Collisional cross sections and momentum distributions in astrophysical plasmas: Dynamics and statistical mechanics link

F. Ferro[1,2,*] and P. Quarati[1,3,†]
[1] *Dipartimento di Fisica, Politecnico di Torino, I-10129 Torino, Italy*
[2] *INFN–Sezione di Torino, I-10125 Torino, Italy*
[3] *INFN–Sezione di Cagliari, I-09042 Monserrato, Italy*
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We show that in stellar core plasmas, the one-body momentum distribution function is strongly dependent, at least in the high velocity regime, on the microscopic dynamics of ion elastic collisions and therefore on the effective collisional cross sections if a random force field is present. We take into account two cross sections describing ion-dipole and ion-ion screened interactions. Furthermore, we introduce a third unusual cross section to link statistical distributions and a quantum effect originated by the energy-momentum uncertainty owing to many-body collisions. We also propose a possible physical interpretation in terms of a tidal-like force. We show that each collisional cross section gives rise to a slight peculiar correction on the Maxwellian momentum distribution function in a well defined velocity interval. We also find a possible link between microscopic dynamics of ions and statistical mechanics in interpreting our results in the framework of nonextensive statistical mechanics.

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I. INTRODUCTION

In a dense neutral plasma such as, for instance, the astrophysical plasma of a stellar core with mean potential energy of about the same order of magnitude as thermal energy [1,2], a random microscopic electric field (usually called an electric microfield) arises at any spatial point. Its origin does not rely on correlations between particles in the plasma, and indeed it is also present in ideal plasmas of statistically independent particles; rather, it is due to local thermal fluctuations in the position of ions [3]. Commonly, the microfield strength is not negligible, being on the same order of magnitude as the Coulomb field of a unit charge at the characteristic Wigner-Seitz radius. As Romanovsky and Ebeling [4] pointed out, the dynamic enhancement of nuclear fusion rates due to electric random fields is large only in very dense stars like white dwarfs; on the contrary, its importance inside the Sun's core is presently believed to be limited.

We show that a random field of generic nature (random electric or magnetic microfields belong, among others, to this category) may play a crucial role, as it influences the upper tail of the one-body stationary momentum distribution function of ions in a dense neutral stellar plasma, leading to slight deviations from a pure Maxwellian distribution. Furthermore, different elastic collisional cross sections among interacting ions may significantly influence the tail of the distribution, and each one provides corrections in a characteristic velocity range only.

We wish to stress here that in astrophysical plasmas, many different collisional processes can be active at the same time, provided that we consider different velocity intervals. Besides, for instance, the usual pure Coulomb interaction (described by the well-known Rutherford cross-section formula), many different screening potentials have been proposed for many years in order to provide effective models for several astrophysical conditions [5]. Therefore, screening and many-body effects, whose importance relies on the fact that they strongly enhance thermonuclear reaction rates, are also important, lying on a kinetic framework, as they modify the collisional cross section between ions [6].

In this paper, we analytically derive the one-body distribution function of momentum starting from a kinetic equation in which we set three cross sections of interest in dense and weakly nonideal plasmas. Our calculations are based on the existence of a random force field F, which can be justified either by the theory of electric microfields (as outlined in Refs. [3,4]) or the theory of dissipative random forces in the approach of the Langevin equation for Brownian motion (see, for example, Ref. [7]), and can be originated from density fluctuations in the plasma. We are comforted in this line of research also by Einstein's criticism of the Boltzmann probability relation, based on the argument that statistical mechanics may only be justified in terms of classical or quantum microscopic dynamics [8].

In the first part of our paper (Secs. II, III, and IV), we use a kinetic approach to describe the motion of particles submitted to a generic random force field with finite relaxation time (i.e., not δ-correlated). The stationary solution can be expressed in terms of collisional cross sections and collisional frequencies; therefore, we may establish a link between the type of particle collisions and the form of stationary distribution functions that can differ from the equilibrium Maxwellian function. We then define parameter q characterizing the deformation factor of our distribution and we calculate it in terms of known physical quantities. Interpreting this deformation on the physical ground of nonextensive statistical mechanics and Tsallis statistics as a special case [9], parameter q can be expressed in terms of dynamical quanti-

*Electronic address: fabrizio.ferro@polito.it
†Electronic address: piero.quarati@polito.it

ties such as cross section, ion-correlation parameter, and plasma parameter. A dynamical realization of nonextensive statistics and, in general, of superstatistics has been completed by Beck [10] by using stochastic differential equations with spatiotemporally fluctuating parameters.

