Complexes of stationary domain walls in the resonantly forced Ginsburg-Landau equation

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The parametrically driven Ginsburg-Landau equation has well-known stationary solutions—the so-called Bloch and Néel, or Ising, walls. In this paper, we construct an explicit stationary solution describing a bound state of two walls. We also demonstrate that stationary complexes of more than two walls do not exist.

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I. INTRODUCTION

In this paper we derive a class of explicit and physically meaningful solutions to an equation that has been under scrutiny, in various contexts, for more than 40 years. The equation is the resonantly driven Ginsburg-Landau; in its most general form it reads

$$\psi_t = (\mu + i\nu)\psi + (g_1 + ic_1)\psi_{xx} - (g_3 + ic_3)|\psi|^2\psi - h(\psi^*)^{n-1}. \tag{1}$$

Equation (1) describes a one-dimensional chain of coupled weakly nonlinear self-sustained oscillators in the continuum approximation. The chain is subjected to periodic forcing at the frequency $\Omega \approx n\omega_0$, where ω_0 is the frequency of the undriven spatially homogeneous linear oscillations. The complex variable $\psi = \psi(x,t)$ is the slowly varying amplitude of the resulting nonlinear oscillations with the period $T = 2\pi n/\Omega$.

Before specializing Eq. (1) to the particular case that will concern us in this paper, we briefly comment on the physical meaning of its coefficients. First of all, the parameter $\mu > 0$ measures the distance to the supercritical Hopf bifurcation at which a (spatially homogeneous) stable limit cycle appears. For the stability of the self-sustained oscillation one also needs $g_3 > 0$. The real constant ν is proportional to the frequency detuning, $\nu \propto \Omega/n - \omega_0$. In the derivation of (1), it is assumed that both the linear growth rate μ and the detuning ν are small compared to the frequency ω_0. (See, e.g., Ref. [1].) Next, the parameter c_3 describes the nonlinear frequency shift, while the derivative term accounts for the interaction of the neighboring oscillators in the chain, with the real and imaginary parts of the coefficient $g_1 + ic_1$ pertaining to the dissipative and reactive types of coupling, respectively. Finally, h is proportional to the amplitude of the forcing. The $(n-1)$st power of ψ^* in Eq. (1) results from resonance forcing of order $n:1$ [2]. The $1:1$, $2:1$, $3:1$, and $4:1$ resonances have been studied most extensively in the literature. In this work, we focus on the case $n=2$.

The analyses of Eq. (1) usually start with considering the gradient, or variational, limit [3,4]. The variational limit corresponds to the assumption that the coupling is purely dissipative ($c_1 = 0$), frequency detuning ν is zero, and the homogeneous oscillations are isochronous ($c_3 = 0$). With these assumptions, a suitable rescaling of t, x, ψ, and h produces

$$\psi_t = \frac{1}{2}\psi_{xx} - |\psi|^2\psi + \psi - h\psi^*. \tag{2}$$

In this paper, we consider stationary solutions of Eq. (2); these satisfy

$$\frac{1}{2}\psi_{xx} - |\psi|^2\psi + \psi - h\psi^* = 0. \tag{3}$$

We will take h to be positive in what follows; this can always be achieved by an appropriate phase shift of ψ. Historically, Eq. (3) was first introduced in the context of the anisotropic XY model, which was used to model an easy-axis ferromagnet near the Curie temperature [5,6]. Nonstationary magnetization configurations were considered as solutions to Eq. (2) [7]. The investigations of the more general Eq. (1) (still with $n=2$, though), including analyses of small nonvariational effects, were reported in Refs. [3,8,9]. Apart from the magnetic applications, these studies were motivated by research in liquid crystals [10], experiments with the periodically forced light-sensitive Belousov-Zhabotinsky reaction [11], and work done in optics, in particular with regard to optical parametric oscillators [12] and lasers with intracavity parametric amplification [13]. Equation (1) also appeared as a phenomenological equation for the parametrically excited surface waves in viscous fluids [14] and granular media [15].

The nontrivial (spatially nonhomogeneous) solutions which attracted interest in the context of each of these fields are domain walls, or kinks, also known as dark solitons in nonlinear optics. The domain wall is a localized interface between two different homogeneous backgrounds (known as domains in the magnetic context). The possible backgrounds are described by the constant nonzero solutions of Eq. (3), $\psi = \pm A_-$ and $\psi = \pm iA_+$, where $A_\pm = \sqrt{1 \pm h}$. The first of these is known to be unstable while the second is stable. We will only consider dark solitons propagating over the stable background—asymptotically, all of the solutions considered here will satisfy $|\psi|^2 \to A_+^2$ as $|x| \to \infty$.

The soliton solutions to Eq. (3) with the desired asymptotic behavior can be either topological (with a phase difference of 180° between the two asymptotic values) or nontopological (with no change in phase between the asymptotic fields). Equation (3) admits two explicit topologi-

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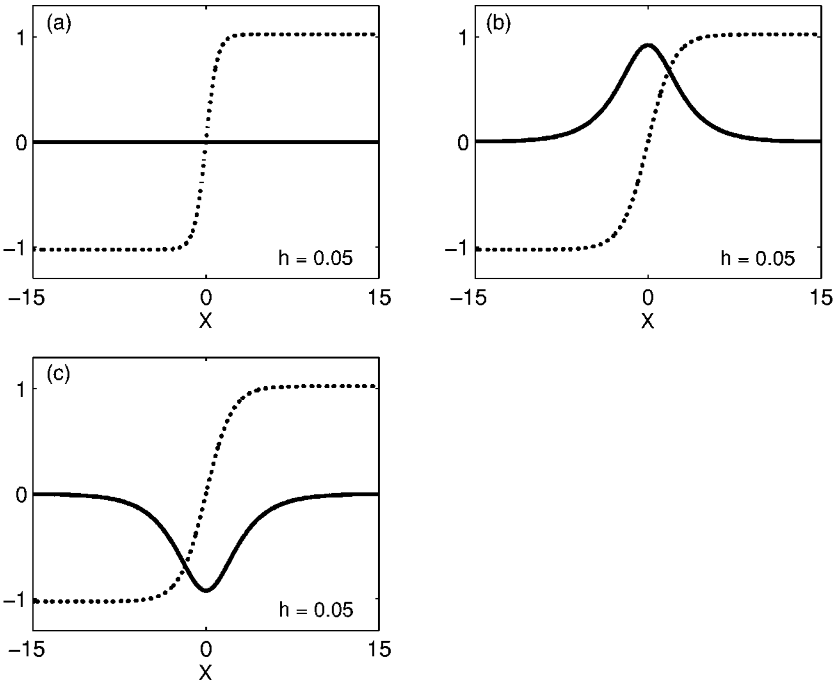

