

## Time-independent approximations for periodically driven systems with friction

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The classical dynamics of a particle that is driven by a rapidly oscillating potential (with frequency  $\omega$ ) is studied. The motion is separated into a slow part and a fast part that oscillates around the slow part. The motion of the slow part is found to be described by a time-independent equation that is derived as an expansion in orders of  $\omega^{-1}$  (in this paper terms to the order  $\omega^{-3}$  are calculated explicitly). This time-independent equation is used to calculate the attracting fixed points and their basins of attraction. The results are found to be in excellent agreement with numerical solutions of the original time-dependent problem.

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Time-dependent systems typically exhibit behavior which is more complicated than that of the corresponding time-independent ones. Moreover, physicists have not yet developed an intuition about the dynamics of time-dependent systems to the level of the one that exists for time-independent ones. As a result, it is difficult to predict the qualitative properties of driven systems even in cases in which it is easy to understand the dynamics of time-independent systems that are similar. In this work, the dynamics of some time-dependent driven systems will be related to the dynamics of time-independent ones. In particular, this will be done for some systems where there is a clear separation of time scales. This will enable qualitative and quantitative analysis of the dynamics of some class of time-dependent systems using the experience with time-independent ones.

There is a vast literature on methods which approximate a dynamical system by a “simpler” system with a smaller number of degrees of freedom. These include, among others, averaging methods [1,2], multiple time-scale analysis [2], and center manifold theory [3]. In fact, the method employed in this article can be viewed as a particular example for the more general method of multiple time-scales analysis [2]. It should be noted that the method used in this work is tailored for equations of the form (1), and therefore its application is much simpler than that of the more general methods. However, it is representative of many physical problems.

The more known case of separation of time scales is the adiabatic one. In this case, the system evolves on a time scale which is much shorter than the time scale of the driving. There are many works which treat different adiabatic approximations [1,2]. On a qualitative level, one can understand the resulting dynamics by treating the driving as it was fixed in time, letting the system evolve, and then changing the values of the parameters of the driving according to their time dependence.

Less known, but not less interesting, is the opposite limit. In this case, the typical time scales of the dynamics of the system in absence of the driving are much longer than the period of the driving. A remarkable effect of such driving is “dynamical stabilization,” in which a particle, that in absence of the driving can escape from some region, may be trapped by the rapidly oscillating field. Examples of this phenomenon include the Kapitza pendulum [4] and the Paul trap [5].

A simple treatment of rapidly driven systems is given in

textbooks [6] following Kapitza’s work on the inverted pendulum [4]. In this approximate calculation the motion is separated into a sum of a slow part and rapid oscillations around it. The rapid oscillations are computed explicitly and their effect on the slow motion is found. The treatment of Landau and Lifshitz turns out to be the leading order of an expansion in  $\omega^{-1}$  (where  $\omega$  is the driving frequency). It was extended to the order  $\omega^{-4}$  in recent works [7,8]. In these works it was demonstrated that for rapidly driven Hamiltonian systems it is possible to obtain a time-independent Hamiltonian that controls the slow motion. It is obtained by a canonical transformation as an expansion in powers of  $\omega^{-1}$ .

In this article, the method developed in [7,8] is extended to rapidly driven classical systems in the presence of friction where the Hamiltonian formalism is inapplicable, since energy is dissipated. The (modest) goal of this article is to demonstrate how one can understand and predict the qualitative properties of the dynamics using very simple approximations. In particular, it will be demonstrated how this method can be used in order to predict qualitatively the general form of the basins of attraction for such systems and how to compute their boundaries from the equation of motion of the slow part.

Newton’s equation of motion, for a rapidly driven system, is given by

$$m\ddot{x} + \alpha\dot{x} = -V'_0(x) - V'_1(x, \omega t), \quad (1)$$

where  $m$  is the mass of the particle and  $\alpha$  is the friction constant. The potential is  $V(x, \omega t) = V_0(x) + V_1(x, \omega t)$ , which is chosen so that the time average of  $V_1$  over a period vanishes. Derivatives with respect to the coordinate and time of  $f(x, t)$  are denoted by  $f'$  and  $\dot{f}$ , respectively, while  $\bar{f}$  denotes the average over a period.