In the second part (Sec. V), we consider the quantum energy-momentum uncertainty due to contemporaneous interaction among many particles of plasma. We use the Kadanoff-Baym ansatz [11] and follow the approach of Galitskiĭ and Yakimets [12], which leads to the appearance of a power-law tail in the momentum distribution function. This function shows an enhanced tail at high momentum depending on collision frequency and collisional cross section. As a consequence, we show a link between nonextensive statistical mechanics and dynamics, by highlighting the type of microscopic elastic collisions acting among particles and how collision frequency is related to momentum distributions. Then we investigate the cross section that reproduces the nonextensive distribution function, limiting ourselves, for simplicity, to the case $q>1$, and we show that the corresponding interaction looks like a tidal-like force superimposed on the two-body attraction, giving a collision cross section $\sigma(\varepsilon_p)\sim\sqrt{\varepsilon_p}$, where ε_p is the relative kinetic energy. The quantum approach that we follow in our discussion relies on equilibrium conditions of the system; therefore, our final result is a real equilibrium (not metastable or stationary) distribution function that differs from the equilibrium Maxwell-Boltzmann distribution. The existence of such an unusual equilibrium function is due to the quantum uncertainty as briefly described in Sec. V.

II. KINETIC EQUATION UNDER A GENERALIZED RANDOM FORCE

A kinetic equation describing electrons in a plasma under an external electric field, in which collisions between electrons and neutral atoms are present, was derived by Chapman and Cowling [13], by Spitzer [14] and by Golant, Žilinskij, and Sacharov [15]. They express the actual one-body velocity distribution function as a formal series,

$$\tilde{f}(\mathbf{v}) = f + f_1 + f_2 + \cdots ,$$

where $f=f(v)$ is the isotropic component (with $v=|\mathbf{v}|$), while f_1, f_2, and so on describe next orders of anisotropy induced by the external field. In addition, $f(v)$ may be only a slight perturbation of the Maxwellian distribution function.

Here, we adopt their equation in order to derive the momentum stationary distribution of ions, but we replace the external electric field with a generalized random force F and focus on isotropic function f only. The elastic collisional cross sections that we are considering describe the interaction among ions and among ions and electric dipoles of polarized neutral compounds of the Wigner-Seitz spheres. All these cross sections will be discussed in the following sections.

Thus, considering the plasma component consisting of ions of mass m, the kinetic equation reads [15]

$$\pm\frac{2}{3}\frac{F^2}{\mu^2\nu^2}\frac{df}{dv} + \kappa\left(vf + \frac{k_BT}{\mu}\frac{df}{dv}\right) = 0, \tag{1}$$

where v is the modulus of the relative velocity between two ions, μ is their reduced mass, $\nu(v)$ is the collisional frequency, k_BT is the thermal energy, and κ, which is defined as

$$\kappa = 2\frac{\mu^2}{m^2},$$

is the energy transfer coefficient (or an average value of it).

Let us discuss briefly the origin of the double sign in the term containing the random force F in Eq. (1). The quantity

$$D_F = \pm\frac{2}{3}\frac{F^2}{\mu^2\nu^2}$$

is the perturbation on the diffusion coefficient of the system due to force F, and the corresponding particle flux is given by

$$J_F = \mp D_F n\frac{df}{dv},$$

n being the ion particle density of the plasma. If F were an electric microfield, i.e., $F=eE$ (with e equal to the electric charge of one ion), the corresponding sign would be positive, thus enhancing the actual diffusivity of the system, while in the opposite case total diffusivity would drop. Therefore, we introduce the double sign since we wish to deal with the most general situation, in which either subdiffusivity or superdiffusivity may be significant.

The analytical solution of Eq. (1) reads

$$f(v) = f(0)\exp\left[-\int_0^v dv'\frac{\mu v'}{k_BT \pm \frac{2}{3}\frac{F^2}{\mu\kappa\nu^2}}\right], \tag{2}$$

where the constant $f(0)$ should be calculated through the normalization condition

$$4\pi\int_0^{+\infty} dv' v'^2 f(v') = 1.$$

Let us now define a characteristic strength of the generalized random field F as

$$F_C = \nu\sqrt{\kappa\mu k_BT}.$$

Then, if the condition

$$F^2 \ll F_C^2 \tag{3}$$

holds, i.e., if the random force is negligible, Eq. (2) gives the Maxwellian distribution function at temperature T. The central point is that in this case the Maxwellian function is a solution of the kinetic equation (1) regardless of any assumption about collisional frequency ν of the plasma.

