# Interactions among periodic waves and solitary waves of the $(N+1)$-dimensional sine-Gordon field

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Exact solutions of the $(n+1)$-dimensional sine-Gordon field equation are studied with help of those of the cubic nonlinear Klein-Gordon fields. The mapping relations among the sine-Gordon field equation and the cubic nonlinear Klein-Gordon fields are pure algebraic. By solving the cubic nonlinear Klein-Gordon equations, many new types of exact explicit solutions such as the periodic-periodic interaction waves, periodic-kink interaction waves, periodic perturbed "snake" shape solitary waves, etc., are displayed both analytically and graphically.

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## I. INTRODUCTION

In the classical and quantum field theory, the sine-Gordon (SG) field is one of the most important examples [1,2]. The SG field is significant not only because of its basic status in field theory but also because of its wide applications in almost all the branches of physics and other scientific fields.

The $(1+1)$-dimensional SG equation first arose in a strictly mathematical context—in differential geometry in the theory of surfaces of constant curvature [3]. The Bäcklund transformation for the SG equation was known before 1882 [4]; the inverse scattering transformation has been given by Ablowitz *et al.* [5].

The earliest physical example is the model of dislocations in solids put forward by Frenkel and Kontorova [6]. The SG equation plays an important role in the theory of long Josephson junctions [7] and in the dynamics of quasi-one-dimensional ferromagnets with easy-plane anisotropy [8]. Other physical applications of the SG equation have been made to liquid crystals [9], spin waves in liquid helium [10], self-induced transparency of a two-level medium in nonlinear optics [11], and the hydrodynamics and even as a model of hadrons [12].

It is also well known that the $(1+1)$-dimensional SG model is equivalent to many other important systems—for instance, the Thirring model, the Coulomb gas system, the ferromagnetic *XY* model, the O(2) sigma model, etc. [13].

Because of the wide applications of the SG model, searching for its exact solutions is of great importance and interest. Some properties and exact solutions for the SG field are known in the literature [14–16].

The $(n+1)$-dimensional sine-Gordon (nSG) equation can be written as

$$\Box\Phi + \frac{m}{g}\sin g\Phi \equiv \sum_{i=1}^{n}\Phi_{x_i x_i} - \Phi_{tt} + \frac{m}{g}\sin g\Phi = 0, \quad (1)$$

which has been also applied in almost all the branches of physics [2] especially for $n=2$ and 3 cases. Though the nSG equation is nonintegrable except for $n=1$, some special types of soliton solutions, such as multiple line (or plane) kink solutions, have been obtained by different methods. In [15] some kinds of exact solutions of the nSG equation have been linked with a single constrained cubic nonlinear Klein-Gordon (CNKG) equation.

In the traditional treatment of nonlinear systems, one usually studies the interaction behaviors among solitons (or solitary waves) in respect that many methods can provide *exact explicit* multiple soliton (or solitary wave) solutions. However, there are few works in the literature to study the interactions among (elliptic) periodic waves and/or between the periodic waves and solitary waves because of the difficulties to find *exact and explicit* multiple (elliptic) periodic wave solutions and/or periodic-solitary wave solutions though one knows a single solitary wave solution can be considered as a limit case of a single periodic wave solution.

Consequently, the first problem we try to treat in this paper is the following: *Are there any exact explicit multiple periodic wave solutions and periodic-solitary wave solutions for the nSG equation?*

Recently, we have found that for high-dimensional integrable systems, there are much richer structures of the localized excitations and periodic wave solutions than in lower dimensions thanks to the intrusion of some arbitrary lower dimensional arbitrary functions [17]. Then the second question we try to answer is the following: *Can we find rich solution structures (with arbitrary functions) for high-dimensional nonintegrable nonlinear systems like the nSG?*

The paper is organized as follows. In Sec. II, we rewrite and extend the mapping relation of special solutions between the nSG and the single constrained CNKG field. In Sec. III, the mapping relation is extended to link some more special solutions of the nSG equation with two and three constrained coupled and noncoupled CNKG fields. Some concrete exact solutions such as the periodic-periodic, periodic-kink, and periodic-periodic-kink interaction solutions are graphically displayed. The last section includes a short summary and some simple discussions.

 

## II. MAPPING RELATION AMONG SPECIAL SOLUTIONS OF THE nSG AND THOSE OF ONE CONSTRAINED CNKG FIELD

To find some special types of exact solutions of the nSG equation (1), many interesting results have been given by various authors especially in 1+1 dimensions [14,15]. In order to find more exact solutions of Eq. (1), we try to establish some mapping relations among the special solutions of the nSG equation (1) and the so-called CNKG equation or $\lambda\phi^4$ model:

$$\Box\phi \equiv \sum_{i=1}^{n} \phi_{x_i x_i} - \phi_{tt} = \lambda\phi + \mu\phi^3. \tag{2}$$

*Theorem 1*. If $\phi$ is a solution of Eq. (2) with the constrained condition

$$(\tilde{\nabla}\phi)^2 \equiv \sum_{i=1}^{n} \phi_{x_i}^2 - \phi_t^2 = \frac{1}{2}(\lambda + \mu - m)\phi^2 + \frac{\mu}{2}\phi^4 + \frac{1}{2}(\lambda + m), \tag{3}$$

then

$$\Phi = \frac{2\alpha\pi}{g} \pm \frac{4}{g}\tan^{-1}\phi, \quad \alpha = 0, \pm 1, \pm 2, \ldots, \tag{4}$$

is a solution of the nSG equation (1).