FIG. 1. The Néel wall (a), and the left- and right-handed Bloch walls [(b) and (c) respectively]. The solid line corresponds to the real part, and the dotted line to the imaginary part. Here $h=0.05$; we plot the two walls for a small value of the parameter in order to accentuate the difference in their widths.

cal solutions. The first is the Néel wall [5,6,16,17]

$$\psi_{\mathrm{N}}(x) = iA_+ \tanh(A_+ x), \tag{4}$$

so named because its magnitude $|\psi|$ vanishes at the origin, at which point the phase becomes discontinuous. (In magnetism, a Néel wall is a domain wall with vanishing magnitude of the magnetization vector at its center.) It is also called the Ising wall, the name appealing to the Ising model which was used to emulate the easy-axis ferromagnet. The Néel wall exists for all positive h.

The second topological solution has the form

$$\psi_{\mathrm{B}}(x) = iA_+ \tanh(Bx) \pm C \,\mathrm{sech}(Bx), \tag{5}$$

where $B=\sqrt{4h}$ and $C=\sqrt{1-3h}$ [6,18,19]. This solution is usually referred to as a Bloch wall (which, in magnetism, is a domain wall connecting the two domains smoothly, with the magnetization vector remaining nonzero everywhere). The Bloch wall exists in two chiralities, distinguished by the sign of the real part in Eq. (5). For convenience, we will refer to the solution with positive (negative) real part as the left-handed (right-handed) Bloch wall (see Fig. 1). Regardless of the chirality, the Bloch walls only exist for $h<\frac{1}{3}$.

In addition to the Bloch and Néel walls, Eq. (3) possesses nontopological solitons. One such solution is known explicitly, for $h=\frac{1}{15}$ [18,20],

$$\psi = iA\left[1-\frac{3}{2}\mathrm{sech}^2(Bx)\right] + 3B\tanh(Bx)\mathrm{sech}(Bx). \tag{6}$$

Here $B=\sqrt{4h}=\sqrt{4/15}$. In addition, a class of bubblelike solutions [including Eq. (6) as a particular case] was found numerically [21]. As is usual with numerical solutions, the structure of the solitonic bubbles has not been completely understood. It has also remained unclear whether they form a continuous family (families) and if they do, what is their range of existence (in h). Finally, a pertinent question is of the generality of these solutions: can the resonantly driven Ginsburg-Landau equation support localized, stationary structures other than the walls and bubbles? The aim of this paper is to address all of these issues.

Here, for each $h<\frac{1}{3}$, we construct a two-parameter family of *explicit* bubblelike solutions and demonstrate that they can be interpreted as stationary bound states, or complexes, of a Bloch and Néel wall. Physically, the constructed solutions represent phase domains of arbitrary length, which are 180° out of phase with the background field. We also prove that these solutions exhaust the list of possible stationary complexes—there can be no stationary bound states of more than two walls.

It is appropriate to mention here that Eq. (3) appeared previously in several other, unrelated, physical contexts. Written in terms of the real and imaginary parts of ψ, it coincides with the stationary equation for the so-called Montonen-Sarker-Trullinger-Bishop (MSTB) model of field theory [18,19]. More recently, it occurred as a stationary limit of the parametrically driven nonlinear Schrödinger [21] equation. The latter equation arises in a large variety of physical applications including nearly resonant optical parametric oscillators [22], easy-plane ferromagnets in magnetic fields [21], and surface waves in wide, vertically vibrated channels of inviscid fluid [17,23]. Accordingly, we expect the additional solutions to admit physical interpretations in these fields as well.

The paper is organized as follows. In Sec. II we derive a family of explicit two-soliton solutions and cast it in a symmetric form allowing easy visualization. These solutions will be interpreted as bound states of a Bloch and Néel wall. Several families of singular solutions appear as by-products in this construction; these will be used later for auxiliary purposes. After that (Sec. III) we show that there exist no other bounded solutions of Eq. (3) which would asymptotically approach the stable $(\pm iA_+)$ background. This means that apart from the constant solution, the only nonsingular solutions to Eq. (3) are the Bloch and Néel walls, and the Bloch-Néel complexes constructed in this paper. Finally, some concluding remarks are made in Sec. IV.

II. EXACT SOLUTIONS FOR THE BLOCH-NÉEL BOUND STATE

A. The Hirota construction

In order to construct a bound state of two walls explicitly, we employ the Hirota bilinear formalism (see, e.g., Ref. [24]). Letting

$$\psi = \frac{G}{F}, \tag{7}$$

where G is complex and F a real function of x, Eq. (3) is cast in the bilinear form

$$F[D_x^2 G \cdot F + (2-\lambda)GF - 2hG^* F] - G[D_x^2 F \cdot F - \lambda F^2 + 2|G|^2] = 0. \tag{8}$$

Here D_x is the Hirota operator defined on ordered products of functions:

$$D_x^n\, a \cdot b \equiv (\partial_x - \partial_y)^n\, a(x)b(y)|_{x=y}.$$

In Eq. (8) we have added the term $\lambda G F^2$ to the second brackets and subtracted it from the first one. The constant λ will be chosen later.

By making the substitution (7) we increased the number of unknowns while the number of equations remained unchanged. We can use this freedom to set the first and the second terms in Eq. (8) to zero (separately):

$$D_x^2\, u \cdot F + (2A_-^2 - \lambda)uF = 0, \tag{9}$$

$$D_x^2\, v \cdot F + (2A_+^2 - \lambda)vF = 0, \tag{10}$$

$$D_x^2\, F \cdot F - \lambda F^2 + 2(u^2 + v^2) = 0, \tag{11}$$

where we let $G = u + iv$ and decomposed the first brackets in Eq. (8) into its real and imaginary parts. Now we look for a solution to the system (9)–(11) as a series

$$u = \epsilon u_1 + \epsilon^2 u_2 + \cdots, \quad v = v_0 + \epsilon v_1 + \epsilon^2 v_2 + \cdots,$$

$$F = 1 + \epsilon F_1 + \epsilon^2 F_2 + \cdots,$$

where ϵ is a formal expansion parameter. An explicit solution will arise if the series truncates at a finite power of ϵ.