On the contrary, if the condition in Eq. (3) fails, the form of the solution $f(v)$ is determined by the explicit dependence of the collisional frequency ν on relative velocity v. The $\nu(v)$ frequency is itself a function of the collisional cross section $\sigma(v)$, as

$$\nu(v) = nv\sigma(v). \tag{4}$$

Thus, in this case, Eq. (2) leads to a Maxwellian provided that we choose the $\sigma(v)=\alpha_0 v^{-1}$ cross section (α_0 being a suitable constant), and that we renormalize the temperature of the plasma in the following fashion:

$$k_B T_{\text{eff}} = k_B T \pm \frac{2}{3}\frac{F^2}{\kappa\mu n^2 \alpha_0^2}, \tag{5}$$

where T_{eff} is an effective temperature which will be of central importance in our subsequent discussion.

III. ANALYTICAL SOLUTION OF THE KINETIC EQUATION

In the following, we discuss the effect of three different cross sections, σ_0, σ_1, and σ_2, whose explicit functional dependence on relative velocity, together with that of their collisional frequencies, is, respectively,

$$\sigma_0(v) = \alpha_0 v^{-1},$$

$$\nu_0(v) = n\alpha_0, \tag{6}$$

$$\sigma_1(v) = \alpha_1,$$

$$\nu_1(v) = n\alpha_1 v, \tag{7}$$

$$\sigma_2(v) = \alpha_2 v,$$

$$\nu_2(v) = n\alpha_2 v^2, \tag{8}$$

where α_0, α_1, and α_2 are dimensional constants. In Sec. IV, we shall discuss the physical meaning of the previous cross sections in dense astrophysical plasmas.

We state the hypothesis of absence of interference between the three collision types, namely we assume that total collisional frequency ν could be cast in the following approximate fashion:

$$\nu^2 = \nu_0^2 + \nu_1^2 + \nu_2^2,$$

because different types of collision act significantly only in separate velocity intervals as is evident from the functional dependencies reported in Eqs. (6), (7), and (8).

Let us now express the solution of Eq. (2) as

$$f(v) = f(0)\exp[-I(v)], \tag{9}$$

where we have defined the integral function

$$I(v) = \int_0^v \frac{\mu v' dv'}{k_B T \pm \frac{2}{3}\frac{F^2}{\kappa\mu n^2\alpha_0^2}\frac{1}{1 + c_1 v'^2 + c_2 v'^4}}$$

$$= \frac{\mu}{k_B T}\int_0^v \frac{v' dv'}{1 + \tau\frac{1}{1 + c_1 v'^2 + c_2 v'^4}}, \tag{10}$$

with $c_1=(\alpha_1/\alpha_0)^2, c_2=(\alpha_2/\alpha_0)^2$, and $\tau=T_{\text{eff}}/T-1$, according to Eqs. (5), (6), (7), and (8).

From Eq. (10), we immediately obtain

$$I(v) = \frac{\mu v^2}{2k_B T} - \frac{\mu}{2k_B T}\tau I_1(v), \tag{11}$$

where

$$I_1(v) = \int_0^{v^2} du \frac{1}{c_2 u^2 + c_1 u + \tau + 1}. \tag{12}$$

Let us now define the following parameter:

$$K = -\frac{c_1^2}{4c_2} + \tau + 1,$$

whose sign is physically relevant, as we are about to show.