*Proof*. Substituting Eq. (4) into the nSG equation (1), we have

$$\Box\phi - \frac{2\phi}{1+\phi^2}(\tilde{\nabla}\phi)^2 + m\frac{\phi(1-\phi^2)}{1+\phi^2} = 0. \tag{5}$$

Let the function $\phi$ be a solution of the CNKG equation (2); Eq. (5) just becomes Eq. (3). Theorem 1 then is proved.

Equation (5) is completely equivalent to the original nSG equation. So when some of the special solutions of Eq. (5) are obtained, then the related solutions of the nSG equation immediately follow from Eq. (4). To get some special solutions of Eq. (5), putting some constraints on the function $\phi$ is necessary. Here, we select $\phi$ as a solution of the $\lambda\phi^4$ because the $\lambda\phi^4$ model is quite familiar to many physicists and easier to get some exact explicit solutions. Actually, some theorems to find new exact solutions of some special constrained CNKG equations and a long list solution table have been given in [15].

In theorem 1, two free parameters $\lambda$ and $\mu$ have been included. The different selections of the free parameters will lead to different types of periodic wave solutions. In principle, infinitely many free parameters can be included in the special solutions of high-dimensional partial differential equations (PDE's). Mathematically, some types of special solutions of a PDE may be integrated out along some suitable lower-dimensional characteristic manifold and then some lower-dimensional arbitrary functions can be included in the special solutions of the investigated model. Physically, the entrance of a free parameter into the solutions of a model implies the existence of a symmetry or a conserved quantity though the concrete meaning of the conserved quantity usually is not very clear because of the existence of infinitely many conserved quantities for a nonlinear system with infinitely many freedoms. For instance, for the nSG equation there are some special integrable reductions—say, the simple reduction

$$\Phi = \varphi(\xi, \tau), \quad \xi = \sum_{i=1}^{n} k_i x_i + \omega_1 t, \quad \tau = \sum_{i=1}^{n} p_i x_i + \omega_2 t, \tag{6}$$

with

$$\sum_{i=1}^{n} k_i^2 - \omega_1^2 = 1, \quad \sum_{i=1}^{n} p_i^2 - \omega_2^2 = -1, \quad \sum_{i=1}^{n} k_i p_i - \omega_1\omega_2 = 0, \tag{7}$$

solves the nSG equation via the well-known integrable 2SG equation

$$\varphi_{\xi\xi} - \varphi_{\tau\tau} + \frac{m}{g}\sin g\varphi = 0, \tag{8}$$

which is known to possess infinitely many conserved quantities.

For some types of exact solutions of the nSG equations infinitely many free parameters can be included in some different ways: namely, (i) by including an *arbitrary* function in the constrained equation, for instance, changing the constrained equations (2) and (3) as

$$\Box\phi = F(\phi), \quad F \text{ arbitrary}, \tag{9}$$

$$(\tilde{\nabla}\phi)^2 = \frac{1+\phi^2}{2\phi}F(\phi) + \frac{m}{2}(1-\phi^2); \tag{10}$$

(ii) by including *arbitrary* functions via solving the fixed constrained equations (see details later for the constrained CNKG system).

An equivalent special case of theorem 1 with

$$m = \mu - \lambda \tag{11}$$

and various exact explicit solutions of the CNKG equation (2) with the constraints (3) and (11) can be found in [15].

Here we give a further quite general explicit solution (with an *arbitrary function*) related to theorem 1:

$$\Phi = \frac{2\alpha\pi}{g} \pm \frac{4}{g}\tan^{-1}\sqrt{k}\,\mathrm{sn}\frac{\sqrt{|m|}V}{k+1}, \tag{12}$$

where

$$\begin{aligned} V = {} & \sum_{i=1}^{n} k_{0i}x_i + a\left(\sum_{i=1}^{n} k_{1i}^2\right)^{-1/2}\left(\sum_{i=1}^{n} k_{0i}k_{1i}\right)t \\ & + f\left(\sum_{i=1}^{n} k_{1i}x_i + a\sqrt{\sum_{i=1}^{n} k_{1i}^2 t}\right) \\ \equiv {} & \xi_0 + f(\xi), \end{aligned} \tag{13}$$

$f(\xi)$ is an arbitrary function of $\xi$, and the free parameters $k_{0i}$, $k_{1i}$, $(i=0,1,\ldots,n)$ are linked by $(a^2=1)$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} k_{0i}^2 k_{1j}^2 - \left(\sum_{j=1}^{n} k_{0i}k_{1i}\right)^2 - \frac{m}{|m|}\sum_{i=1}^{n} k_{1i}^2 = 0, \qquad (14)$$

while the constant $k$ is the modulus of the Jacobi elliptic function $\mathrm{sn}(z) \equiv \mathrm{sn}(z,k)$.