The slow and fast motion are separated with the help of the ansatz

$$x(t) = X(t) + \xi(X, \dot{X}, \tau), \quad (2)$$

where  $\xi$  is periodic in the fast time variable  $\tau \equiv \omega t$ , with a vanishing average over a period, and  $X(t)$  is the slow part of the motion. The fast time  $\tau$  is treated as an independent variable. One ensures that  $X(t)$  is indeed slow by choosing  $\xi$  in such a way that the equation of motion for  $X$  does not depend

explicitly on time. This can be done, at least approximately, by expanding  $\xi$  in a power series in  $\omega^{-1}$ , using

$$\xi = \sum_{n=1}^{\infty} \frac{1}{\omega^n} \xi_n. \quad (3)$$

Then, the functions  $\xi_n$  are determined, order by order (in  $\omega^{-1}$ ), from the condition that the remaining equation of motion for  $X$  is time-independent. This procedure leads to new time-independent terms in the equation of motion for  $X(t)$ . (These terms cannot be canceled by a choice of  $\xi_n$  that will stay bounded for large times.) An explicit derivation of the first few terms for a similar problem can be found in [8]. For an equation of motion of the form (1), the perturbation theory results in the following explicit solution for  $\xi$  [given in terms of  $X(t)$ ]:

$$\begin{aligned} \xi(X, \dot{X}, \omega t) \simeq & -\frac{1}{m\omega^2} \int^{(2)\tau} [V_1'] + \frac{2\dot{X}}{m\omega^3} \int^{(3)\tau} [V_1''] \\ & + \frac{\alpha}{m^2\omega^3} \int^{(3)\tau} [V_1'] + O(\omega^{-4}). \end{aligned} \quad (4)$$

Substitution of Eq. (4) in Eq. (1) leads to the slow equation of motion, for  $X(t)$ ,

$$\begin{aligned} m\ddot{X} = & -\alpha\dot{X} - V_0'(X) - \frac{1}{m\omega^2} \overline{\int^{\tau} [V_1''] \int^{\tau} [V_1']} \\ & + \frac{\alpha}{m\omega^3} \overline{\int^{\tau} [V_1'] \int^{(2)\tau} [V_1']} + O(\omega^{-4}). \end{aligned} \quad (5)$$

The symbol  $\int^{\tau}[\dots]$  denotes an integral over  $\tau$ , defined *only* for periodic functions of  $\tau$  with vanishing average, which is performed in such a way that the resulting function of  $\tau$  is also periodic with vanishing average. This integral is easy to compute using the Fourier expansion of the integrand (see [8] for details). Multiple application of the integral ( $j$  times) is denoted by  $\int^{(j)\tau}[\dots]$ . The overline denotes an average over a period of  $\tau$ . Therefore, Eq. (5) does not depend explicitly on time (and its solution will *not* exhibit oscillations with the external frequency  $\omega$ ).

The leading-order ( $\omega^{-2}$ ) correction to the motion in absence of the driving can be seen as resulting from the effective potential

$$V_{eff}(X) = V_0(X) + \frac{1}{2m\omega^2} \overline{\left( \int^{\tau} [V_1'(X, \tau)] \right)^2} \quad (6)$$

that acts on the slow motion. However, at higher orders, terms appear which do not seem to result from a potential. Note that in spite of the rapid oscillations, the friction, and hence the energy dissipation, is associated only with the slow motion (at the order  $\omega^{-2}$ ).

Equations (4) and (5) are the results of a high-frequency perturbation theory. They result in a mapping of a time-dependent problem into a time-independent one. In any given order in  $\omega^{-1}$ , this theory reproduces the result of the mathematical theory of separation of time scales, but it is much simpler. One of the goals of this article is to demonstrate how these equations can be used to understand the

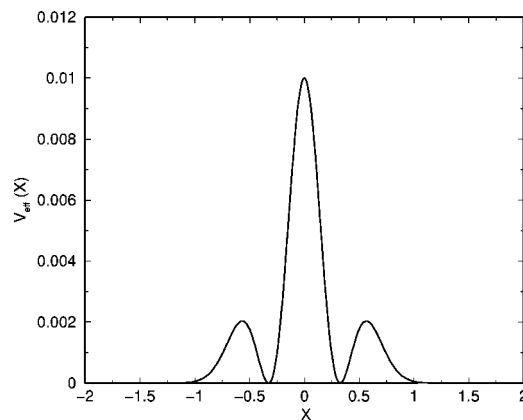


FIG. 1. The effective potential for the slow motion. The values of the parameters in Eq. (8) are  $A=m=1$ ,  $\beta=4$ ,  $k=2$ , and  $\omega=10$ .

dynamics of such driven systems. It will be demonstrated by a simple example.

