Universality of rescaled curvature distributions

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We applied a recently proposed rescaling of curvatures of eigenvalues of parameter-dependent random matrices to experimental data from acoustic systems and to a theoretical result. It is found that the data from four different experiments, ranging from isotropic plates to anisotropic three-dimensional objects, and the theoretical result always agree with the universal curvature distribution, if only the curvatures are rescaled such that the average of their absolute values is unity.

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I. INTRODUCTION

The usefulness of the study of statistical properties of eigenvalues and eigenvectors of quantum systems has been demonstrated in many areas of physics [1]. It has become clear that these properties follow universal patterns that can be modeled by probability distributions extracted from ensembles of random Hamiltonians [2] of the same class of the system under study. In particular, time-reversal invariant physical systems whose underlying classical motion is chaotic have correlated spectra with statistical properties consistent with the predictions of the Gaussian orthogonal ensemble (GOE) random matrix model. This result has been confirmed by analyses numerically performed for two degrees of freedom systems (one particle in a two-dimensional box), but also by experimentally studying resonances of microwave cavities with adequate boundaries or of quartz or metallic blocks (elastodynamics) (see, for instance, contributions by Richter [3], Stöckmann [4], Ellegaard [5], and Schaadt [6]).

In a pioneering work Wilkinson started the investigation of the dynamical aspect of the same problem, namely, how the correlations manifest themselves in the flow of the levels when an external parameter is varied [7]. A random matrix model has then been constructed to investigate the statistical properties of the first (velocities) and the second (curvatures) derivatives of GOE eigenvalues with respect to an adequately defined parameter. It has been found that the velocities are Gaussian distributed while the curvatures, after an appropriate rescaling, follow the simple distribution given by [8,9]

$$P(k) = \frac{1}{2(1+k^2)^{3/2}}. \tag{1}$$

Large curvatures are produced by close encounters and they probe repulsion between neighboring pair of levels. A simple estimation [10] shows that the power decay $|k|^{-3}$ predicted for large curvatures by the above equation follows from the linear behavior of the repulsion between GOE eigenvalues. Another feature of this distribution is that the curvatures were normalized in such a way that the average of their modulus is equal to one, i.e., $\langle|k|\rangle=1$ with the quantity $\langle|k|\rangle$ playing the same role the average density plays in the usual level statistics. One and the other give in the respective dynamical and static cases, respectively, the scaling necessary to be able to talk about universality.

The difficult task of checking experimentally this prediction was undertaken by the experimental group at the Center for Chaos and Turbulence (CAT) of the Niels Bohr Institute. They studied the dependence on different external parameters of the spectrum of frequencies of four systems ranging from isotropic plates to anisotropic three-dimensional objects. The statistical properties for fixed value of the parameters were verified to follow the GOE statistics. The measured curvature distributions have shown slight but significant deviations from the above theoretical prediction.

By analyzing the data relative to a quartz block in which the temperature was the parameter [11], it has been shown however in Ref. [10] that the agreement with the universal prediction is remarkable if one rescales the curvature such that it satisfies the requirement that their average is equal to one, as is implied by Eq. (1). We extend here this procedure to the others systems studied by the CAT group. We show that rescaling the data the results of all four experiments agree remarkably with the universal curvature distribution.

The same procedure is also successfully applied to a theoretical curvature data obtained with the so-called Robnik

TABLE I. Overview of the parameters that govern each of the experiments. The first column specifies the system under study. The following columns contain the quality factor value of the system at a typical frequency, the typical level number, the number of levels included in the study, the number of parameters values, the ratio of the resonance width to the mean level spacing, the range covered on the normalized parameter axis, the ratio of the typical wavelength to the size of the system, and the number of independent points, roughly, in frequency-parameter space for calculation of statistical quantities like the curvature distribution.

System	Q	N	n_ε	n_x	Γ/Δ	$x_{\max}$	L/λ	$n_\varepsilon(x+1)$
(a)	10^4	2×10^2	171	63	30	3.3	5	735
(b)	10^4	3×10^2	275	63	30	2.0	7	825
(c)	10^5	1×10^3	466	59	25	0.8	6	839
(d)	10^5	3×10^3	709	101	15	1.8	7	1276

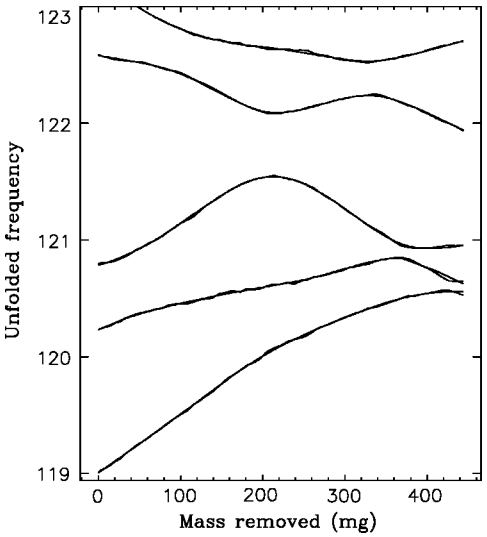

FIG. 1. Unfolded eigenfrequencies of the quartz block plotted as function of the mass removed from one surface of the block with polishing powder.

billiard (a quadratic conformal map of the unit disk) by varying its shape parameter [12]. The complete disagreement with Eq. (1) shown by the initial data has been reduced in Ref. [13] by the introduction of an appropriate unfolding procedure of the parameter. We show that the agreement becomes better if the curvatures are rescaled.