If $K<0$, Eq. (12) gives, apart from an unimportant numerical term,

$$I_1 = \frac{1}{2\sqrt{|K|c_2}}\left[\ln\left(\frac{2c_2v^2 + c_1 - 2\sqrt{|K|c_2}}{2c_2v^2 + c_1 + 2\sqrt{|K|c_2}}\right) + \text{const}\right],$$

which in turn, through Eqs. (11) and (9), yields our first result (for $K<0$),

$$f(v) \propto \exp\left(-\frac{\mu v^2}{2k_B T}\right)\left(\frac{2c_2v^2 + c_1 - 2\sqrt{|K|c_2}}{2c_2v^2 + c_1 + 2\sqrt{|K|c_2}}\right)^{\mu\tau/4k_B T\sqrt{|K|c_2}}.$$

In this case, cross section σ_1 dominates and the generalized random force field F does not play any role in the region of interest for fusion reaction rate calculations in astrophysical plasmas, as the perturbation from the Maxwellian function vanishes in the limit $v \to +\infty$ (nevertheless, it can be of interest in studies of some atomic processes such as radiative recombination, whose cross section increases as v goes to zero and which therefore has rates sensibly modified by slight corrections at low velocity).

As far as astrophysical plasmas are concerned, a more interesting physical situation occurs when $K>0$, namely, if the condition

$$\frac{T_{\text{eff}}}{T} > \frac{\nu_1^4}{4\nu_0^2\nu_2^2} = \frac{\alpha_1^4}{4\alpha_0^2\alpha_2^2} \tag{13}$$

holds. Condition (13) is fulfilled providing force field F is

$$F > n\sqrt{\frac{3}{2}\kappa\mu k_B T\left(\frac{\alpha_1^4 - 4\alpha_0^2\alpha_2^2}{\alpha_2^2}\right)}, \tag{14}$$

in the case of superdiffusivity, or instead

$$F < n\sqrt{\frac{3}{2}\kappa\mu k_B T\left(\frac{4\alpha_0^2\alpha_2^2 - \alpha_1^4}{\alpha_2^2}\right)}, \tag{15}$$

when considering subdiffusivity.

From Eq. (12), we obtain

$$I_1(v) = \frac{1}{K}\int_0^{v^2} du\left[\left(\sqrt{\frac{c_2}{K}}u + \frac{c_1}{2\sqrt{Kc_2}}\right)^2 + 1\right]^{-1}$$

$$= \frac{1}{\sqrt{Kc_2}}\left[\arctan\left(\sqrt{\frac{c_2}{K}}v^2 + \frac{c_1}{2\sqrt{Kc_2}}\right) - \arctan\frac{c_1}{2\sqrt{Kc_2}}\right]. \tag{16}$$

Starting from Eqs. (11), (12), and (16), we can express $I(v)$ as a formal series of powers of v^2,

$$I(v)=\frac{\mu v^2}{2k_BT_{\mathrm{eff}}}+\delta\left(\frac{\mu v^2}{2k_BT_{\mathrm{eff}}}\right)^2+\gamma\left(\frac{\mu v^2}{2k_BT_{\mathrm{eff}}}\right)^3+\cdots,$$

where

$$\delta=\pm\frac{2}{3}\frac{F^2}{\kappa\mu^2n^2}\frac{\alpha_1^2}{\alpha_0^4},$$

and

$$\gamma=\pm\frac{8}{9}\frac{F^2k_BT}{\kappa\mu^3n^2}\frac{\alpha_2^2}{\alpha_0^4}\left(1-\frac{\alpha_1^4}{\alpha_0^2\alpha_2^2}\right)+\frac{16}{27}\frac{F^4}{\kappa^2\mu^4n^4}\frac{\alpha_2^2}{\alpha_0^6},$$

both being $|\delta|,|\gamma|\ll 1$.

Therefore, the final form of the one-body distribution function under generalized random fields reads

$$f(v)\propto\exp\left[-\frac{\mu v^2}{2k_BT_{\mathrm{eff}}}\right]\exp\left[-\delta\left(\frac{\mu v^2}{2k_BT_{\mathrm{eff}}}\right)^2\right]\times\exp\left[-\gamma\left(\frac{\mu v^2}{2k_BT_{\mathrm{eff}}}\right)^3\right],$$

which corresponds to an energy probability factor

$$f(\varepsilon_p)\propto\exp\left[-\frac{\varepsilon_p}{k_BT_{\mathrm{eff}}}\right]\exp\left[-\delta\left(\frac{\varepsilon_p}{k_BT_{\mathrm{eff}}}\right)^2\right]\times\exp\left[-\gamma\left(\frac{\varepsilon_p}{k_BT_{\mathrm{eff}}}\right)^3\right],\tag{17}$$

where $\varepsilon_p=p^2/(2\mu)$ is the center-of-mass kinetic energy, given linear momentum $\mathbf{p}=\mu\mathbf{v}$.

