Emergence of large-scale vorticity during diffusion in a random potential under an alternating bias

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Conventional wisdom indicates that the presence of an alternating driving force will not change the long-term behavior of a Brownian particle moving in a random potential. Although this is true in one dimension, here we offer direct evidence that the inevitable local symmetry breaking present in a two-dimensional random potential leads to the emergence of a local ratchet effect that generates large-scale vorticity patterns consisting of steady-state net diffusive currents. For small fields the spatial correlation function of the current follows a logarithmic distance dependence, while for large external fields both the vorticity and the correlations gradually disappear. We uncover the scaling laws characterizing this unique pattern formation process, and discuss their potential relevance to real systems.

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Driven diffusion in random media is a much studied problem impacting a number of research areas, ranging from relaxation phenomena in spin glasses [1], to dislocation motion in disordered crystals [2], transport in porous media [3], and turbulent diffusion [4]. These and related studies have offered convincing evidence that the interplay between the annealed randomness of the diffusion process and the quenched randomness of the media gives rise to unexpected scaling phenomena [5]. If the external driving force has a symmetric and time-periodic form, the motion of a particle is the result of the combined effect of thermally activated diffusion and a periodic component, induced by the coupling to the applied field. At time scales much larger than the period of the driving force one expects the influence of the alternating field to be negligible, the dynamics of the system being described by the classical Brownian motion. Indeed, while the particle moves back and forth along the direction of the alternating field, a stroboscopic view of the system, obtained by taking pictures only at times that are integer multiples of the external field period, is expected to show a randomly diffusing particle, as if the periodic external field was absent. In contrast to this intuitive picture, we offer convincing evidence that in the presence of quenched randomness an external alternating field can fundamentally change the nature of the diffusive dynamics.

Indications that an alternating bias may influence the nature of diffusive motion in a random potential comes from the recent advances in thermal ratchets [6] and driven chemical reactions [7]. In equilibrium, a particle moving in a one-dimensional (1D) periodic asymmetric potential for which the $x \rightarrow -x$ symmetry is broken displays a simple diffusive behavior. However, if the particle is also driven by an alternating field, it drifts in the direction defined by the asymmetry of the potential [7]. The emergence of this nonequilibrium steady-state net current is called the *ratchet effect*, and the average drift velocity of the particle is the *ratchet velocity*. The ratchet velocity is expected to vanish, however, when the particle moves in a *random* potential, since a random potential obeys inversion symmetry (in a statistical sense). Indeed, if identical energy wells are separated by sufficiently high energy barriers, the ratchet velocity is exactly zero, irrespective of the barrier height distribution [8].

On the other hand, in two-dimensional (2D) systems any *finite* region of a random potential exhibits some degree of broken symmetry, forcing the particle to follow different trajectories during the two half periods of the alternating driving force. In this paper we show that the interplay between this local symmetry breaking in a random 2D potential landscape and the alternating drive leads to the unexpected appearance of a highly correlated steady-state net current field, characterized by a large-scale vorticity.

We investigate the driven dynamics of a single particle moving in a random uncorrelated potential on a 2D square Euclidean lattice with $N=L\times L$ sites, where each site is connected to its nearest neighbors by bonds of unit length. Random potential barriers are assigned to each bond and periodic boundary conditions are assumed. A schematic illustration of the system's geometry is shown in Fig. 1. The diffusing particle is also driven by an external alternating field, applied along the x-direction. We define the local probability currents $J^x_{i,j}$ [$J^y_{i,j}$] on site (i,j) as the currents flowing across the potential barriers between the lattice sites (i,j) and $(i+1,j)$ [$(i,j+1)$]. As shown in Fig. 1, a particle located at site (i,j) at moment t can hop to any of the *four* nearest-neighbor lattice sites by overcoming the potential barriers, $E^\gamma_{i,j}$, assigned to each bond. To simplify the notations we denote by $E^r_{i,j}$, $E^l_{i,j}$, $E^u_{i,j}$, and $E^d_{i,j}$ the local potential barriers associated with the right $E^r_{i,j}=E_{(i,j)\rightarrow(i+1,j)}$, left $E^l_{i,j}=E_{(i,j)\rightarrow(i-1,j)}$, up $E^u_{i,j}=E_{(i,j)\rightarrow(i,j+1)}$, and down $E^d_{i,j}=E_{(i,j)\rightarrow(i,j-1)}$ jumps, such that $E^r_{i,j}\equiv E^l_{i+1,j}$ and $E^u_{i,j}\equiv E^d_{i,j+1}$. The barrier heights are chosen to be random and quenched, given by $E^\gamma_{i,j}=E_0+U\eta^\gamma_{i,j}$, where γ stands for r, l, u, and d; E_0 is a constant; and $\eta^r_{i,j}\equiv\eta^l_{i+1,j}$ and $\eta^u_{i,j}\equiv\eta^d_{i,j+1}$ are uncorrelated random numbers uniformly dis-

