_z \hat{X} \rangle_{t+\tau} + \Omega \text{Re}\{c_{z,+}^{(2)} \langle \hat{\sigma}_+ \rangle_{t+\tau}\} \\
 &\quad + \text{Re}\{c_{z,+}^{(2)} \langle \hat{\sigma}_+ \hat{X} \rangle_{t+\tau}\}, \quad (\text{G2})
 \end{aligned}$$

where the averages are taken at the time $t + \tau$ immediately before the second pulse, and where we used the definition (61) of the parametrization coefficients. For clarity, we recall that definition here,

$$\mathcal{P}_k \hat{\sigma}_a = \sum_{b=\pm,z} c_{a,b}^{(k)} \hat{\sigma}_b, \quad a = \pm, z, \quad k = 1, 2. \quad (\text{G3})$$

In order to calculate W_2 , we thus have to determine $\hat{X}(t + \tau)$, $\hat{\sigma}_+(t + \tau)$, and $\hat{\sigma}_z(t + \tau)$. Recall from Eq. (54) that, e.g.,

$$\hat{X}(t + \tau) = \mathcal{E}_t \mathcal{P}_1 \mathcal{E}_\tau \hat{X}, \quad (\text{G4})$$

where \mathcal{E}_t is the free evolution (super)operator defined in Eq. (53). One infers from Eqs. (8) and (9) that

$$\mathcal{E}_\tau \hat{X} = \hat{\eta}(\tau) - \hat{\sigma}_z G(\tau), \quad (\text{G5})$$

$$\mathcal{E}_t \hat{\eta}(\tau) = \hat{\eta}(t + \tau) - \hat{\sigma}_z [G(\tau) - G(t + \tau)], \quad (\text{G6})$$

and then

$$\begin{aligned}
 \hat{X}(t + \tau) &\equiv \mathcal{E}_t \mathcal{P}_1 \mathcal{E}_\tau \hat{X} \\
 &= \hat{\eta}(t + \tau) + [G(\tau) - G(t + \tau)] \hat{\sigma}_z - G(\tau) \mathcal{E}_t \mathcal{P}_1 \hat{\sigma}_z \quad (\text{G7})
 \end{aligned}$$

The formula for $\hat{\sigma}_z(t + \tau)$ is more straightforward,

$$\hat{\sigma}_z(t + \tau) = \mathcal{E}_t \mathcal{P}_1 \hat{\sigma}_z, \quad (\text{G8})$$

$$\langle \hat{\sigma}_z(t + \tau) \rangle = c_{z,z}^{(1)} \langle \hat{\sigma}_z \rangle, \quad (\text{G9})$$

where we noted that in

$$\langle \mathcal{E}_t \mathcal{P}_1 \hat{\sigma}_k \rangle = \sum_{n=\pm,z} c_{k,n}^{(1)} \langle \mathcal{E}_t \hat{\sigma}_n \rangle = c_{k,z}^{(1)} \langle \hat{\sigma}_z \rangle, \quad k = \pm, z, \quad (\text{G10})$$

only one term contributes, since $\langle \hat{\sigma}_{\pm} \rangle = \langle e^{i\chi \hat{\sigma}_z} \hat{\sigma}_{\pm} \rangle = 0$ due to the initial conditions (17) and (41).

In the same way, one calculates

$$\langle \hat{\sigma}_z \hat{X} \rangle_{t+\tau} = [G(\tau) - G(t + \tau)] c_{z,z}^{(1)} - G(\tau), \quad (\text{G11})$$

$$\begin{aligned}
 \hat{\sigma}_+(t + \tau) &\equiv \mathcal{E}_t \mathcal{P}_1 \mathcal{E}_\tau \hat{\sigma}_+ = e^{-iF(\tau) + i\Omega \tau} \mathcal{E}_t \mathcal{P}_1 \hat{\Pi}_+(0, \tau) \hat{\sigma}_+ \\
 &= e^{-iF(\tau) + i\Omega \tau + i\chi \hat{\sigma}_z} \hat{\Pi}_+(t, t + \tau) \mathcal{E}_t \mathcal{P}_1 \hat{\sigma}_+, \quad (\text{G12})
 \end{aligned}$$