Corresponding to the solution of Eq. (12), the solution of the constraint CNKG equation reads

$$\phi = \sqrt{k}\ \mathrm{sn}\frac{\sqrt{|m|}V}{k+1}, \qquad (15)$$

with the parameter selections

$$\lambda = \frac{|m|(1+k^2)\sum_{i=1}^{n}\sum_{j=1}^{n}(k_{0i}k_{0j}k_{1i}k_{1j} - k_{0i}^2 k_{1j}^2)}{(1+k)^2\sum_{i=1}^{n} k_{1i}^2}, \qquad (16)$$

$$\mu = \lambda + m. \qquad (17)$$

The special solution (12) denotes some particular types of resonant solutions of two traveling waves moving in the directions which are perpendicular to the planes (or lines for $n=2$):

$$\sum_{i=1}^{n} k_{1i}x_i = 0, \quad \sum_{i=1}^{n} k_{0i}x_i = 0,$$

respectively.

If we take

$$f(\xi) = \frac{1}{3}\sqrt{\xi^2+1}, \qquad (18)$$

then the solution (12) denotes a type of periodic-periodic wave interaction solutions for the modulus $k \neq 1$. For this type of solutions, in one region ($\xi \gg 1, \sqrt{\xi^2+1} \approx \xi$), one periodic wave is dominant and can be approximately expressed by

$$\Phi_1 \approx \frac{2\alpha\pi}{g} \pm \frac{4}{g}\tan^{-1}\sqrt{k}\ \mathrm{sn}\frac{\sqrt{|m|}(\xi/3+\xi_0)}{k+1}, \quad \xi \gg 1, \qquad (19)$$

while in another region ($\xi < 0, |\xi| \ll 1, \sqrt{\xi^2+1} \approx -\xi$) the other periodic wave becomes dominant with the approximate expression

$$\Phi_2 \approx \frac{2\alpha\pi}{g} \pm \frac{4}{g}\tan^{-1}\sqrt{k}\ \mathrm{sn}\frac{\sqrt{|m|}(\xi_0-\xi/3)}{k+1}, \quad \xi < 0, \quad |\xi| \gg 1. \qquad (20)$$

Figure 1(a) is a (2+1)-dimensional special structure of this type of solution with the upper sign "+" of Eq. (12) and the parameter selections

$$\alpha = 0, \quad k_{11} = k_{02} = \omega_0 = 3, \quad \omega_1 = 5, \quad k_{12} = 4,$$
$$m = g = k_{01} = 1, \quad k = 0.9, \qquad (21)$$

at time $t=0$.