II. EXPERIMENTAL SETUP

Using an HP 3589A spectrum / network analyzer and piezoelectric transducers, we measure ultrasound transmission spectra of solid objects. There are three such transducers, of which one is transmitter and two others are receivers. The temperature of the system is kept constant to within 0.005 °C using a temperature controller, such that the eigenfrequencies are not affected by the fluctuations in room temperature. The pressure of the air surrounding the sample can be controlled and kept at a low value where air damping of the vibration is insignificant. During measurements, the system is resting only on three tiny spikes, making the vibrations as close to free as possible. For a more detailed description of this setup see Ref. [14].

III. EXPERIMENTS

Parametric experiments were carried out on four different systems. The first experiment (a) was carried out on an aluminum plate cut in the shape of a quarter stadium with a slit cut on one face to break the mirror symmetry through the middle plane of the plate. This ensures that flexural and in-plane modes are completely mixed, see [15] for details. The parameter was the size of the straight line section in the stadium. The second experiment (b) was carried out on an aluminum plate cut in the shape of one quarter of the Sinai-stadium, but this time no slit was cut to mix modes. Instead, the two mode classes were separated experimentally using the air pressure, and the flexural modes were selected for analysis. The eigenfrequencies were perturbed by gradually cutting off material from the plate [16]. The third experiment (c) was carried out on a three-dimensional (3D) system, a block of monocrystalline quartz cut in the shape of one eighth of a 3D Sinai billiard [17]. The block was carefully polished with fine grinding powder in order to perturb the eigenfrequencies by removing mass. The fourth experiment (d) was carried out on similar block, but this time the temperature was used as the external parameter. In Table I the most important information about the experiments is given. Finally, the fifth system (e) was a theoretical result from Robnik billiard performed by Leboeuf and Sieber [13].

IV. FITTING THE SPAGHETTI

The eigenfrequencies of measured resonance curves are determined using an interactive software tool that we have developed in the programming language IDL. The user points out a resonance on the screen, and the software fits the data with a "skew Lorentzian," see [18] for details, whose parameters include the eigenfrequency and the width. Once the eigenfrequencies are determined, the unfolding procedure is performed by fitting their cumulative density with a third order polynomial from which the new spectrum with a unitary average density is extracted.

In Fig. 1 we plot the unfolded frequencies as function of the mass removed. As one can see, the "spaghetti strings" corresponding to raw data contain noise. Since we are interested in slopes and curvatures, we must deal with this noise. We find that some standard noise reduction schemes, like smoothing, fitting and low pass filtering, cannot do a sufficiently good job. In our case, however, the following two points are bound to complicate any attempt to reduce noise: For some level, say ϵ_i, no prediction for the function $\epsilon_i(T)$ exists, i.e., we cannot just fit some function to the spaghetti string. In fact, the level motion is known to be very complicated. We know that the saghetti strings have curvatures on all scales. In particular since we are interested in the tail of the curvature distribution, it is of great importance that the large curvatures are not altered by the noise reduction. We decided to use the algorithm explained in the following. Consider a point on some spaghetti string $\epsilon_i(T)$, corresponding to the parameter value $T=T'$. Fit a polynomial to the $2n+1$ points that include the point in question and the n neighboring points on both sides. If the polynomial is $P(T)=aT^2+bT+c$, the slope and curvature of the spaghetti string at the considered point is simply $2aT'+b$ and $2a$, respectively. The parameter n is a constant of the algorithm and must be determined such that $2n+1$ is larger than the typical size of the noise but still small enough to satisfy locality. An improved version was introduced, where n is determined by the curvature in a converging procedure. The idea is that for a point on the spaghetti string where the curvature is large, one should use a small n to make sure that the polynomial really follows the sharp curve of the spaghetti string, whereas for a point of small curvature one should use a larger n to avoid fitting the noise. The algorithm starts out setting $n=N$ where N is an intermediate value and then fits $P(T)=aT^2+bT+c$ to the $2n+1$ points in question. The resulting