where we used Eq. (F2), $[\hat{\sigma}_z, \hat{\Pi}_{\pm}] = 0$, and where by definition [from Eq. (F2)]

$$\chi(\tau, t) = \int_0^\tau ds [G(s) - G(t + s)] = F(t) + F(\tau) - F(t + \tau). \quad (\text{G13})$$

Now let us recall Eq. (25),

$$\langle \hat{\Pi}_{\pm}(t_1, t_2) \rangle = \exp[-\xi(t_2 - t_1) + iF(t_2 - t_1)], \quad (\text{G14})$$

because it is used in averaging the RHS of Eq. (G12),

$$\langle \hat{\sigma}_+(t + \tau) \rangle = c_{+,z}^{(1)} e^{i\Omega \tau - \xi(\tau)} \langle e^{i\chi \hat{\sigma}_z} \hat{\sigma}_z \rangle, \quad (\text{G15})$$

where we additionally employed the reasoning which led us to Eq. (G10),

The last term we have to calculate is $\langle \hat{\sigma}_+ \hat{X} \rangle_{t+\tau}$. Directly multiplying Eqs. (G7) and (G12), one gets

$$\langle \hat{\sigma}_+ \hat{X} \rangle_{t+\tau} = e^{-iF(\tau)+i\Omega\tau+i\chi\hat{\sigma}_z} \hat{\Pi}_+(t, t+\tau) \mathcal{E}_i \mathcal{P}_1 \hat{\sigma}_+ \hat{\eta}(t+\tau) \quad (\text{G16})$$

$$+ [G(\tau) - G(t+\tau)] e^{-iF(\tau)+i\Omega\tau+i\chi\hat{\sigma}_z} \hat{\Pi}_+(t, t+\tau) \mathcal{E}_i \{ (\mathcal{P}_1 \hat{\sigma}_+) \hat{\sigma}_z \} \quad (\text{G17})$$

$$- G(\tau) e^{-iF(\tau)+i\Omega\tau+i\chi\hat{\sigma}_z} \hat{\Pi}_+(t, t+\tau) \mathcal{E}_i \{ \mathcal{P}_1 \hat{\sigma}_+ \hat{\sigma}_z \}. \quad (\text{G18})$$

Following Eq. (G3), we now expand $\mathcal{P}_1 \hat{\sigma}_+$ in Eqs. (G17) and (G18). With the same reasoning as for Eq. (G10), we need to keep in these expansions only terms proportional to $c_{+,z}^{(1)}$ (since $\langle \hat{\sigma}_\pm \rangle = 0$ according to the initial conditions (17) and (41)]. After further simplifications with help of Eq. (E7) we obtain

$$\langle \hat{\sigma}_+ \hat{X} \rangle_{t+\tau} = c_{+,z}^{(1)} e^{i\Omega\tau - \xi(\tau)} \{ i \dot{\xi}(\tau) \langle e^{i\chi\hat{\sigma}_z} \hat{\sigma}_z \rangle + [G(\tau) - G(t+\tau)] \langle e^{i\chi\hat{\sigma}_z} \rangle \}. \quad (\text{G19})$$

The final formula for the work reads

$$\begin{aligned} \frac{1}{\hbar} W_2 &= \frac{1}{2} (1 - c_{z,z}^{(1)}) (1 - c_{z,z}^{(2)}) G(\tau) + \frac{1}{2} (1 - c_{z,z}^{(2)}) c_{z,z}^{(1)} \\ &\times [\Omega \langle \hat{\sigma}_z \rangle - G(t+\tau)] + \Omega \operatorname{Re} \{ c_{z,+}^{(2)} \langle \hat{\sigma}_+ \rangle_{t+\tau} \} \\ &+ \operatorname{Re} \{ c_{z,+}^{(2)} \langle \hat{\sigma}_+ \hat{X} \rangle_{t+\tau} \}. \end{aligned} \quad (\text{G20})$$

Note that in the limit $t \rightarrow \infty$ (which means $t \gg 1/\Gamma$), one has $G(t+\tau) \rightarrow G$, where G is defined in Eq. (C11), and $\chi \rightarrow \chi_2(\tau)$ defined in Eq. (87).