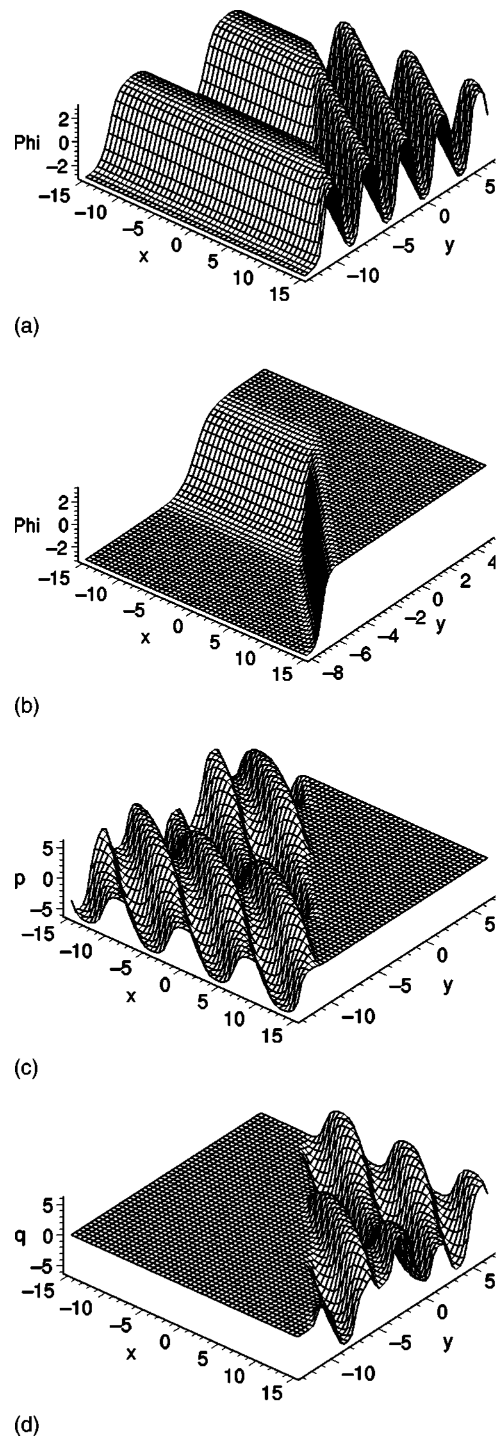


FIG. 1. (a) A typical periodic solitoff structure of the 2SG equation expressed by Eq. (12) with Eqs. (18) and (21) at $t=0$. (b) A special two-kink-like solitoff solution which is a limit case of (a) for the modulus $k$ of the Jacobi elliptic function: $k \to 1$. (c) A plot of the function $p \equiv \Phi - \Phi_1$ where $\Phi$ is same as (a) and $\Phi_1$ is given by Eq. (19) with Eqs. (18) and (21) at $t=0$. (d) A plot of the function $q \equiv \Phi - \Phi_2$ similar to (c). All the quantities used in this paper are set to be dimensionless to fit possible different applications.

When $k \to 1$, Eq. (12) with Eq. (18) tends to a two-solitoff solution. A solitoff is defined as a half straight-line soliton [18]. Figure 1(b) shows the structure of a special two-solitoff

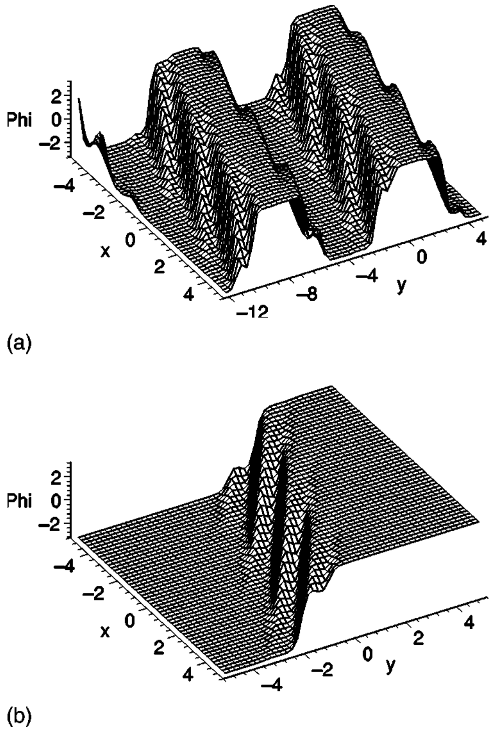


FIG. 2. (a) The periodic-periodic wave interaction structure of the 2SG equation expressed by Eq. (12) with Eqs. (22) and (21) at $t=0$. (b) A periodic line kink soliton structure which is a limit case of (a) for $k\to 1$.

solution expressed by Eq. (12) with Eq. (18) and the parameter selections are the same as in Eq. (21) except for $k=1$.

To display the correctness of the approximate expressions (19) and (20), $p\equiv\Phi-\Phi_1$ and $q\equiv\Phi-\Phi_2$ are plotted in Figs. 1(c) and 1(d), respectively. The flat parts of Figs. 1(c) and 1(d) tell us that the expressions (19) and (20) are quite well approximations of Eq. (12) with Eqs. (13) and (18) at their valid regions.

The structures of the periodic traveling waves shown by Eq. (12) may be quite rich because of the existence of the arbitrary function $f(\xi)$. For instance, Fig. 2(a) shows another particular periodic-periodic wave interaction structure by selecting the function $f(\xi)$ as

$$f(\xi)=\sin\xi, \tag{22}$$

while the other parameters are same as in Fig. 1(a).

Figure 2(b) shows a straight-line kink soliton with a periodic traveling wave deformation. The parameter and function selections of Fig. 2(b) are same as those in Fig. 2(a) except that the modulus $k$ is taken as the limiting value $k=1$.

## III. MAPPING RELATION AMONG SPECIAL SOLUTIONS OF THE nSG AND THOSE OF TWO CONSTRAINED CNKG FIELDS

In order to search for more exact solutions of the nSG equation, we may use two or more solutions of the CNKG equations. By taking two special exact solutions of the CNKG equations, we have the following.

*Theorem 2*. If $\phi_1$ and $\phi_2$ are solutions of the CNKG models

$$\Box\phi_1=\lambda_1\phi_1+\mu_1\phi_1^3,\quad \Box\phi_2=\lambda_2\phi_2+\mu_2\phi_2^3, \tag{23}$$

under the constraint conditions

$$(\tilde{\nabla}\phi_1)^2=g_1(\phi_1,\phi_2),\quad (\tilde{\nabla}\phi_2)^2=g_2(\phi_1,\phi_2), \tag{24}$$

and

$$(\tilde{\boldsymbol{\nabla}}\phi_1)\cdot(\tilde{\boldsymbol{\nabla}}\phi_2)\equiv\sum_{i=1}^{n}\phi_{1x_i}\phi_{2x_i}-\phi_{1t}\phi_{2t}=g_{12}(\phi_1,\phi_2), \tag{25}$$

with $g_1(\phi_1,\phi_2)\equiv g_1$, $g_2(\phi_1,\phi_2)\equiv g_2$, and $g_{12}(\phi_1,\phi_2)\equiv g_{12}$ being functions of $\{\phi_1,\phi_2\}$ and related by