It is noteworthy that our result in Eq. (17) may be related, at least for small deformations, to the nonextensive distribution function at the same effective temperature T_{eff} [9],

$$f(\varepsilon_p)\propto\left[1-(1-q)\frac{\varepsilon_p}{k_BT_{\mathrm{eff}}}\right]^{1/(1-q)},\tag{18}$$

where q is called the entropic parameter. As can be straightforwardly shown after some manipulations, in the low deformation limit $(q-1)\varepsilon_p/(k_BT_{\mathrm{eff}})\to 0$, Eq. (18) reduces to Eq. (17), provided that $\delta=(1-q)/2$. Thus, this condition establishes a link between the entropic parameter q and our parameter δ which comes from microfield strength and cross sections. We point out that in the same limit, other distributions of generalized statistics also have the same behavior, as explained in Ref. [23].

We recall that in the recent past we have shown that if the generalized random force is due to an electric microfield distribution, parameter δ can be related to the nonextensive (Tsallis) entropic parameter q and the following analytical expression can be derived:

$$\delta=\frac{1-q}{2}=12\Gamma^2\alpha^4,$$

where Γ is the plasma parameter and α is a dimensionless parameter accounting for ion correlations in the ion-sphere model $(0.4<\alpha<1)$ [16].

From Eq. (17) it follows that there exist three different intervals of relative velocity in which the evaluated corrections due to the random field are sufficiently large to be noteworthy. First of all, we observe that if $\varepsilon_p\sim k_BT_{\mathrm{eff}}$, the dominant factor is $\exp[-\varepsilon_p/(k_BT_{\mathrm{eff}})]$, namely the Maxwellian factor characterized by cross section σ_0. The exponential factor with the δ parameter, corresponding to collisional cross section σ_1, becomes not negligible with respect to the Maxwellian only when $\varepsilon_p\sim k_BT_{\mathrm{eff}}/|\delta|$; it is also often called the Druyvenstein factor. Finally, the third term $\exp[-\gamma(\varepsilon_p/k_BT_{\mathrm{eff}})^3]$ arises when $\varepsilon_p\sim|\delta/\gamma|k_BT_{\mathrm{eff}}$; as we shall briefly describe in Sec. V, it can be related to quantum energy-momentum uncertainty effects in dense astrophysical plasmas.

In conclusion of this section, let us summarize that if the random force field is absent or negligible, in spite of the presence of whatever kind of collisional cross sections, all stationary states which are solutions of the kinetic equation have an analytical expression that coincides with the equilibrium Maxwellian distribution function. Therefore, the nonextensive statistical description, at least in a classical framework, requires as a first condition that particles be subjected to a sufficiently strong random force field and, as a second condition, that a constant collisional cross section (or depending on higher positive powers of velocity) should act among the particles of the system.

IV. COLLISIONAL CROSS SECTIONS IN ASTROPHYSICAL PLASMAS

We wish to discuss the physical meaning of the collisional processes related to σ_0,σ_1, and σ_2, defined in Sec. III and inserted into the kinetic equation in order to derive the one-body distribution function.

Cross section $\sigma_0(v)$, defined in Eq. (6), is the most important one as it originates the well-known Maxwellian distribution function even in the presence of a generalized random field. Our first unavoidable requirement is that the solution of the kinetic equation [Eq. (1)], at first order, shall be the Maxwellian function, because the actual kinetic solution for $F=0$ is indeed the Maxwellian, and we are dealing only with slight corrections.

Following Ref. [17], we can state that starting from an interaction force that depends on distance as r^{-s}, the corresponding cross section is $\sigma(v)\propto v^{-4/(s-1)}$. Therefore, in the case of cross section $\sigma_0(v)\propto v^{-1}$, the underlying force goes like r^{-5}, while the potential energy is proportional to r^{-4}, and we can interpret it as the cross section due to the interaction between an ionic charge and an electric dipole induced by the ion on the neutral system of charges composing a Debye sphere [15].

On the contrary, if we considered a pure Coulomb interaction due to a force $F_C\propto r^{-2}$ (with $s=2$), it would give a collisional cross section proportional to v^{-4}; however, this case is not physically suitable in the presence of an intensive random force field because of induced divergences in the distribution function at low velocities. Krook and Wu showed in the past [18] that collisional cross sections going like v^{-1} and v^{-3} always produce a Maxwellian distribution after a sufficiently long time; however, their system is not subjected to a random force field.