Consider a particle which, apart from the friction, is under the influence of an oscillating field given by

$$V_1(x, \omega t) = A e^{-\beta x^2} \sin(kx) \cos(\omega t + \phi). \quad (7)$$

This simple system is of interest since the time average of the potential vanishes,  $V_0(x)=0$ . Therefore, the influence of the oscillatory field is dominant even at high frequencies, in contrast to systems with  $V_0(x) \neq 0$ . According to the high-frequency perturbation theory, the slow part of the motion of the particle can be (approximately) viewed as motion, with friction, in the effective potential

$$V_{eff}(X) = \frac{A^2}{4m\omega^2} e^{-2\beta X^2} [k \cos(kX) - 2\beta X \sin(kX)]^2. \quad (8)$$

This effective potential is depicted in Fig. 1. It exhibits several minima. These are of interest since a particle moving in a time-independent potential, in the presence of friction, will be found at one of these minima after a sufficiently long time, in spite of the fact that the linear stability of these points is time-dependent in the original system. While the potential (8), depicted in Fig. 1, has several minima, two of those, at  $x \approx \pm 0.3266$ , are more pronounced. Based on these observations, one can predict that after a sufficiently long time the particle will be found in the vicinity of one of the minima also for the time-dependent system.

In this article, we are interested mainly in the basins of attraction of the two minima located at  $x \approx \pm 0.3266$ , that is, the initial values  $x_0 \equiv x(0)$ ,  $v_0 \equiv \dot{x}(0)$  which evolve to these minima for long times. These basins of attraction, for different values of the frequency, are depicted in Fig. 2. The basins of attraction exhibit some qualitative properties which are of interest. At lower frequencies ( $\omega=6, 10$ ) these basins are separated, while for higher frequencies ( $\omega=20, 40$ ) they have a common boundary. This qualitative behavior can be understood with the help of the effective potential (Fig. 1) for the approximate time-independent system.

Consider the trajectory (of the effective time-independent system) which starts with the initial values  $X_0=0$ ,  $\dot{X}_0=\epsilon > 0$