Substituting into Eqs. (9)–(11), the order ϵ^0 gives

$$\partial_x^2 v_0 + (2A_+^2 - \lambda)v_0 = 0, \tag{12}$$

$$2v_0^2 - \lambda = 0. \tag{13}$$

We now choose $\lambda = 2A_+^2$. Then, Eqs. (12) and (13) give $v_0 = A$. (Here, and in the rest of the paper, A stands for A_+.) Next, at the order ϵ^1 we obtain

$$(-\partial_x^2 + 4h)u_1 = 0, \tag{14}$$

$$\partial_x^2(AF_1 + v_1) = 0, \tag{15}$$

$$(-\partial_x^2 + 2A^2)F_1 = 2Av_1. \tag{16}$$

Equation (14) yields $u_1 = e^{\theta_1}$, where $\theta_1 = 2h^{1/2}(x - x_1)$ and x_1 is an arbitrary constant. From Eq. (15) we infer $v_1 = -AF_1$. Substituting this into Eq. (16) we get

$$(-\partial_x^2 + 4A^2)F_1 = 0,$$

whence $F_1 = e^{\theta_2}$, where $\theta_2 = 2A(x - x_2)$ and x_2 is another arbitrary constant.

The order ϵ^2 gives three equations:

$$(-D_x^2 + 4h)(u_2 \cdot 1 + u_1 \cdot F_1) = 0, \tag{17}$$

$$D_x^2(v_0 \cdot F_2 + v_1 \cdot F_1 + v_2 \cdot 1) = 0, \tag{18}$$

$$(-D_x^2 + 2A^2)(F_1 \cdot F_1 + 2F_2 \cdot 1) = 2(u_1^2 + v_1^2 + 2v_0 v_2). \tag{19}$$

Substituting for u_1 and F_1 in Eq. (17) this equation becomes

$$(-\partial_x^2 + 4h)u_2 = 4AC_- e^{\theta_1 + \theta_2}, \tag{20}$$

where

$$C_- = A - 2h^{1/2}. \tag{21}$$

Ignoring its homogeneous solution, we get

$$u_2 = -\frac{A - 2\sqrt{h}}{A + 2\sqrt{h}} e^{\theta_1 + \theta_2} \equiv -\frac{C_-}{C_+} e^{\theta_1 + \theta_2}. \tag{22}$$

Next, substituting $v_0 = A$ and $v_1 = -AF_1$ into Eq. (18), we obtain $v_2 = -AF_2$, after which Eq. (19) becomes

$$(-\partial_x^2 + 4A^2)F_2 = e^{2\theta_1},$$

whence, ignoring again the homogeneous solution,

$$F_2 = \frac{1}{4C_+ C_-} e^{2\theta_1}. \tag{23}$$

The singularity arising for $A^2 = 4h$ can be removed by letting $x_1 = \infty$ in θ_1.

To the cubic order in ϵ we get

$$(-D_x^2 + 4h)(u_1 \cdot F_2 + u_2 \cdot F_1 + u_3 \cdot 1) = 0, \tag{24}$$

$$D_x^2(v_0 \cdot F_3 + v_1 \cdot F_2 + v_2 \cdot F_1 + v_3 \cdot 1) = 0, \tag{25}$$

$$(-D_x^2 + 2A^2)(F_3 \cdot 1 + F_2 \cdot F_1) = 2(u_1 u_2 + v_1 v_2 + v_0 v_3). \tag{26}$$

Substituting for u_1, u_2, F_1, and F_2, Eq. (24) becomes

$$(-\partial_x^2 + 4h)u_3 = 0,$$

whence $u_3 = 0$. On the other hand, Eqs. (25) and (26) reduce to the system

$$\partial_x^2(AF_3 + v_3) = 2A\frac{C_-}{C_+} e^{2\theta_1 + \theta_2}, \tag{27}$$

$$(-\partial_x^2 + 4A^2)(v_3 - AF_3) = 0, \tag{28}$$

which gives $v_3 = AF_3$ and

$$F_3 = \frac{C_-}{4C_+^3} e^{2\theta_1 + \theta_2}. \tag{29}$$

Next, the order ϵ^4 yields

$$(-D_x^2 + 4h)(u_4 \cdot 1 + u_2 \cdot F_2 + u_1 \cdot F_3) = 0, \tag{30}$$

$$D_x^2(v_4 \cdot 1 + v_3 \cdot F_1 + v_2 \cdot F_2 + v_1 \cdot F_3 + v_0 \cdot F_4) = 0, \tag{31}$$

$$\begin{aligned}(-D_x^2 + 2A^2)&(2F_4 \cdot 1 + 2F_3 \cdot F_1 + F_2^2) \\ &= 2(u_2^2 + v_2^2 + 2v_1 v_3 + 2v_0 v_4),\end{aligned} \tag{32}$$

where we have taken into account that $u_3 = 0$. Substituting for all the variables in Eq. (30) we obtain

$$(-\partial_x^2 + 4h)u_4 = 0,$$

whence $u_4 = 0$. Equations (31) and (32) reduce to the homogeneous system

$$\partial_x^2(v_4 + AF_4) = 0,$$

$$(-\partial_x^2 + 2A^2)F_4 = 2Av_4.$$

We choose the trivial solution, $v_4 = F_4 = 0$.

The order ϵ^5 is the last order that we have to do "by hand"; in dealing with all higher orders (ϵ^n with $n \geqslant 6$) we will simply invoke the machinery of mathematical induction. To ϵ^5, we get

$$(-D_x^2 + 4h)(u_2 \cdot F_3 + u_5 \cdot 1) = 0, \tag{33}$$

$$D_x^2(v_0 \cdot F_5 + v_2 \cdot F_3 + v_3 \cdot F_2 + v_5 \cdot 1) = 0, \tag{34}$$

$$(-D_x^2 + 2A^2)(F_5 \cdot 1 + F_2 \cdot F_3) = 2(Av_5 + v_2 v_3), \tag{35}$$

where we have taken into account that $u_3 = u_4 = v_4 = F_4 = 0$. Substituting for u_2 and F_3, Eq. (33) gives

$$(-\partial_x^2 + 4h)u_5 = 0,$$

whence $u_5 = 0$. Recalling that $v_2 = -AF_2$ and $v_3 = AF_3$, Eq. (34) becomes

$$\partial_x^2(v_5 + AF_5) = 0,$$

whereas making use of $(-D_x^2 + 2A^2)F_2 \cdot F_3 = 2v_2 v_3$ in Eq. (35) we get

$$(-\partial_x^2 + 2A^2)F_5 - 2Av_5 = 0.$$

The last two equations are satisfied by letting $v_5 = F_5 = 0$.