$$\begin{aligned}hf(f^2-h^2)m&+(h^2+f^2)[\phi_1(\lambda_1+\phi_1^2\mu_1)(bc\phi_2^2-a)\\&+\phi_2(\phi_2^2\mu_2+\lambda_2)(b\phi_1^2-ca)]+2(bc\phi_2^2-a)(b\phi_2 f+h)g_1\\&+2(b\phi_1^2-ca)(ch+b\phi_1 f)g_2-4hf(ab+c)g_{12}=0,\end{aligned} \tag{26}$$

where

$$f\equiv a+b\phi_1\phi_2,\quad h\equiv\phi_1+c\phi_2,$$

then

$$\Phi=\frac{2\alpha\pi}{g}\pm\frac{4}{g}\tan^{-1}\frac{h}{f},\quad \alpha=0,\pm1,\pm2,\ldots, \tag{27}$$

is a solution of nSG (1).

*Proof*. Substituting Eq. (27) into Eq. (1) yields

$$\begin{aligned}hf(f^2-h^2)m&+(h^2+f^2)[(bc\phi_2^2-a)\Box\phi_1+(b\phi_1^2-ca)\Box\phi_2]\\&-4hf(ab+c)(\tilde{\boldsymbol{\nabla}}\phi_1)\cdot(\tilde{\boldsymbol{\nabla}}\phi_2)+2(bc\phi_2^2-a)(b\phi_2 f+h)\\&\times(\tilde{\nabla}\phi_1)^2+2(b\phi_1^2-ca)(ch+b\phi_1 f)(\tilde{\nabla}\phi_2)^2=0.\end{aligned} \tag{28}$$

Now substituting the constraint conditions (23)–(25) into Eq. (28) leads to the relation (26) and then the theorem 2 is proved.

To obtain some explicit solutions from theorem 2, we have to solve the coupled constraint system (23)–(25). Here we introduce some further restrictions

$$\begin{aligned}\phi_1&=\phi_1(V_1(x_1,\ldots,x_n,t))\equiv\phi_1(V_1),\\ \phi_2&=\phi_2(V_2(x_1,\ldots,x_n,t))\equiv\phi_2(V_2),\end{aligned} \tag{29}$$

$$\phi_{1V_1}^2=\lambda_1\phi_1^2+\frac{\mu_1}{2}\phi_1^4+C_1,\quad \phi_{2V_2}^2=\lambda_2\phi_2^2+\frac{\mu_2}{2}\phi_2^4+C_2, \tag{30}$$

with $V_1$ and $V_2$ being arbitrary solutions of the simple constraint equations

$$\begin{aligned}\Box V_1=\Box V_2&=(\tilde{\boldsymbol{\nabla}}V_1)\cdot(\tilde{\boldsymbol{\nabla}}V_2)=0,\\ (\tilde{\nabla}V_1)^2=G_1,&\quad(\tilde{\nabla}V_2)^2=G_2,\end{aligned} \tag{31}$$

where $G_1$ and $G_2$ are constants.

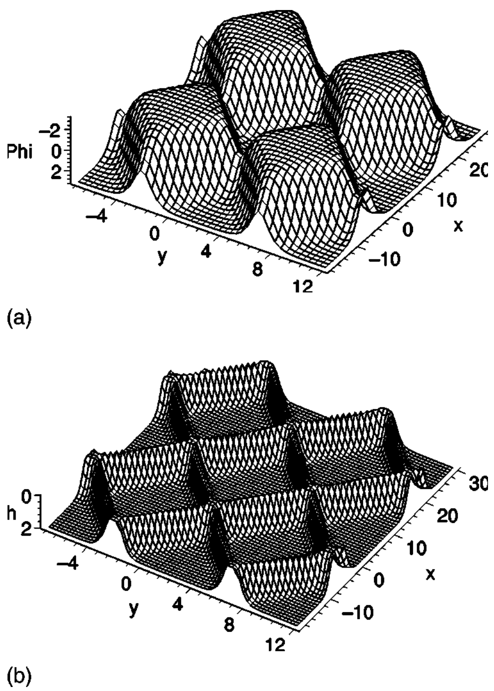


FIG. 3. (a) The periodic-periodic wave interaction solution expressed by Eq. (45) with Eqs. (42)–(44) and the parameter selections (46) at time $t=0$. (b) A plot of the potential energy density related to (a).