Equation (G20) can be put into dimensionless form as announced by Eq. (88). To this end note from Eqs. (A13) and (A14) that $\xi(\tau)$ and $(1/\Gamma)\dot{\xi}(\tau)$ can be expressed via dimensionless quantities $\tau\Gamma$, γ , and $T/(\hbar\Gamma)$. In the same way we note from Eqs. (38) and (39) that $(1/\Gamma)G(\tau)$ and $F(\tau)$ are expressed via γ and $\tau\Gamma$.

APPENDIX H: DERIVATIONS FOR THREE PULSES (SPIN-ECHO SETUP)

Now we consider three pulses— \mathcal{P}_1 , \mathcal{P}_π , and \mathcal{P}_2 —which are applied, respectively, at times t , $t+\tau$, and $t+2\tau$. The pulses \mathcal{P}_1 and \mathcal{P}_2 are kept arbitrary, while \mathcal{P}_π is the π pulse defined by Eq. (103).

The work done for the pulse \mathcal{P}_2 is defined by the same formula (G2), where now all the averages are taken at the time $t+2\tau$ immediately before the application of \mathcal{P}_2 . Our

calculations in the following will be relatively brief, since in essence they follow the pattern of calculations in the previous appendix.

For $\hat{\sigma}_+(t+2\tau)$, we get

$$\begin{aligned} \hat{\sigma}_+(t+2\tau) &\equiv \mathcal{E}_i \mathcal{P}_1 \mathcal{E}_\tau \mathcal{P}_\pi \mathcal{E}_\tau \hat{\sigma}_+ = e^{-iF(\tau)+i\omega\tau} \mathcal{E}_i \mathcal{P}_1 \mathcal{E}_\tau \hat{\Pi}_+(0, \tau) \hat{\sigma}_- \\ &= e^{-4iF(\tau)+iF(2\tau)} \mathcal{E}_i \mathcal{P}_1 \hat{\Pi}_+(\tau, 2\tau) \hat{\Pi}_-(0, \tau) \hat{\sigma}_- \\ &= e^{-4iF(\tau)+iF(2\tau)} \mathcal{E}_i \{ \hat{\Pi}_+(\tau, 2\tau) \hat{\Pi}_-(0, \tau) \} \mathcal{E}_i \{ \mathcal{P}_1 \hat{\sigma}_- \} \\ &= e^{-4iF(\tau)+iF(2\tau)} \hat{\Pi}_+(t+\tau, t+2\tau) \hat{\Pi}_-(t, t+\tau) \\ &\quad \times e^{-i\chi_3 \hat{\sigma}_z} \mathcal{E}_i \{ \mathcal{P}_1 \hat{\sigma}_- \}, \end{aligned} \quad (\text{H1})$$

where we used Eq. (F2) and defined

$$\chi_3(\tau, t) = 2F(\tau) - F(2\tau) - 2F(t+\tau) + F(t) + F(t+2\tau). \quad (\text{H2})$$

Taking in this equation the limit $t \gg 1/\Gamma$ and using Eq. (39), we return to the quantity $\chi_3(\tau)$ as defined by Eq. (111).