Finally, we prove that all coefficients u_n, v_n, and F_n with $n \geqslant 6$ are also equal to zero. We assume that $u_{n-1} = v_{n-1} = F_{n-1} = 0$ and show that this entails $u_n = v_n = F_n = 0$. Consider, first, Eqs. (10) and (11). Setting to zero the coefficients of ϵ^n, we obtain

$$D_x^2\left(v_0 \cdot F_n + \sum_{k=1}^{n-1} v_k \cdot F_{n-k} + v_n \cdot 1\right) = 0, \tag{36}$$

$$\begin{aligned}(-D_x^2 + 2A^2)&\left(\sum_{k=1}^{n-1} F_k \cdot F_{n-k} + 2F_n \cdot 1\right) \\ &= 2\sum_{k=1}^{n-1}(u_k u_{n-k} + v_k v_{n-k}) + 4v_0 v_n.\end{aligned} \tag{37}$$

Since $u_{n-k} = 0$ for all $3 \leqslant n-k \leqslant n-1$ and $F_{n-k} = v_{n-k} = 0$ for all $4 \leqslant n-k \leqslant n-1$, the sum involving u_{n-k} in the right-hand side of Eq. (37) begins with $k = n-2$ (rather than with $k=1$), while all sums involving F_{n-k} and v_{n-k} begin with $k = n-3$. On the other hand, since $n-2 \geqslant 4$, all u_k in the sum in Eq. (37) are equal to zero. In a similar way, all v_k and F_k in the sums in Eqs. (36) and (37) equal zero—except v_{n-3} and F_{n-3} for $n = 6$. Therefore, for $n \geqslant 7$ Eqs. (36) and (37) become a pair of homogeneous equations for v_n and F_n; hence we can set $v_n = F_n = 0$. For $n = 6$, we get

$$D_x^2(v_0 \cdot F_6 + v_3 \cdot F_3 + v_6 \cdot 1) = 0,$$

$$(-D_x^2 + 2A^2)(F_3 \cdot F_3 + 2F_6 \cdot 1) = 2(v_3^2 + 2v_0 v_6).$$

Using $v_3 = AF_3$, this also reduces to a homogeneous system for v_6 and F_6.

Finally, Eq. (9) gives, to the order ϵ^n,

$$(-D_x^2 + 4h)\left(\sum_{k=1}^{n-1} u_k \cdot F_{n-k} + u_n \cdot 1\right) = 0.$$

Since $n-3 \geqslant 3$, we have $u_k = 0$ for $k \geqslant n-3$. On the other hand, all $F_{n-k} = 0$ for $k \leqslant n-3$ and so all terms in the sum equal zero. Hence $u_n = 0$. ∎

B. The explicit solution and its interpretation

Thus we have constructed an explicit solution of the form $\psi = (u + iv)F^{-1}$, where u, v, and F are polynomials of e^{θ_1} and e^{θ_2} with real coefficients. It will be shown later that if $C_- < 0$, the solution has a singularity. [We recall that C_- is given by Eq. (21).] For now, we assume that $C_- > 0$ and cast the solution in a more symmetric form.

First of all, the parameter ϵ can be absorbed into e^{θ_1} and e^{θ_2} through the redefinition of the arbitrary constants x_1 and x_2. Next, we define $\chi_1 = \theta_1 - \alpha - \beta$ and $\chi_2 = \theta_2 - 2\beta$, where

$$e^{2\alpha} = 4C_+ C_- = 4(A^2 - 4h),$$

$$e^{2\beta} = \frac{C_+}{C_-} = \frac{A + 2h^{1/2}}{A - 2h^{1/2}}. \tag{38}$$

The new phases χ_1 and χ_2 still involve arbitrary constants x_1 and x_2. We can choose these constants in such a way that, up to an overall translation,

$$\chi_1 = 2h^{1/2}(x - s), \quad \chi_2 = 2A(x + s), \tag{39a}$$

where s is the only free parameter remaining. The numerator and denominator of the solution

$$\psi = (u + iv)F^{-1} \tag{39b}$$

are then given by the following expressions:

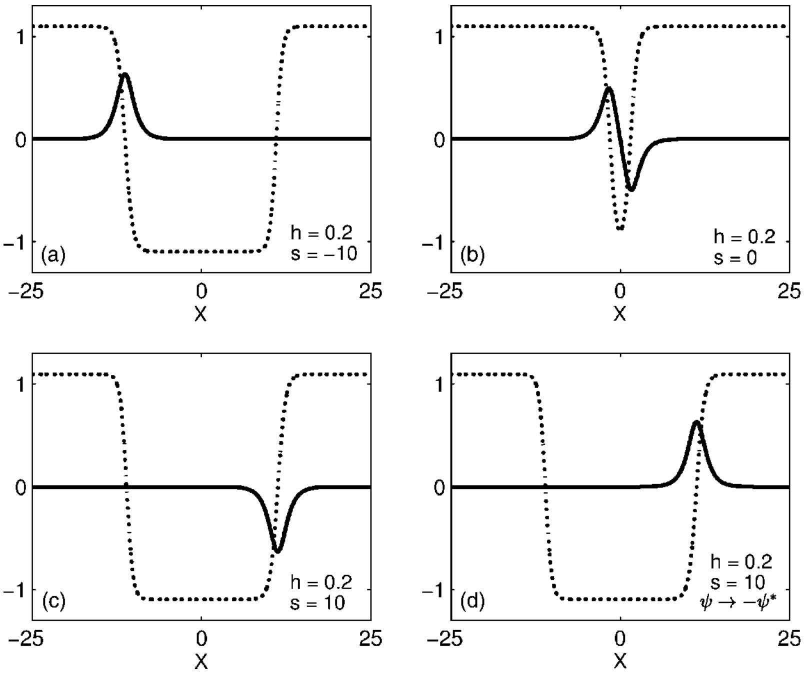

FIG. 2. The bubble solution (39) for $s=$ (a) -10, (b) 0, and (c) 10. The panel (d) corresponds to the bubble of opposite chirality [i.e., with $\psi(x)\to-\psi^*(x)$] ; like (c), it is plotted for $s=10$. The solid line corresponds to the real part, and the dotted line to the imaginary part.