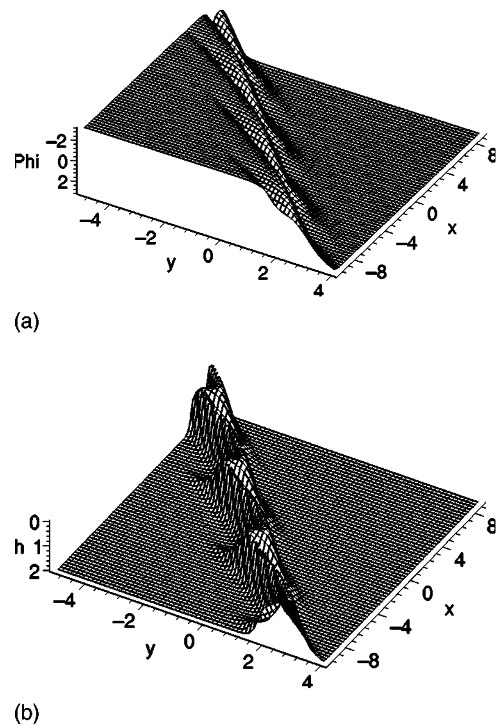


FIG. 4. (a) A plot of Eq. (47) with Eqs. (48), (49), and (47) at $t=0$. (b) The structure of the potential energy density $h\equiv 1-\cos(g\Phi)$ related to (a).

Under the constraints (29)–(31), theorem 2 is simplified to the following.

*Theorem 3*. $\Phi$ expressed by Eq. (27) with the constraints (29)–(31) is a solution of (1) iff (if and only if) either

$$c=ab,\quad a^2=-\frac{\lambda_1}{\mu_1},\quad G_2=0,\quad G_1=\frac{m}{2\lambda_1},\quad C_1=\frac{\lambda_1^2}{2\mu_1} \tag{32}$$

or

$$a^4=\frac{2C_1}{\mu_1},\quad G_1=-\frac{m(b^2\lambda_2+\mu_2)}{a^2(3\mu_2+b^2\lambda_2)\mu_1+(\mu_2-b^2\lambda_2)\lambda_1},$$

$$b^4=\frac{\mu_2}{2C_2},\quad G_2=-\frac{mb^2(a^2\mu_1+\lambda_1)}{a^2(3\mu_2+b^2\lambda_2)\mu_1+(\mu_2-b^2\lambda_2)\lambda_1}, \tag{33}$$

are satisfied.

In derivation of theorem 3, another situation

$$C_2=\frac{\lambda_2^2}{2\mu_2},\quad b^2=-\frac{\mu_2}{\lambda_2},\quad G_1=0,\quad G_2=\frac{m}{2\lambda_2}$$

has been ruled out because it is equivalent to the first case of theorem 3.

A special situation of theorem 3 reads

$$V_1=f_1\left(\sum_{i=1}^{n}k_{1i}x_i+\omega_1 t\right)+\sum_{i=1}^{n}k_{0i}x_i+\omega_0 t, \tag{34}$$

$$V_2=f_2\left(\sum_{i=1}^{n}k_{2i}x_i+\omega_2 t\right)+\sum_{i=1}^{n}k_{3i}x_i+\omega_3 t, \tag{35}$$

$$\sum_{i=1}^{n}k_{j_1i}k_{j_2i}-\omega_{j_1}\omega_{j_2}=-G_1\delta_{j_10}\delta_{j_20}-G_2\delta_{j_13}\delta_{j_23},$$
$$j_1,j_2=0,1,2,3, \tag{36}$$

where

$$\delta_{ij}=\begin{cases}0, & i\neq j,\\ 1, & i=j,\end{cases}$$

and $f_1(\xi_1)$ and $f_2(\xi_2)$ are arbitrary functions.

The solutions given by Eq. (27) with Eqs. (29), (30), (33) [or (32)], and (34)–(36) denote the interaction solutions of three or four traveling periodic or kink waves. Here are three more special explicit examples in (2+1) dimensions and (3+1) dimensions.

(a) If we take

$$\phi_1=\sqrt{n_1}\,\mathrm{sn}(V_1,n_1),\quad \phi_2=\sqrt{n_2}\,\mathrm{sn}(V_2,n_2),\quad a=b=c=1, \tag{37}$$

$$V_1=k_1x+l_1y+\omega_1 t,\quad V_2=k_2x+l_2y+\omega_2 t, \tag{38}$$

$$\lambda_1=-(1+n_1^2),\quad \mu_1=2n_1,\quad C_1=n_1,$$
$$\lambda_2=-(1+n_2^2),\quad \mu_2=2n_2,\quad C_2=n_2, \tag{39}$$

$$G_1 = k_1^2 + l_1^2 - \omega_1^2, \quad G_2 = k_2^2 + l_2^2 - \omega_2^2, \tag{40}$$

where $n_1$ and $n_2$ are the modulus of the Jacobi sn functions and $\omega_1$, $\omega_2$, $k_1$, $k_2$, $l_1$, and $l_2$ are determined by

$$\omega_1^2 = k_1^2 + l_1^2 - \frac{m(n_2-1)^2}{(n_2+1)^2(n_1^2+1) + (2n_2^2 - 12n_2 + 2)n_1}, \tag{41}$$

$$\omega_2^2 = k_2^2 + l_2^2 - \frac{m(n_1-1)^2}{(n_2+1)^2(n_1^2+1) + (2n_2^2 - 12n_2 + 2)n_1}, \tag{42}$$

$$k_1k_2 + l_1l_2 - \omega_1\omega_2 = 0, \tag{43}$$

then a special (2+1)-dimensional periodic-periodic wave solution of Eq. (1) can be immediately read off from the second case of theorem 3:

$$\Phi = \frac{2\alpha\pi}{g} \pm \frac{4}{g}\arctan\frac{\sqrt{n_1}\,\mathrm{sn}(k_1x + l_1y + \omega_1t, n_1) + \sqrt{n_2}\,\mathrm{sn}(k_2x + l_2y + \omega_2t, n_2)}{1 + \sqrt{n_1n_2}\,\mathrm{sn}(k_1x + l_1y + \omega_1t, n_1)\mathrm{sn}(k_2x + l_2y + \omega_2t, n_2)}. \tag{44}$$