With help of Eq. (E11) and the reasoning of Eq. (G10), one has

$$\langle \hat{\sigma}_+(t+2\tau) \rangle = c_{-,z}^{(1)} e^{-4\xi(\tau)+\xi(2\tau)} \langle e^{-i\chi_3 \hat{\sigma}_z} \hat{\sigma}_z \rangle. \quad (\text{H3})$$

In the same way as for Eq. (H1), we have

$$\hat{\sigma}_z(t+2\tau) \equiv \mathcal{E}_i \mathcal{P}_1 \mathcal{E}_\tau \mathcal{P}_\pi \mathcal{E}_\tau \hat{\sigma}_z = -\mathcal{E}_i \mathcal{P}_1 \hat{\sigma}_z, \quad (\text{H4})$$

while applying Eqs. (G5) and (G6) one derives

$$\begin{aligned} \hat{X}(t+2\tau) &\equiv \mathcal{E}_i \mathcal{P}_1 \mathcal{E}_\tau \mathcal{P}_\pi \mathcal{E}_\tau \hat{X} \\ &= \hat{\eta}(t+2\tau) + [2G(\tau) - G(2\tau)] \mathcal{E}_i \mathcal{P}_1 \hat{\sigma}_z \\ &\quad + [G(2\tau) - G(t+2\tau)] \hat{\sigma}_z. \end{aligned} \quad (\text{H5})$$

The only nontrivial relation in calculating $\langle \hat{\sigma}_+ \hat{X} \rangle_{t+2\tau}$ is

$$\begin{aligned} &\langle \hat{\Pi}_+(t+\tau, t+2\tau) \hat{\Pi}_-(t, t+\tau) \eta(t+2\tau) \rangle \\ &= [2i\dot{\xi}(\tau) - i\dot{\xi}(2\tau) - 2G(\tau) + G(2\tau)] e^{-4\xi(\tau)+\xi(2\tau)}, \end{aligned} \quad (\text{H6})$$

which is obtained in the same way as Eqs. (E7) and (E11). The easiest way to check this relation is to follow to the derivation of Eq. (E5), that is, to differentiate Eq. (E11) over t_4 , to put $t_4=t+2\tau$, $t_3=t_2=t+\tau$, $t_1=t$, and then to change the sign of all G factors in the final expression.

If the reader has followed us so long, he/she can continue alone, since the remaining calculations are fairly straightforward.

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- [40] There is also a third relaxation time T_2^* which has a different origin. It only appears when dealing with an ensemble of non-interacting spins each having Hamiltonian (2) with a randomly distributed energy ε (dephasing). The influence of T_2^* is studied in Sec. VII.
- [41] Note that the commutator of the quantum noise is a c-number: $[\hat{\eta}(t), \hat{\eta}(s)] = -2i \sum_k g_k^2 \sin \omega_k(t-s) = -2i \hbar \dot{G}(t-s)$.
- [42] It is useful to note that this process of bringing spin in contact to the bath need by itself not be connected with any fundamental energy cost. Imagine, for example, a sudden switching of the interaction Hamiltonian $\hat{H}_I = \frac{1}{2} \hat{X} \hat{\sigma}_z$. Since in the equilibrium state of the bath $\langle \hat{X} \rangle = 0$, the work done for the realization of this switching is zero.
- [43] Recalling Wick's theorem (for $\langle \eta \rangle = 0$): Any correlation of an odd number of $\hat{\eta}$'s vanishes. A correlation of an even number of $\hat{\eta}$'s is equal to the sum of products of pair correlations, the sum being taken over all pairings. For example,
- $$\begin{aligned} \langle T \hat{\eta}(t_1) \hat{\eta}(t_2) \hat{\eta}(t_3) \hat{\eta}(t_4) \rangle &= \langle T \hat{\eta}(t_1) \hat{\eta}(t_2) \rangle \langle T \hat{\eta}(t_3) \hat{\eta}(t_4) \rangle \\ &+ \langle T \hat{\eta}(t_1) \hat{\eta}(t_3) \rangle \langle T \hat{\eta}(t_2) \hat{\eta}(t_4) \rangle \\ &+ \langle T \hat{\eta}(t_1) \hat{\eta}(t_4) \rangle \langle T \hat{\eta}(t_2) \hat{\eta}(t_3) \rangle. \end{aligned}$$
- Note that the similar Wick decomposition of $\langle T \hat{\eta}(t_1) \cdots \hat{\eta}(t_{2k}) \rangle$ will be a sum of $(2k-1)!! = (2k-1)(2k-3) \cdots 3$ terms. Wick's theorem is related to the fact that the commutator of the quantum noise is a c-number; see Ref. [41].
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- $$\begin{aligned} \frac{\partial}{\partial \delta} e^{i\delta \hat{H}/\hbar} \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] \\ = \frac{i}{\hbar} [\hat{H} - e^{i\delta \hat{H}/\hbar} \hat{H}(t+\delta) e^{-i\delta \hat{H}/\hbar}] e^{i\delta \hat{H}/\hbar} \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right] \\ = -\frac{i}{\hbar} e^{i\delta \hat{H}/\hbar} \hat{H}_F(t+\delta) e^{-i\delta \hat{H}/\hbar} e^{i\delta \hat{H}/\hbar} \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_t^{t+\delta} ds \hat{H}(s) \right], \end{aligned}$$