$$u = e^{\alpha+\beta+\chi_1}(1 - e^{\chi_2}), \tag{39c}$$

$$v = A(1 - e^{2\beta+\chi_2} - e^{2\beta+2\chi_1} + e^{2\chi_1+\chi_2}), \tag{39d}$$

$$F = 1 + e^{2\beta+\chi_2} + e^{2\beta+2\chi_1} + e^{2\chi_1+\chi_2}. \tag{39e}$$

Asymptotically, $\psi\to iA$ as both $x\to\infty$ and $x\to-\infty$, and hence Eq. (39) has the form of a bubble. If we perform the reflection $x\to-x$, and at the same time replace s with $-s$, the real and imaginary parts of the solution change according to $u/F\to-u/F$, $v/F\to v/F$. Denoting the solution (39) by $\psi(x;s)$, we therefore have the following symmetry:

$$\psi(-x;-s) = -\psi^*(x;s).$$

This implies, in particular, that the solution (39) with $s=0$ has an odd real and even imaginary part. When $h=\frac{1}{15}$, the $s=0$ solution reproduces the explicit solution that has been known before, Eq. (6). [More precisely, it is equivalent to Eq. (6) with $x\to-x$.]

The solution (39) describes a stationary complex of a Bloch and a Néel wall, with the parameter s characterizing their separation. This is easily seen by examining Eq. (39) in the limit of large s. First, let s be large and positive. For $x\sim s$, we have $e^{\chi_2}\gg e^{2\chi_1}\sim 1$. In this region, the solution (39) reduces to

$$\psi = iA\tanh X - \sqrt{1-3h}\,\mathrm{sech}\,X,$$

with $X=2\sqrt{h}(x-s)-\beta$, which is a right-handed Bloch wall centered at $x_0=s+\frac{1}{2}\beta h^{-1/2}$. In the region $x\sim -s$, we find that Eq. (39) becomes

$$\psi = -iA\tanh[A(x+s)+\beta],$$

which is a Néel wall centered at $x_0=-s-\beta A^{-1}$. On the other hand, if we let s be large and negative, then we find a Néel wall on the right (centered at $x_0=-s+\beta A^{-1}$) and a right-handed Bloch wall on the left (centered at $x_0=s-\frac{1}{2}\beta h^{-1/2}$).

These conclusions are illustrated by Fig. 2 which depicts the real and imaginary parts of Eq. (39) for several representative values of s. We also note that the transformation $x\to-x$, $s\to-s$ which was noted above to change the sign of the real part while leaving the imaginary part intact, simply flips the chirality of the Bloch wall [see Fig. 2(d)].

There is yet another way of seeing that Eq. (39) represents a bound state of two walls, and this time we do not have to assume that s is large. Consider the integral

$$I = \int (A^2 - |\psi|^2)\, dx \tag{40a}$$

which gives an integral measure ("area") of solutions with $|\psi(x)|\to A$ as $|x|\to\infty$. This integral usually has some physical meaning in the nonlinear Schrödinger-based interpretations of Eq. (3). For example, if Eq. (3) is regarded as a stationary reduction of the Landau-Lifshitz equation for the easy-plane ferromagnet in the external magnetic field [21], the integral (40a) gives the total number of magnons in the excited state of the ferromagnet. In the Ginsburg-Landau based applications of Eq. (3), the integral (40a) does not usually have any special meaning, but nevertheless can be used as a scalar characteristic of stationary solutions. Letting $\psi=(u+iv)F^{-1}$, Eq. (40a) becomes

$$I = \int (A^2F^2 - u^2 - v^2)\frac{dx}{F^2}. \tag{40b}$$

For the Bloch wall, this integral equals $I_B=2B$ while for the Néel wall we get $I_N=2A$. As for the solutions (39), the evaluation of the integral (40) becomes trivial if we notice that, due to Eq. (11), the integrand is a total derivative:

$$(A^2F^2 - u^2 - v^2)F^{-2} = \partial_x(\partial_x F/F). \tag{41}$$

Evaluating $\partial_x F/F$ at $x=\pm\infty$ we conclude that each member of the family of bubbles (39) has the "area" $I=2(A+B)$—which is exactly the sum of I_B and I_N.

We conclude this section by mentioning that bound states of a pair of stationary dark solitons have also been found in the (integrable) Manakov system [25].

C. Other solutions

If we had chosen $F_1=-e^{\theta_2}$ instead of e^{θ_2} at the order ϵ, we would have arrived at a different solution. Assuming $h<\frac{1}{3}$ (so that $C_->0$) and defining α, β, χ_1, and χ_2 as in Eqs. (38) and (39a), this solution can be written as Eq. (39b) with

$$u = e^{\alpha+\beta+\chi_1}(1 + e^{\chi_2}), \tag{42a}$$

$$v = A(1 + e^{2\beta+\chi_2} - e^{2\beta+2\chi_1} - e^{2\chi_1+\chi_2}), \tag{42b}$$

$$F = 1 - e^{2\beta+\chi_2} + e^{2\beta+2\chi_1} - e^{2\chi_1+\chi_2}. \tag{42c}$$

It is not difficult to check that this solution is singular. Indeed, the function $F(x)$ is continuous, and has opposite signs at the two infinities: $F\to 1$ as $x\to-\infty$ and $F\to-\infty$ as $x\to\infty$. Therefore, it must pass through zero at least once. Since u cannot vanish for finite x, we have $u/F=\infty$ wherever $F=0$. Although singular solutions are not physically meaningful, there will be some indirect use for them in the next section.