Figure 3(a) shows a special (2+1)-dimensional structure of the two periodic waves expressed by Eq. (44) with Eqs. (41)–(43) and the parameter selections

$$\alpha = k_1 = l_2 = \omega_2 = 0, \quad m = g = l_1 = 1, \quad n_1 = n_2 = \frac{9}{10}, \quad k_2 = \frac{10}{\sqrt{721}}, \quad \omega_1 = \frac{207}{\sqrt{49749}}, \tag{45}$$

at $t=0$. Figure 3(b) is a plot of the potential energy density $h \equiv 1 - \cos(g\Phi)$ related to Fig. 3(a).

(b) A particular periodic-kink wave interaction solution for the (2+1)-dimensional sine-Gordon (2SG) equation possesses the form

$$\Phi = \frac{2\pi}{g} \pm \frac{4}{g}\arctan\frac{\mathrm{dn}(k_1x + l_1y + \omega_1t, k)\tanh(k_2x + l_2y + \omega_2t) + a}{a\tanh(k_2x + l_2y + \omega_2t) + \mathrm{dn}(k_1x + l_1y + \omega_1t, k)}, \tag{46}$$

where $\omega_1$, $\omega_2$, $k_1$, $k_2$, $l_1$, and $l_2$ are determined by

$$\omega_1^2 = k_1^2 + l_1^2, \quad \omega_2^2 = k_2^2 + l_2^2 - \frac{m}{4}, \tag{47}$$

$$k_1k_2 + l_1l_2 - \omega_1\omega_2 = 0, \tag{48}$$

while the modulus $k$ of the Jacobi dn function and the constant $a$ remain free.

Figure 4(a) is a plot of a concrete example of Eq. (46) with Eqs. (47) and (48) and the parameter selections

$$\alpha = 0, \quad k_1 = l_1 = m = g = 1, \quad k_2 = \frac{1}{2}, \quad k_2 = \omega_2 = \frac{1}{2}\frac{1}{\sqrt{2}-1}, \quad a = \frac{9}{10}, \quad \omega_1 = \sqrt{2}, \quad k = \frac{2}{5}, \tag{49}$$

at $t=0$. Figure 4(b) shows the structure of the potential energy density $h \equiv 1 - \cos(g\Phi)$ related to Fig. 4(a).

(c) A particular multiple periodic wave interaction solution for the (3+1)-dimensional sine-Gordon (3SG) equation possesses the form

$$\Phi = \frac{2\alpha\pi}{g} \pm \frac{4}{g}\arctan\frac{\mathrm{sn}(f_1(\xi) + k_2x, a^{-2}) + ab\,\mathrm{sn}(f_2(\xi) + k_1y + \omega_1t, b^2)}{a + b\,\mathrm{sn}(f_1(\xi) + k_2x, a^{-2})\,\mathrm{sn}(f_2(\xi) + k_1y + \omega_1t, b^2)}, \tag{50}$$

with $f_1(\xi)$ and $f_2(\xi)$ being two arbitrary functions of $\xi$,

$$\xi \equiv \sqrt{k_1^2 - \omega_1^2}y + \omega_1z + k_1t, \tag{51}$$

while the five solution parameters $k_1$, $k_2$, $\omega_1$, $a$, and $b$ have to satisfy two conditions

$$k_1^2 = \frac{(a^2-1)^2k_2^2}{(b^2-1)^2a^4} + \omega_1^2, \tag{52}$$

$$ma^4(b^2-1)^2 - k_2^2[(ab+1)^2 + (a+b)^2][(ab-1)^2 + (a-b)^2] = 0. \tag{53}$$

Actually, the mapping relations of this section can be extended more generally such that more CNKG fields can be used to find more complicated periodic and kink interaction modes. Here we only write down a further special (2+1)-dimensional example

$$\Phi = \frac{2\alpha\pi}{g} + \frac{4}{g}\arctan\frac{a\tanh\eta - c\,\mathrm{sn}(k\xi,n_1) - [d + b\,\mathrm{sn}(k\xi,n_1)\tanh\eta]\,\mathrm{sn}(\xi,n_2)}{-a + c\,\mathrm{sn}(k\xi,n_1)\tanh\eta + [b\,\mathrm{sn}(k\xi,n_1) + d\tanh\eta]\,\mathrm{sn}(\xi,n_2)}, \qquad (54)$$

where

$$\xi = \sqrt{\omega_1^2 - p_1^2}x + p_1 y + \omega_1 t, \qquad \eta = \frac{\omega_1\omega_2 - p_1 p_2}{\sqrt{\omega_1^2 - p_1^2}}x + p_2 y + \omega_2 t, \qquad (55)$$