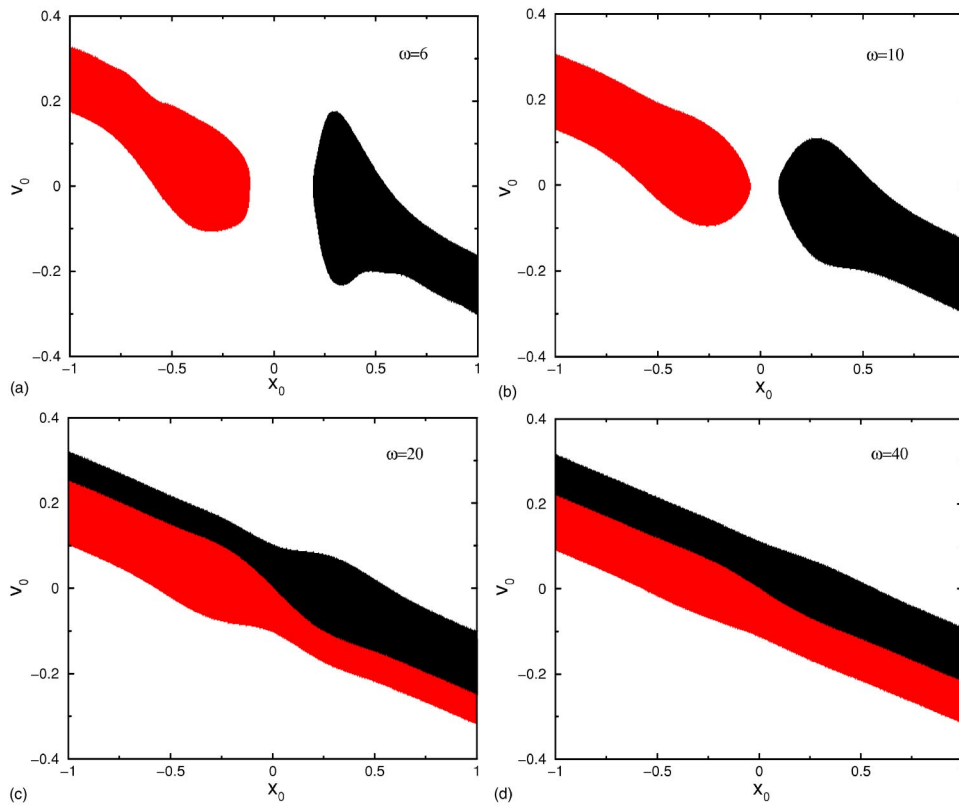


FIG. 2. Basins of attraction of  $x(t \rightarrow \infty) \approx \pm 0.3266$  for the time-dependent system (7). Results for several frequencies are presented, while  $A=m=1$ ,  $\beta=4$ ,  $k=2$ , and  $\phi=0$ .

(that is, arbitrarily close to  $X_0=\dot{X}_0=0$ ). If this trajectory, after a long time, manages to pass over the first maxima of  $V_{eff}$  with positive  $X$ , then any other trajectory which starts, say, with  $X_0 < 0$  and a (large enough) positive velocity will also

do so. In this case, the basins of attraction of  $x \approx \pm 0.3266$  are separated. In contrast, if this orbit, starting at  $X_0=0$  and  $\dot{X}_0 = \epsilon > 0$ , is trapped at  $x \approx 0.3266$  after a long time, then the basins do have a common boundary which includes the point

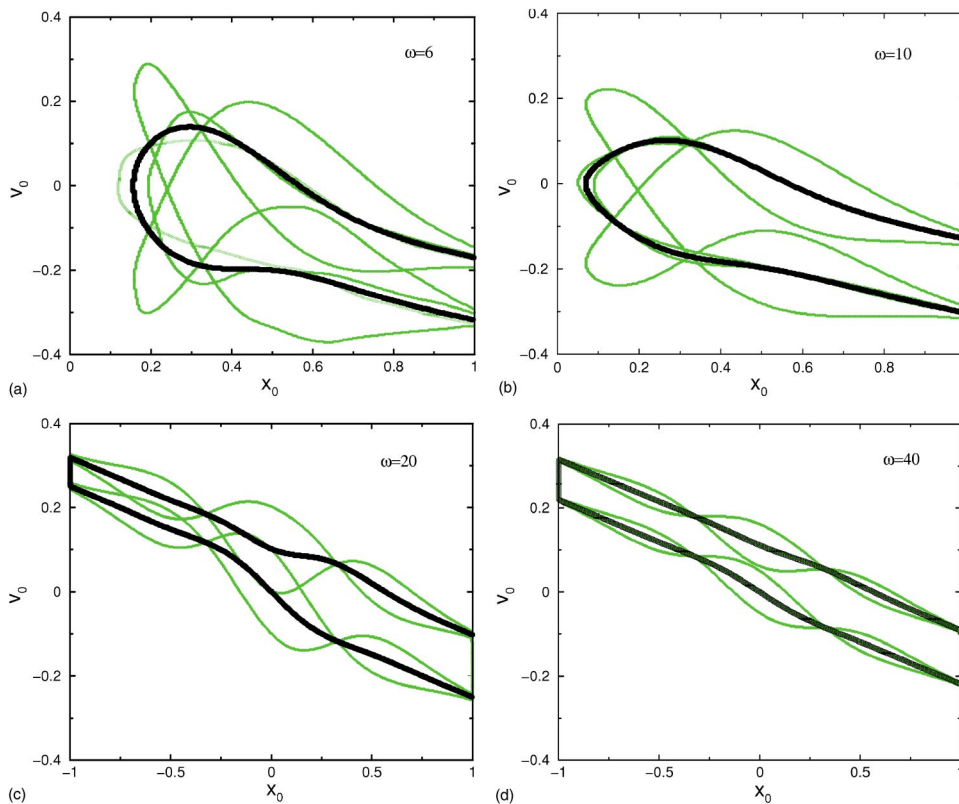


FIG. 3. The boundary of the basin of attraction of  $x \approx 0.3266$  of the time-dependent system, for various phases (thin lines), are compared to the one of the time-independent effective system (heavy line), for different frequencies. The values of the phase are  $\phi=0, \pi/2, \pi$ , and  $3\pi/2$ .