Assume now $h>\frac{1}{3}$. Here we have $C_-=A-2h^{1/2}<0$ and the functions (39) and (42) are no longer solutions, because the parameters α and β, as defined by Eq. (38), are not real. We can, however, define α and β by

$$e^{2\alpha} = -4C_+C_- = 4(4h - A^2),$$

$$e^{2\beta} = -\frac{C_+}{C_-} = \frac{2h^{1/2}+A}{2h^{1/2}-A}, \tag{43}$$

and, instead of Eqs. (39c)–(39e) and (42a)–(42c), arrive at the following two solutions:

$$u = e^{\alpha+\beta+\chi_1}(1 \pm e^{\chi_2}), \tag{44a}$$

$$v = A(1 \mp e^{2\beta+\chi_2} + e^{2\beta+2\chi_1} \mp e^{2\chi_1+\chi_2}), \tag{44b}$$

$$F = 1 \pm e^{2\beta+\chi_2} - e^{2\beta+2\chi_1} \mp e^{2\chi_1+\chi_2}. \tag{44c}$$

If the top sign is chosen in Eq. (44), an argument similar to the one following Eq. (42) demonstrates that F must pass through zero whereas u is strictly positive; once again, u/F is singular. If the bottom sign is chosen, F is positive in the asymptotic regions $x\to\pm\infty$, and this type of argument would not work. However, at the point $x=-s-\beta A^{-1}$, we have $\chi_2=-2\beta$ and so Eq. (44c) (with the bottom sign chosen) becomes

$$F(x) = e^{2\chi_1}(e^{-2\beta} - e^{2\beta}). \tag{45}$$

By Eq. (43), $e^{2\beta}>1$, which means that the contents of the parentheses in Eq. (45) are negative. Thus for this value of x, $F<0$. But since F is positive in both asymptotic regions and is continuous, it follows that F must vanish at least at two points. Since the bottom sign has been chosen, Eq. (44b) implies that $v\geqslant A>0$, so when $F=0$, the quotient v/F is infinite.

Thus, in summary, of all the families of solutions found in this section, only family (39) is regular, and only for $h<\frac{1}{3}$. The other solutions are singular and not physically relevant.

III. "COMPLETENESS" OF THE LIST OF SOLUTIONS

It is convenient to rescale Eq. (3) so that the asymptotic values are equal to ±1. Letting $\psi(x)=iA\Psi(Ax)$, Eq. (3) becomes

$$\Psi_{xx} + \frac{2}{A^2}\Psi - 2|\Psi|^2\Psi + 2\left(1-\frac{1}{A^2}\right)\Psi^* = 0. \tag{46}$$

We will demonstrate that solutions (4), (5), and (39) (when suitably rescaled) are the *only* bounded solutions of Eq. (46) that asymptotically approach one of the stable flat backgrounds $\Psi=\pm1$ as $x\to-\infty$. We will consider solutions of Eq. (46) as trajectories in a four-dimensional phase space and show that for any direction in which a trajectory can leave the fixed point (representing the flat background), we already have a solution leaving in that direction. By uniqueness, no other trajectories will be allowed to exist. For definiteness, we confine ourselves to the case of trajectories that approach $\Psi=1$ as $x\to-\infty$; a similar argument is valid for solutions approaching $\Psi=-1$.

For convenience of presentation, we start by listing all known solutions to the rescaled equation (46). Besides the solutions derived in the previous section, we also include solutions which are the unbounded counterparts of the isolated Bloch and Néel walls. In all of these, we make explicit the translational invariance of Eq. (46).

For $h<\frac{1}{3}$, apart from the flat background $\Psi_0=1$, the list consists of the Néel wall $\Psi_\mathrm{N}=-\tanh(x-x_0)$ and its singular counterpart

$$\tilde{\Psi}_\mathrm{N} = -\coth(x-x_0); \tag{47}$$

the Bloch walls

$$\Psi_\mathrm{B} = -\tanh[B(x-x_0)] \pm iC\,\mathrm{sech}[B(x-x_0)], \tag{48}$$

where $B=2(A^2-1)^{1/2}A^{-1}$ and $C=(4-3A^2)^{1/2}A^{-1}$, and, finally, the Bloch-Néel complex (the bubble) which we write together with its unbounded counterpart:

$$\Psi_\mathrm{BN} = \frac{u+i\sigma v}{1 \pm e^{2\beta+\chi_2} + e^{2\beta+2\chi_1} \pm e^{2\chi_1+\chi_2}}. \tag{49a}$$

Here

$$u = 1 \mp e^{2\beta+\chi_2} - e^{2\beta+2\chi_1} \pm e^{2\chi_1+\chi_2}, \tag{49b}$$

$$v = 2(1+B)e^{\chi_1}(1 \mp e^{\chi_2}), \tag{49c}$$

$e^{2\beta}=(1+B)(1-B)^{-1}$, $\chi_1=B(x-x_0-s)$, and $\chi_2=2(x-x_0+s)$. The real parameters s and x_0 are arbitrary; s characterizes the separation distance and x_0 describes uniform translations. The sign factor $\sigma = \pm 1$ determines the chirality of the Bloch wall bound in the complex. [Note that σ is not correlated with the sign factor in Eqs. (49b) and (49c).] If the top sign is chosen, Eq. (49) gives the rescaled regular solution (39). If the bottom sign is chosen, we have the rescaled singular solution (42).

For $h > \frac{1}{3}$, the functions Ψ_0, Ψ_{N}, and $\tilde{\Psi}_{\mathrm{N}}$ defined in the previous paragraph remain as solutions. Equations (48) and (49), on the other hand, are no longer solutions. Equation (48) is replaced by

$$\tilde{\Psi}_{\mathrm{B}} = -\coth[B(x-x_0)] \pm iC\,\mathrm{cosech}[B(x-x_0)], \tag{50}$$

where B stays as it was previously defined whereas C is now given by $C = (3A^2-4)^{1/2}A^{-1}$. Equation (49) is replaced by

$$\tilde{\Psi}_{\mathrm{BN}} = \frac{u + i\sigma v}{1 \mp e^{2\beta+\chi_2} - e^{2\beta+2\chi_1} \pm e^{2\chi_1+\chi_2}}, \tag{51a}$$

where

$$u = 1 \pm e^{2\beta+\chi_2} + e^{2\beta 2\chi_1} \pm e^{2\chi_1+\chi_2}, \tag{51b}$$

$$v = 2(1+B)e^{\chi_1}(1 \mp e^{\chi_2}), \tag{51c}$$

and all parameters are defined as for Eq. (49) except $e^{2\beta} = (B+1)(B-1)^{-1}$. Both (50) and (51) are singular solutions [for all choices of signs in Eq. (51)].

We now turn to Eq. (46) and write it as a dynamical system for a particle on the plane:

$$\eta_{xx} - 2(\eta^2 + \xi^2)\eta + 2\eta = 0, \tag{52a}$$

$$\xi_{xx} - 2(\eta^2 + \xi^2)\xi + (2 - B^2)\xi = 0, \tag{52b}$$

where $\Psi = \eta + i\xi$ with η and ξ real, and $B^2 = 4(A^2-1)A^{-2}$ [as defined for Eq. (48)]. This is a Hamiltonian system, with the Hamiltonian

$$\mathcal{H} = \frac{1}{2}[\eta_x^2 + \xi_x^2 - (\eta^2 + \xi^2 - 1)^2 - B^2\xi^2]. \tag{53}$$

There is also a second, independent, integral of motion

$$\mathcal{I} = (\xi\eta_x - \eta\xi_x)^2 + B^2[\eta_x^2 - \eta^2\xi^2 - (\eta^2 - 1)^2]. \tag{54}$$

[In the derivation of Eq. (54) we were guided by the results of Ref. [26] which considers a similar system.] Thus all trajectories of the system (52) are confined to lie on a two-dimensional surface, defined by the constraints (53) and (54).