Cross section $\sigma_1(v)$, defined in Eq.(7), was introduced by Ichimaru [19] in the framework of an ion-sphere model for nonideal and weakly coupled plasmas with a Γ parameter of order unity, and with a small number of ions inside the Debye sphere. In these physical conditions, the collisional cross section, directly derived from the pure Coulomb one, is constant according to the approximations of the model and it can be cast in the simple form

$$\sigma_1(v) \approx 2\pi(\alpha a)^2,$$

where α is the correlation factor already introduced in Sec. III and a is the interparticle distance. The correction due to σ_1 on the probability function of energy is of order $\exp[-\delta\varepsilon_p^2/(k_B T_{\mathrm{eff}})^2]$, and shows the same behavior as the so-called nonextensive corrective factor (see Ref. [20]).

Cross section $\sigma_2(v)$ will be described in the next section.

V. EFFECTS OF QUANTUM ENERGY-MOMENTUM UNCERTAINTY ON THE EQUILIBRIUM DISTRIBUTION FUNCTION

Here we introduce simple arguments to synthetically explain the meaning and justify the use of cross section $\sigma_2(v)$ defined in Eq. (8) and, at the same time, to show a possible link between quantum energy-momentum uncertainty and nonextensivity.

Quantum energy-momentum uncertainty in weakly nonideal dense stellar plasmas influences thermonuclear rates. In fact, in classical physics, energy ε and momentum $\mathbf{p}$ (or $\varepsilon_p = \mathbf{p}^2/2\mu$, with μ reduced mass) are linked by the dispersion relation $\delta(\varepsilon,\varepsilon_p)=\delta(\varepsilon-\varepsilon_p)$. Nevertheless, if the particles cannot be considered free, ε and ε_p are independent variables. For instance, an ion tunneling the Coulomb barrier during a thermonuclear fusion reaction can interact simultaneously with many other particles. In this case, the dispersion relation is given by the function $\delta_\gamma(\varepsilon,\varepsilon_p)$ defined using the ansatz of Kadanoff and Baym [11]. Under equilibrium conditions, and this time without any random force field, the energy and momentum generalized distribution function can be written as

$$f(\varepsilon,\varepsilon_p) = \frac{n(\varepsilon)}{\pi}\delta_\gamma(\varepsilon,\varepsilon_p),$$

with

$$\delta_\gamma(\varepsilon,\varepsilon_p) = \frac{\mathrm{Im}\Sigma^R(\varepsilon,\varepsilon_p)}{(\varepsilon-\varepsilon_p-\mathrm{Re}\Sigma^R)^2+(\mathrm{Im}\Sigma^R)^2},$$

where $n(\varepsilon)$ is the particle number distribution and $\Sigma^R(\varepsilon,\varepsilon_p)$ is the mass operator for the one-particle retarded Green function.

Galitskiĭ and Yakimets [12] (see also Refs. [21,22]) derived that the quantum energy-momentum indeterminacy and a nonzero value of $\mathrm{Im}\Sigma^R$ lead to the nonexponential tail of the energy probability factor $f(\varepsilon_p)$.

We limit ourselves to the case of a dispersion relation of Lorentz type,

$$\delta_\gamma(\varepsilon,\varepsilon_p) = \frac{1}{\pi}\frac{\gamma(\varepsilon,\varepsilon_p)}{(\varepsilon-\varepsilon_p)^2+\gamma^2(\varepsilon,\varepsilon_p)},$$

with

$$\gamma(\varepsilon,\varepsilon_p) = \hbar\nu_{\mathrm{coll}}^T(\varepsilon,\varepsilon_p) = \hbar n\sigma_t(\varepsilon_p)\left(\frac{2\varepsilon}{\mu}\right)^{1/2},$$

where $\nu_{\mathrm{coll}}^T(\varepsilon,\varepsilon_p)$ is the total collision frequency and $\sigma_t(\varepsilon_p)$ is the collisional cross section.