$X_0 = \dot{X}_0 = 0$ . Consider the effective potential depicted in Fig. 1. A particle located near  $X=0$  will feel an effective force due to it. This effective force will accelerate the particle while the friction will decelerate it. If the friction is dominant, the particle will be trapped in the first minimum and it will be found near  $x=0.3266\dots$  after a long time. In contrast, if the effective force is strong enough, one can expect that the particle will be able to pass the first maximum of  $V_{eff}(X)$  at positive  $X$  and therefore will not be near  $x \approx 0.3266$  after a long time. In the example used in the present numerical investigation, the friction does not scale with the frequency while the effective potential (and the resulting force) scales as  $\omega^{-2}$ . Therefore, one can expect to find a transition from separated basins of attraction to basins with a common boundary as the frequency is increased. This is consistent with the numerical results presented in Fig. 2.

It was demonstrated that some qualitative properties of the time-dependent system are described by properties of an appropriate approximate time-independent system. It is of interest to see whether one can obtain also quantitative results using the high-frequency perturbation theory. The boundaries of one of the basins of attraction of  $x \approx 0.3266$  of the time-dependent system are compared to the ones resulting from the approximated time-independent system in Fig. 3. The motion of the time-dependent system depends also on the phase of the oscillating force at  $t=0$ . It is important to note that this comparison is fairly naive since the initial values  $x_0$  and  $v_0$  are only approximately equal to  $X_0$  and  $\dot{X}_0$  of the time-independent system.

Figure 3 shows that the boundary of the basin of attraction of the time-dependent system fluctuates around the corresponding boundary of the effective time-independent system when the phase of the field is varied. This is not surprising since it is clear that the basins of attraction must depend on this phase.

As was mentioned earlier, the comparison in Fig. 3 is naive, in the sense that the coordinates  $x(0)$ ,  $v(0)$  are *not* the same as the coordinates  $X(0)$ ,  $\dot{X}(0)$  of the time-independent system. However, the results presented in Fig. 3 still demonstrate that at high frequencies, the basins of attraction of the time-dependent system can be approximated by those of time-independent effective ones. The size of the fluctuations seems to decrease when the frequency is increased.

To obtain a better quantitative correspondence between the results obtained using the high-frequency perturbation theory and the numerical results for the time-dependent system, one has to account for the difference between the slow coordinates  $X, \dot{X}$  and  $x, v$ . The connection between  $x(t)$  and  $X(t)$  [and also  $\dot{x}(t)$ ] is given by Eq. (2). One can use the high-frequency perturbation theory to obtain an expansion for  $\xi$  and then substitute  $t=0$ . This leads to an equation for the initial value  $x_0=x(0)$  in terms of  $X_0=X(0), \dot{X}_0=\dot{X}(0)$ . Similarly, by differentiating with respect to time at  $t=0$ , one obtains  $v_0=v(0)$ . The calculation results in

$$x_0 = X_0 + \frac{A}{m\omega^2} e^{-\beta X_0^2} \cos(\phi) f(X_0) + O(\omega^{-3}),$$