$n_1$, $n_2$, $p_1$, $p_2$, $k$, and $\omega_1$ are arbitrary constants and $\omega_2$ is given by

$$\omega_2 = \frac{\sqrt{m(\omega_1^2 - p_1^2)}}{2p_1} + \frac{p_2\omega_1}{p_1}. \qquad (56)$$

Figure 5 plots a particular periodic-kink wave expressed by Eq. (54) with Eqs. (55) and (56) and

$$a = -23, \quad b = 7, \quad c = 13, \quad d = 2, \quad g = 1, \quad p_2 = 0,$$
$$p_1 = \frac{4}{3}, \quad \omega_1 = 3, \quad k = 3, \quad m = \frac{2304}{65},$$
$$n_1 = \frac{999}{1000}, \quad n_2 = \frac{9}{10}, \qquad (57)$$

at $t=0$. Figure 5(a) is a direct plot of the field $\Phi$ of Eq. (54) while Fig. 5(b) shows the structure of the potential energy density $h \equiv 1 - \cos(g\Phi)$.

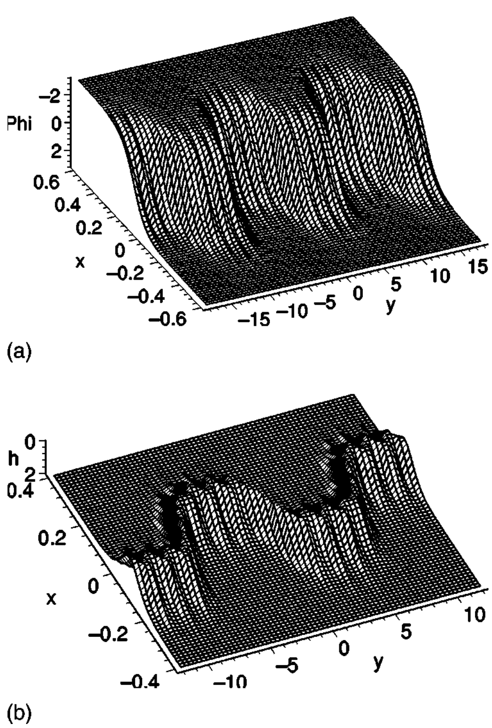


FIG. 5. (a) A plot of Eq. (55) with Eqs. (56) and (57) at $t=0$. (b) The structure of the potential energy density related to (a).

From Fig. 5(b), one can see that a "snake" shape solitary wave is perturbed by two periodic waves with different periods but moving in the same direction, perpendicular to the line $\xi=0$.

## IV. SUMMARY AND DISCUSSION

In summary, by means of the special solutions of the CNKG systems such as Eq. (2) with Eq. (3) and Eq. (23) with Eqs. (24)–(26), a diversity of exact explicit solutions of the nSG system (1) are obtained simply from some pure algebraic mapping relations (4) and (27).

It is known that for the (2+1)-dimensional integrable models, there are many more abundant localized structures than in (1+1)-dimensional case because some types of arbitrary functions can be included in the explicit solution expressions [19,20]. Though the nSG equation is nonintegrable unless $n=1$, the localized structures may also be quite rich due to the existence of arbitrary functions—say, those in Eqs. (13) and (34)–(36). In this paper, some special types of explicit multiple wave interaction solutions including periodic-periodic waves, periodic-kink waves, and periodic-periodic-kink waves are explicitly given both analytically and graphically.

Even for two periodic interaction waves there may be many kinds of patterns. In this paper, three kinds of (2+1)-dimensional periodic-periodic patterns are explicitly revealed. The first type of periodic-periodic wave solution [Eq. (12) with Eqs. (13) and (18)] is the generalization of two solitoff solutions. Though the multisolitoff solution has been known in the literature for many (2+1)-dimensional models, its elliptic generalization is first found in this paper. The second type of periodic-periodic wave solution [Eq. (12) with Eqs. (13) and (22)] is a generalization of a single periodic perturbed line soliton while the third type of periodic-periodic wave solution [Eq. (44) with Eqs. (41)–(43)] is an alternative generalization of a single nonperturbed straight-line kink soliton solution.

For many (2+1)-dimensional and (3+1)-dimensional integrable models [17] as well as some special types of high-dimensional nonintegrable ones [21], there are some types of localized excitations decaying in all directions such as dromions and ring solitons, which, however, we still have not yet found for the nSG system. Consequently, how to find the localized solutions which decay in all directions should be studied further. Moreover, due to the wide applications of the nSG equation in physics, it is more interesting to find some possible applications of these exact solutions. However, lacking theoretical studies and experiments related to the high-dimensional SG, we could not further say something about the real physical meanings of our exact solutions. We hope that in future experimental studies some kinds of exact wave solutions obtained here can be realized in some fields such as those listed in the Introduction.

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