Let us take the example of a pure Coulomb interaction. We have that

$$\gamma(\varepsilon,\varepsilon_p) = \frac{2\pi\hbar n e^4}{\varepsilon_p^2}\left(\frac{2\varepsilon}{\mu}\right)^{1/2},$$

and the momentum distribution becomes

$$f(\varepsilon_p) = \int d\varepsilon f(\varepsilon,\varepsilon_p) = \int d\varepsilon \frac{n(\varepsilon)}{\pi}\delta_\gamma(\varepsilon,\varepsilon_p)$$
$$= \frac{2}{\pi^{3/2}}\frac{\sqrt{\varepsilon_p}}{(k_B T)^{3/2}}[f_{MB}(\varepsilon_p)+f_Q(\varepsilon_p)], \qquad (19)$$

with

$$f_{MB}(\varepsilon_p) = \exp\left(-\frac{\varepsilon_p}{k_B T}\right)$$

and

$$f_Q(\varepsilon_p) = \frac{2}{3\pi}C(k_B T)^{3/2}\frac{1}{\varepsilon_p^4},$$

where C is a constant depending on density n.

At high momenta, the last term in Eq. (19) can be many orders of magnitude greater than the first one and represents an enhancement of the tail, with important consequences in the calculations of nuclear fusion rates.

We wish to verify if, by using a certain elastic collision cross section, we can obtain from the quantum uncertainty effect the nonextensive Tsallis distribution, limiting ourselves, for simplicity, to the case of entropic parameter $q > 1$ [9].

Following (and adapting to our present needs energy fluctuations instead of temperature fluctuations) the approach outlined by Beck and Cohen [23], we may state that any non-Maxwellian energy probability function should be cast in the form of a Laplace transformation of the function $\delta_f(E,\varepsilon_p)$ which describes the nonideality of the system [24],

$$f(\varepsilon_p) = \int_0^{+\infty} dE\, \exp\left(-\frac{E}{k_B T}\right)\delta_f(E,\varepsilon_p).$$

The function $\delta_f(E,\varepsilon_p)$ must be assumed to be a gamma (or χ^2) function, in order to ensure that $f(\varepsilon_p)$ is a nonextensive (Tsallis) distribution [10,23].

Let us compare this integral with the integral of Eq. (19), which can be written explicitly as $f(\varepsilon_p)=\int d\varepsilon\, \exp(-\varepsilon/k_B T)\delta_\gamma(\varepsilon,\varepsilon_p)$. Quantum uncertainty and nonextensivity are two different and distinct causes of deformation of the Maxwell-Boltzmann distribution. Nevertheless, they give the same effect if the microscopic interaction among the particles (i.e., the collisional cross section) is of a particular nature as we discuss below.

Let us turn our attention to the physical property of interest, which is "width" D_f of the $\delta_f(E,\varepsilon_p)$ distribution; it can be shown that for the nonextensive distribution function, the width is $D_{NE} \sim \varepsilon_p^2$, while for the quantum uncertainty the $D_Q \sim \sigma^2(\varepsilon_p)\varepsilon_p$ relation holds, where $\sigma(\varepsilon_p)$ is the collisional cross section. If we now impose that superextensivity and quantum uncertainty give the same physical effect on distribution functions, we should require that $\sigma(\varepsilon_p) \propto \sqrt{\varepsilon_p}$ or, in terms of relative velocity, $\sigma(v) = \sigma_2(v) \propto v$. Thus, the cross section σ_2 that we used in Sec. III is strongly related to both quantum and nonextensive statistical effects. The nonextensive and the Galitskiĭ-Yakimets distributions result given by the same expression.

Let us recover the behavior of the interaction force responsible for cross section $\sigma(\varepsilon_p) \sim \sqrt{\varepsilon_p}$. We can write its dependence on the relative coordinate r of the two interacting particles as [17]

$$F(r) = f_0\left(\frac{r}{R_0}\right)^{-s},$$

where f_0 is a dimensional constant, R_0 is a characteristic distance of the two-body center of mass with respect to a given origin, and s is a negative or positive integer.

Defining the collisional cross section as

$$\sigma = \pi d^2$$

with

$$d \sim \left(f_0 \frac{\mu}{|\mathbf{p}|^2}\right)^{1/(s-1)},$$

in order to have the requested behavior of $\sigma(\varepsilon_p) \propto \sqrt{\varepsilon_p}$, we must set $s=-3$. Let us recall that the case of $s=-3$ is, from the point of view of the orbit differential equation of motion, one of the integrable cases, with solutions given in terms of elliptical functions [25].