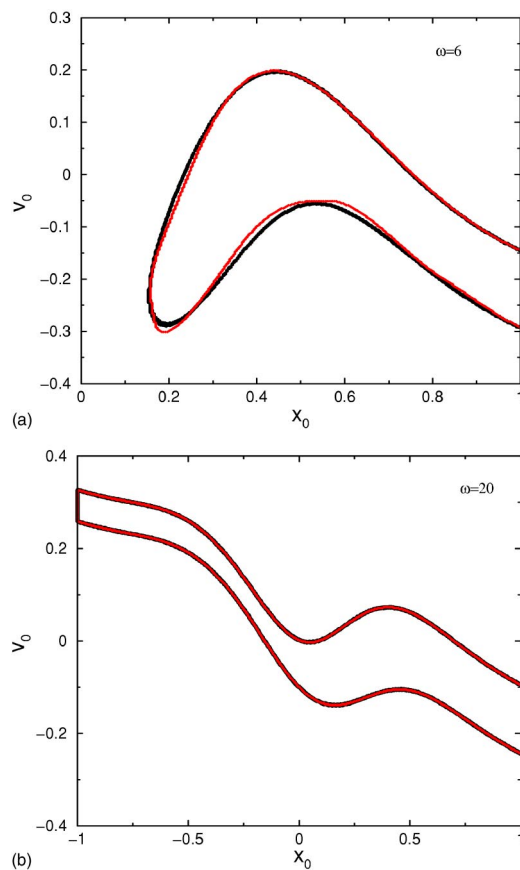


FIG. 4. Comparison between the boundary of the basin of attraction of  $x \approx 0.3266$  of the time-independent system (heavy line) and the boundary of the time-dependent one (thin line), for different frequencies, where  $\phi = \pi/2$  while the rest of the parameters are those of Fig. 2.

$$v_0 = \dot{X}_0 - \frac{A}{m\omega} e^{-\beta X_0^2} \sin(\phi) f(X_0) - \frac{A}{m\omega^2} e^{-\beta X_0^2} \dot{X}_0 \cos(\phi) g(X_0) - \frac{\alpha A}{m^2 \omega^2} e^{-\beta X_0^2} \cos(\phi) f(X_0) + O(\omega^{-3}), \quad (9)$$

where  $f(X_0) = k \cos(kX_0) - 2\beta X_0 \sin(kX_0)$ , while  $g(X_0) = [4\beta^2 X_0^2 - k^2 - 2\beta] \sin(kX_0) - 4\beta X_0 k \cos(kX_0)$ . From Eq. (9), it is clear that the fluctuations in  $v_0$  are of order  $\omega^{-1}$  when the phase is varied, while the fluctuations in  $x_0$  scale only as  $\omega^{-2}$ . This is in agreement with the fluctuations presented in Fig. 3.

To test the perturbation theory more quantitatively, the boundary of the basin of attraction of the effective time-independent system was mapped back to the original coordinates using Eq. (9) and compared to the numerical results obtained for the time-dependent system. To avoid complicated graphs, we only present the comparison for the phase  $\phi = \pi/2$ . The results are presented in Fig. 4. It is clear that the agreement is excellent. Only a small difference is seen for  $\omega=6$ , while for  $\omega=20$  the difference between the boundaries cannot be observed in the plot. This demonstrates that the high-frequency perturbation theory can be used to obtain quantitative results and not only qualitative ones.

In this article, we have used a high-frequency perturbation



theory to describe the motion of a rapidly driven classical particle in the presence of friction. In this perturbation theory, the motion is separated into a slow part and rapid oscillations around it [see Eqs. (2)–(5)]. An equation of motion for the slow coordinate, accurate to order  $\omega^{-3}$ , was obtained. The oscillations of the particle around the slow solution were calculated as well. The slow motion is found to be approximately described, to the order  $\omega^{-2}$ , by the motion of a particle in an effective potential with friction. This suggests that, after a long time, the particle will be found at the minimum of this potential, a fact which is confirmed numerically also for the time-dependent system given by Eq. (7). Due to the dissipation, these minima are surrounded by basins of attraction, which include the initial phase-space points that flow to those minima after a long time. The numerical results, presented in Figs. 2 and 3, demonstrate that some of the qualitative features of the basins of attraction can be understood by considering the simpler time-independent effec-

tive system with the potential of Fig. 1. It was also shown, in Fig. 4, that, by a more careful analysis, one can obtain excellent quantitative agreement between the motion generated by the time-dependent equation of motion and the motion generated by the corresponding time-independent one.

The results presented in this article suggest that the high-frequency perturbation theory can be used to obtain both a qualitative and a quantitative understanding of the dynamics of a classical rapidly driven particle in the presence of friction. The friction effectively dissipates the energy (only) of the slow motion (up to the order  $\omega^{-2}$ ).

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