We consider trajectories that flow out of the fixed point $(\eta, \xi, \eta_x, \xi_x) = (1,0,0,0)$. (On these trajectories, the conserved quantities obey $\mathcal{H} = 0 = \mathcal{I}$.) This fixed point is a saddle, with two positive and two negative real eigenvalues. Thus in a neighborhood of the fixed point on the unstable manifold, η_x will be either positive or negative (as will ξ_x), while oscillatory behavior is not possible. In this neighborhood, the sign of η_x will obviously be the same as the sign of $(\eta - 1)$; similarly, the sign of ξ_x will be the same as that of $(\xi - 0)$. As for the *magnitudes* of η_x and ξ_x, these are determined by η and ξ via the constraints $\mathcal{H} = 0$, $\mathcal{I} = 0$. Thus the variables η and ξ uniquely determine η_x and ξ_x in the vicinity of the fixed point and therefore provide coordinates on the local unstable manifold.

In the vicinity of the fixed point, Eqs. (52a) and (52b) can be written as

$$\eta_{xx} = 4(\eta - 1) + 2\xi^2, \tag{55a}$$

$$\xi_{xx} = B^2\xi. \tag{55b}$$

(Note that we cannot drop the ξ^2 term in the top equation here, as the equation does not include any term linear in ξ.) The solution to Eq. (55) satisfying $\eta \to 1, \xi \to 0$ as $x \to -\infty$ is

$$\eta = 1 + \mathcal{M}e^{2x} - \frac{\mathcal{N}^2}{2(1-B^2)}e^{2Bx}, \tag{56a}$$

$$\xi = \mathcal{N}e^{Bx}. \tag{56b}$$

The real constants $\mathcal{M}$ and $\mathcal{N}$ are arbitrary and provide a parametrization of the local unstable manifold: each pair $(\mathcal{M}, \mathcal{N})$ defines a trajectory on the manifold, and the other way around—for each point $(\eta, \xi, \eta_x, \xi_x)$ on the manifold, we can find a pair $(\mathcal{M}, \mathcal{N})$ such that there is a trajectory connecting $(\eta, \xi, \eta_x, \xi_x)$ to $(1,0,0,0)$ which is described by Eq. (56). Indeed, for the given pair of coordinates (η_0, ξ_0) we can solve Eq. (56) to get

$$\mathcal{M}e^{2x_0} = \eta_0 + \frac{\xi_0^2}{2(1-B^2)} - 1, \quad \mathcal{N}e^{Bx_0} = \xi_0. \tag{57}$$

Due to the translational invariance of the system (52) , replacing x with $x - x_0$ in Eq. (56) simply furnishes a different parametrization of the same trajectory. Hence the required pair $(\mathcal{M}, \mathcal{N})$ is obtained, e.g., by setting $x_0 = 0$ in Eq. (57).

Thus, if we factor the translation invariance out, there is a one-to-one correspondence between $(\mathcal{M}, \mathcal{N})$ and (η, ξ) on the local unstable manifold. We now show that for every pair $(\mathcal{M}, \mathcal{N})$ (with $-\infty < \mathcal{M}, \mathcal{N} < \infty$) we already have an explicit solution in our list, regular or singular, with the asymptotics (56). This will imply that our list is complete and no other solutions with this asymptotic behavior can exist.

First of all, the solution corresponding to $(\mathcal{M}, \mathcal{N}) = (0,0)$ is the flat background $\Psi_0 = 1$. Keeping $\mathcal{N} = 0$, we have two possibilities (for all h): in the case $\mathcal{M} > 0$, Eqs. (56) give the asymptotics of the solution $\tilde{\Psi}_{\mathrm{N}}$, Eq. (47), with x_0 defined by $\mathcal{M} = 2e^{-2x_0}$; similarly, the case $\mathcal{M} < 0$ corresponds to the Néel wall Ψ_{N}, with $\mathcal{M} = -2e^{-2x_0}$.

Now let $\mathcal{M} = 0$ while $\mathcal{N} \neq 0$. Here the two cases $h < \frac{1}{3}$ and $h > \frac{1}{3}$ have to be considered separately. For $h < \frac{1}{3}$, the pair $(0, \mathcal{N})$ corresponds to the Bloch wall (48), with

$$|\mathcal{N}| = 2(1-3h)^{1/2}A^{-1}e^{-Bx_0}. \tag{58}$$

The sign of $\mathcal{N}$ is arbitrary and determines the chirality of the soliton. If $h > \frac{1}{3}$, we recover the solution $\tilde{\Psi}_{\mathrm{B}}$; this solution also occurs with two different chiralities, depending on the sign of $\mathcal{N}$.

Finally, we let $\mathcal{M} \neq 0$, $\mathcal{N} \neq 0$. Assume, first, that $h < \frac{1}{3}$ and consider the solution Ψ_{BN}, Eq. (49). This solution comes in two chiralities $\sigma = \pm 1$. We let $\mathcal{N} > 0$; this selects one of the chiralities $(\sigma = +1)$. (The case $\mathcal{N} < 0$ can be considered in a similar way.) Comparing the asymptotic behavior of the solution (49) to Eqs. (56), we find

$$\mathcal{M} = \mp 2e^{2(\beta - x_0 + s)}$$

and

$$\mathcal{N} = 2(1+B)e^{-B(x_0+s)}.$$

Thus $\mathcal{M}<0$ corresponds to the (regular) Bloch-Néel bound state [top sign in Eq. (49)], and $\mathcal{M}>0$ to its singular counterpart [bottom sign in Eq. (49)]. For $h>\frac{1}{3}$, a similar consideration involving Eq. (51) demonstrates that any pair $(\mathcal{M},\mathcal{N})$ with $\mathcal{M}\neq 0$, $\mathcal{N}\neq 0$ corresponds to a known solution as well. ■

IV. CONCLUDING REMARKS

In this paper, we have explicitly constructed bound states of a pair of domain walls in the resonantly forced Ginsburg-Landau equation. The constructed solutions represent domains (of arbitrary length) that are 180° out of phase with the background field. We have also demonstrated that stationary bound states of more than two dark solitons cannot exist. [We have, in fact, proved that the solutions presented in this paper are the *only* solutions to the reduced scalar equation (3) which approach the stable background at infinity.]