Therefore, the interacting force responsible for the collisions that lead to $\sigma(\varepsilon_p) \sim \sqrt{\varepsilon_p}$ reads

$$F_Q(r) = \begin{cases} f_{Q_0}\left(\dfrac{r}{R_0}\right)^3, & r \leqslant R_0 \\ 0, & r > R_0, \end{cases}$$

where the cutoff is needed in order to avoid divergences of the potential energy.

We may argue that force $F_Q(r)$ can be understood as a tidal-like force [26] if we assume that an attractive central force of intensity f_{Q_0}, centered at a distance R_0 from the center of mass of the two interacting particles separated by a distance r, is superimposed. The tidal-like force acts globally over all the particles of the system. This is the dynamical requirement to recover the nonextensive (Tsallis) distribution in the framework of quantum energy-momentum uncertainty. It is noteworthy and interesting to remark that by applying the virial theorem to this case, we obtain negative kinetic energy, which is, in fact, understandable and admissible by the uncertainty principle.

We derive the analytical expression of q by equaling the complete expressions of D_{NE} and D_Q. We obtain

$$q = 1 + \frac{(\hbar c)^2 n^2 (\Sigma_l)^2}{2^5(\mu c^2)} \frac{R_0^3}{f_{Q_0}},$$

where n is the plasma density and Σ_l is of order of unity [22].

The correction to the unity may be thought to be due to the many-body effect over the two-body interacting system. As an example, let us make the following numerical approximate evaluation of f_{Q_0}: if the correction is on the order of 10%, the density $n \approx 10^{-14}$ fm^{-3}, and $R_0 \approx 10^5$ fm, we obtain, for a proton plasma ($\mu c^2 \approx 460$ MeV),

$$f_{Q_0} \approx 10^{-12} \text{ MeV/fm}.$$

Before concluding this section, we remark that the nonextensive distribution usually describes metastable states or stationary states of nonequilibrium systems. On the contrary, in this case, quantum uncertainty with collisional cross section $\sigma(\varepsilon_p) \sim \sqrt{\varepsilon_p}$ gives a distribution function which belongs to an equilibrium state, although different from the Maxwell-Boltzmann distribution. Other generalized distributions have recently been proposed [27]. For situations with small deformation, our arguments are valid also for these distributions.

VI. CONCLUSIONS

We have set a kinetic equation suitable to describe the stationary states of a weakly nonideal plasma of a stellar core subject to generalized random forces. Provided that a random force satisfying condition (14) for superextensivity or (15) for subextensivity is present, the momentum distribution function can be cast in the simple fashion of Eq. (17) to which, besides the well-known Maxwellian factor, other terms also contribute.

The momentum distribution function is formally identical to the nonextensive distribution (when $q>1$) expanded in powers of $(1-q)$ for slight deformation. An analytical expression of q, the entropic parameter, can be derived in terms of the elastic collision cross sections acting among the particles of the system.

The main point is that each correction factor is due to a particular collisional process between ions, and that each of them contributes in a well-defined interval of relative velocity, as shown at the end of Sec. III. All these corrections are small, nevertheless they are not negligible at high energy, i.e., in the region of ion spectrum of predominant interest for calculations of nuclear reaction rates in astrophysical plasmas.

We have stressed that in physical conditions as, for example, stars with $\Gamma \gtrsim 1$, many collisional processes may be active, even at the same time, and that each one of them is described by a cross section with a dependence over velocity stronger (proportional to v^{-1}, v^0, or even $\propto v$) than the simple Coulomb scattering (proportional to v^{-4}). This fact is intimately related to statistical many-body effects and represents a link between dynamics (the type of two-body elastic collisions) and statistical mechanics (the momentum distribution function of the stationary states involved).

Finally, in the framework of a quantum many-body description of the equilibrium state, considering the energy-momentum uncertainty due to the noncommutativity of position and momentum operators, we have found that if the collisional cross section $\sigma(\varepsilon_p)$ behaves like $\sqrt{\varepsilon_p}$, the distribution function coincides with the nonextensive (Tsallis) distribution function with $q>1$. The requested behavior of the cross section $\sigma(\varepsilon_p)$ is due to an interaction similar to a tidal-like force. Therefore, the analogy between the quantum uncertainty effect on the distribution and the nonextensive effect is achieved provided that an overall attractive interaction is superimposed on the two-body interaction. This again represents a possible link between dynamics and statistical mechanics.

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