$$\begin{aligned} & \frac{\partial}{\partial \delta} \mathcal{T} \exp \left[-\frac{i}{\hbar} \int_0^\delta ds e^{is\hat{H}/\hbar} \hat{H}_F(s+t) e^{-is\hat{H}/\hbar} \right] \\ &= -\frac{i}{\hbar} e^{i\delta\hat{H}/\hbar} \hat{H}_F(t+\delta) e^{-i\delta\hat{H}/\hbar} \mathcal{T} \\ & \quad \times \exp \left[-\frac{i}{\hbar} \int_0^\delta ds e^{is\hat{H}/\hbar} \hat{H}_F(s+t) e^{-is\hat{H}/\hbar} \right]. \end{aligned}$$

[49] In order to get Eq. (55) from Eq. (56), note that the external fields are acting only on the spin and $\partial_t \hat{H}_F(t) = \partial_t \hat{H}(t)$. Then in Eq. (56), we can change the reduced density matrix ρ_S for the full density matrix ρ since the only time dependence of the Hamiltonian lives in the Hilbert space of the spin. Then Eq. (56) can be written as

$$\frac{dW}{dt} = \text{tr} \left(\rho(t) \frac{\partial \hat{H}_F(t)}{\partial t} \right) = \text{tr} \left(\rho(t) \frac{\partial \hat{H}(t)}{\partial t} \right).$$

Now integrate this expression from 0 to τ ,

$$\begin{aligned} \int_0^\tau dt \frac{dW}{dt} &= W(0, \tau) = \int_0^\tau dt \text{tr} \left(\rho(t) \frac{\partial \hat{H}(t)}{\partial t} \right) \\ &= \text{tr}[\rho(\tau) \hat{H}(\tau)] - \text{tr}[\rho(0) \hat{H}(0)] - \int_0^\tau dt \text{tr}[\dot{\rho}(t) \hat{H}(t)]. \end{aligned}$$

Note that the last integral is equal to zero due to the equation of motion (57).

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 [57] Here and after, we do not present the results of the full optimization of the work over the parameters of the involved pulses. The reason is that we do not want to make the pulsing setup too dependent on the details of the model. On the other hand, the results of this full optimization did not show any qualitative difference with the presented ones.
 [58] The assumption that each spin has its bath is a natural one for cases when the spins are sufficiently well separated [37,38].
 [59] The assumption on the Gaussian character of this distribution can be motivated by the central limit theorem, where the randomness of is viewed to be caused by many (nearly) independent small random factors.
 [60] It is known in the literature that one can obtain decoupling of the spin from the means of strong pulses [22,34]. However, recently the restriction of sharp pulses was relaxed while still sucesfully achieving dynamical decoupling [61]. This prompts a hope that also for the present effect the strong and sharp pulse restriction is not necessary.
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