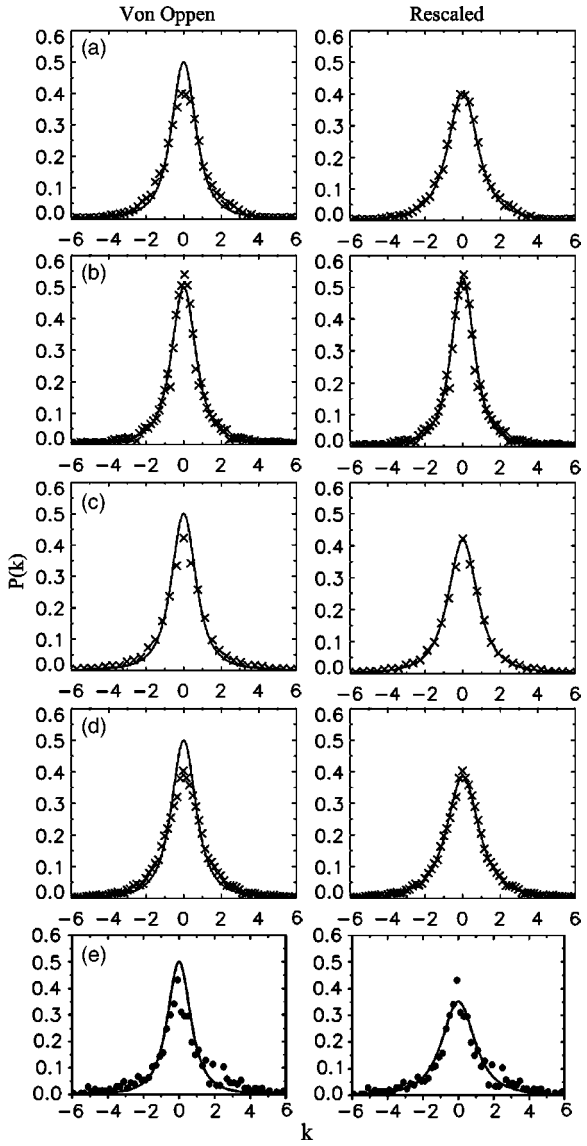

FIG. 2. Curvature distributions for the five systems. In the first column the curves correspond to Eq. (1); in the second column they are the fitted curves using Eq. (4).

curvature $2a$ is translated into a new value of $n=n(2a)$ according to the above mentioned idea. This step is continued until the resulting curvature stays the same, i.e., the value of the curvature has converged to some number. This algorithm is used for every point on every spaghetti string and the resulting slopes and curvatures are stored for analysis. Tests show that the results are independent of the choice of N, which is reassuring.

V. CURVATURES

The normalization of the curvatures start by replacing the physical parameter T by a new dimensionless parameter t such that [13]

TABLE II. The fitted parameters from the method of least squares for the five systems corresponding to the rescaled curves in the second column in Fig. 2.

System	γ	χ^2
Stadium plate–slit cut, (a)	1.23	2.1×10^{-5}
Sinai-stadium plate, (b)	0.95	6.9×10^{-5}
3D Sinai billiard–mass, (c)	1.18	1.5×10^{-5}
3D Sinai billiard–temperature, (d)	1.30	9.3×10^{-6}
Robnik billiard, (e)	1.42	5.3×10^{-4}

$$\frac{dt}{dT} = \sqrt{\left\langle \left(\frac{d\epsilon}{dT}\right)^2 \right\rangle}, \tag{2}$$

where the average is made over the whole set of unfolded eigenfrequencies at a given value of T. The level curvature is then defined in terms of these new scaled variables as

$$k = \frac{1}{\pi}\frac{d^2\epsilon}{dt^2} = \frac{1}{\pi\langle\dot{\epsilon}^2\rangle}\left(\ddot{\epsilon} - \frac{\langle\dot{\epsilon}\ddot{\epsilon}\rangle}{\langle\dot{\epsilon}^2\rangle}\dot{\epsilon}\right), \tag{3}$$

where the dots denote derivatives with respect to the physical parameter T. In Eq. (3), $1/\pi$ is exactly the factor that normalizes the curvatures in the context of random matrix theory. The experimental data were fitted with the one parameter distribution

$$P(k) = \frac{\gamma^2}{2(\gamma^2 + k^2)^{3/2}} \tag{4}$$

with $\gamma=\langle|k|\rangle$. The results of the fitting for the five systems are shown in Fig. 2 and in Table II. The small values of chi-square show that the agreement with the data is not only visual. We do not observe any correlation between the values of γ in Table II and the material of the samples or the external parameter used.

VI. DISCUSSION AND CONCLUSION

In conclusion, we have shown that the rescaling introduced in Ref. [10] makes experimental and theoretical data of level curvatures to agree with the universal distribution Eq. (1). We draw from these results that random matrix theory correctly predicts the functional form of the curvature distribution but the curvature normalization factor. This factor has to be extracted from the data. We remark that this feature has also been observed in the case of another parametric distribution, the level correlation function, whose universality has been experimentally studied using quantum dots [19].

A clue to understand why the random matrix normalization factor $1/\pi$ is not sufficient is maybe provided by the fact that in all the present systems the average $\langle(d\epsilon/dT)^2\rangle$ defining the new unfolded parameter is not a constant during the evolution. As a consequence, in the definition of the scaled curvature Eq. (3) the first derivative is subtracted from the second derivative.

Before concluding, we add that authors have attributed discrepancies with the universal curve to the presence of

symmetries [20,21]. Without ruling out completely this possibility we argue that symmetries should reduce the repulsion between levels and, as a consequence, make their trajectories closer to straight lines increasing and not reducing small curvatures as observed in the systems being here analyzed.

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