Apart from the Ginsburg-Landau equation, Eq. (3) occurred as a stationary reduction of several conservative nonlinear evolution equations modeling some other physical situations. We have already mentioned the MSTB model of field theory [18,19]; the corresponding equation of motion can be written as the complex ϕ^4 equation with the broken U(1) symmetry:

$$\frac{1}{2}\psi_{tt} - \frac{1}{2}\psi_{xx} + |\psi|^2\psi - \psi + h\psi^* = 0. \tag{59}$$

(For the current status of the MSTB and related theories, see Ref. [27].) In addition, Eq. (3) describes stationary solutions of the parametrically driven nonlinear Schrödinger equation:

$$i\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi + \psi - h\psi^* = 0. \tag{60}$$

In fluid dynamics, Eq. (60) governs the amplitude of the oscillation of the water surface in a vertically vibrated channel with large width-to-depth ratio [17,23]. [If, conversely, the channel is deep and narrow, Eq. (60) is still valid, but with the opposite sign in front of the nonlinear term.] The same Eq. (60) arises as an amplitude equation for the upper cutoff mode in the parametrically driven nonlinear lattices [28]. It was also derived for the doubly resonant $\chi^{(2)}$ optical parametric oscillator in the limit of large second-harmonic detuning [22]. In all of these cases, the term $h\psi^*$ represents parametric pumping of some sort. Finally, Eq. (60) describes magnetization waves in a quasi-one-dimensional ferromagnet with a weakly anisotropic easy plane, in a perpendicular stationary magnetic field [21]. In the magnetic context, the $h\psi^*$ term accounts for the anisotropy of the ferromagnetic crystal.

The two-soliton bound state solution we have obtained in this paper admits a transparent interpretation in each of the above physical situations. In the context of vibrated water troughs and chains of coupled pendula, the Bloch-Néel complex describes a patch oscillating 180° out of phase with the rest of the chain or channel. A similar interpretation arises in optical parametric oscillators; there, a bound state of two dark solitons represents a localized fundamental field domain with a 180° phase difference from the rest of the cavity. In the context of ferromagnetism, the stationary complex corresponds to a magnetic bubble, i.e., an "island" of one stable phase in the "sea" of the other one.

The question of stability of the two-soliton bound state is beyond the scope of our current investigation. The answer will obviously depend on whether Eq. (3) is considered as a stationary reduction of the Ginsburg-Landau, Klein-Gordon, or nonlinear Schrödinger (NLS) equation [i.e., Eq. (2), (59), or (60)]. To illustrate the model dependence of stability properties of one and the same solution, it is instructive to bring up the example of a free-standing Néel wall, Eq. (4). Let, for example, $h<\frac{1}{3}$. If the Néel wall is considered as a stationary solution of the Ginsburg-Landau equation (2) or of the Klein-Gordon equation (59), then it is found to be unstable while the Bloch wall is stable [3,8,20,29]. On the contrary, *both* Bloch and Néel walls are stable [21] when considered as stationary solutions of the parametrically driven NLS equation, Eq. (60). We are planning to return to the issue of stability of the bound states in future publications.

Finally, it is appropriate to mention that Eq. (3) appeared in one more optical context, namely, that of birefringent optical fibers. The vector nonlinear Schrödinger equation for pulses traveling in a birefringent fiber was derived by Menyuk [30]:

$$i\left(\frac{\partial E_1}{\partial t} + \delta\frac{\partial E_1}{\partial x}\right) + \frac{1}{2}\frac{\partial^2 E_1}{\partial x^2} - \left(|E_1|^2 + \frac{2}{3}|E_2|^2\right) \times E_1 - \frac{1}{3}E_2^2E_1^*e^{-4iht} = 0, \tag{61a}$$

$$i\left(\frac{\partial E_2}{\partial t} - \delta\frac{\partial E_2}{\partial x}\right) + \frac{1}{2}\frac{\partial^2 E_2}{\partial x^2} - \left(\frac{2}{3}|E_1|^2 + |E_2|^2\right) \times E_2 - \frac{1}{3}E_1^2E_2^*e^{4iht} = 0. \tag{61b}$$

Here δ is proportional to the difference of group velocities of the fast and slow linearly polarized modes (whose envelopes are described by E_1 and E_2), and h measures the mismatch of the corresponding propagation constants. Equations (61a) and (61b) are written in a frame moving with the average of the group velocities; the choice of coefficients corresponds to the fiber in the regime of normal dispersion. If one assumes that the difference of the two group velocities is so small that it can be neglected, while the difference of the propagation constants is nonnegligible (though possibly small), then the substitution

$$E_1 = Ue^{-i(h+1)t}, \quad E_2 = Ve^{i(h-1)t}$$

takes Eq. (61) to

$$i\frac{\partial U}{\partial t}+\frac{1}{2}\frac{\partial^2 U}{\partial x^2}-\left(|U|^2+\frac{2}{3}|V|^2\right)U-\frac{1}{3}V^2U^*+(1+h)U=0, \tag{62a}$$

$$i\frac{\partial V}{\partial t}+\frac{1}{2}\frac{\partial^2 V}{\partial x^2}-\left(\frac{2}{3}|U|^2+|V|^2\right)V-\frac{1}{3}U^2V^*+(1-h)V=0. \tag{62b}$$

The above assumption ($\delta=0, h\neq 0$) can be justified in the case of the *bright* solitons, i.e., in the anomalous dispersion regime. In that case, Blow, Doran, and Wood demonstrated the existence of bound pairs of bright solitons where each soliton is polarized along a different birefringence axis [31]. Later, these solutions were explicitly constructed by Tratnik and Sipe [32]. Whether the assumption $\delta=0$, $h\neq 0$ can be justified in the case of the *dark* solitons is an open question which is beyond the scope of our investigation. Here, we simply note that for time-independent real fields U and V, Eqs. (62) reduce to our Eq. (3) with $\psi=U+iV$. This fact was used by Christodoulides [33] who obtained the Bloch and Néel walls for a weakly birefringent fiber (under the above assumption). Our solution (39) is a bound state of the "dark" and "bright-dark" vector solitons of Christodoulides.

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