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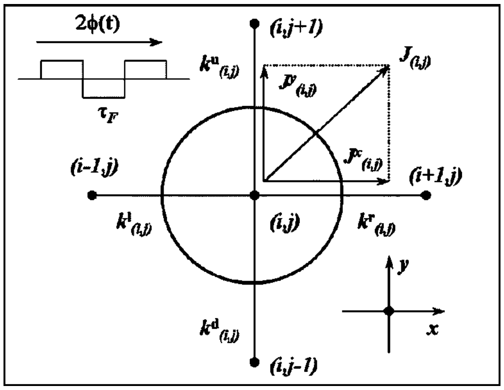

FIG. 1. Schematic diagram illustrating the parameters of the studied two-dimensional driven diffusion problem. A particle on the site (i,j) hops to its neighboring sites with hopping rates $k_{i,j}^{\gamma}$, defined by both the potential barriers between these sites and by the alternating field. The time-dependent behavior of the external field is shown on the top left.

tributed between 0 and 1. Throughout the paper we measure the energies in units of $k_B T$, where k_B stands for the Boltzmann constant and T is the absolute temperature.

In addition to the thermally activated diffusion, the particle is also driven by an external alternating force, $2\phi(t)$, applied along the x direction (see Fig. 1). Since the energy barriers are placed halfway between adjacent sites, the alternating force modulates their height by $-\phi(t)$ in the positive x direction and by $+\phi(t)$ in the negative one. Thus, for large enough barriers the hopping rates of a particle occupying site (i,j) are given by

$$k_{i,j}^{r} = \nu \exp\{-[U\eta_{i,j}^{r} - \phi(t)]\},$$

$$k_{i,j}^{l} = \nu \exp\{-[U\eta_{i,j}^{l} + \phi(t)]\},$$

$$k_{i,j}^{u} = \nu \exp\{-U\eta_{i,j}^{u}\},$$

$$k_{i,j}^{d} = \nu \exp\{-U\eta_{i,j}^{d}\}, \qquad (1)$$

where $\nu = \nu_0 \exp\{-E_0\}$ and the attempt frequency ν_0 is also a constant.

Next we turn to an ensemble description, in which the probability of the particle occupying site (i,j) is denoted by $P_{i,j}$, and the time evolution of this probability distribution is described by the set of master equations

$$\frac{\partial P_{i,j}}{\partial t} = -J_{i,j}^{x} + J_{i-1,j}^{x} - J_{i,j}^{y} + J_{i,j-1}^{y}, \qquad (2)$$

where

$$J_{i,j}^{x} = k_{i,j}^{r} P_{i,j} - k_{i+1,j}^{l} P_{i+1,j},$$

$$J_{i,j}^{y} = k_{i,j}^{u} P_{i,j} - k_{i,j+1}^{d} P_{i,j+1} \qquad (3)$$

are the x and y components of the local probability currents $\vec{J}_{i,j}$, as defined above.

For simplicity, we restrict ourselves to symmetric square-wave external fields, i.e., when the field $\phi(t)$ alternates between $+F$ and $-F$ at constant time intervals τ_F. We also assume that τ_F is much larger than the relaxation time of the entire system, estimated as $\tau_{\mathrm{relax}} \approx L^2 \max(1/k_{i,j}^{\gamma})$. In this case, for the first half period of the alternating driving force the probability distribution $P_{i,j}$ and currents $\vec{J}_{i,j}$ relax to their steady-state values $P_{i,j}(+F)$ and $\vec{J}_{i,j}(+F)$, while for the second half they relax to $P_{i,j}(-F)$ and $\vec{J}_{i,j}(-F)$. The magnitude of these currents can be determined from the stationary solution of the master equations for $\phi(t) = +F$ and $\phi(t) = -F$, respectively. From direct analogy with the ratchet velocities, we can then define the *net currents* as

$$\vec{\mathcal{J}}_{i,j} = \frac{1}{2}[\vec{J}_{i,j}(+F) + \vec{J}_{i,j}(-F)]. \qquad (4)$$

The appearance of nonzero net currents is a property of systems with locally broken symmetry, their magnitude being given by the second- and higher-order terms in the response function [9].

We solved Eqs. (1)–(3) numerically using the conjugate gradient method, modified for sparse matrices [10]. In Figs. 2(a)–2(d) we show the steady-state local net current patterns obtained for a system with linear size $L=50$, randomness parameter $U=0.5$, and external field amplitudes $F=0.01$, 0.05, 0.1, and 0.9. In the absence of collective effects we would expect no net current in the system, or at best, small-scale random currents, reflecting the local symmetry breaking. In contrast, Fig. 2 indicates the emergence of highly nontrivial spatially extended steady-state net current fields, characterized by long-range correlations and large-scale vorticity. The large-scale vorticity is most apparent for small external fields [$F \ll U$, Fig. 2(a)]. As F increases the correlations gradually disappear and the net current field converges to a nearly random structure [Fig. 2(d)] [9].

To quantify the scaling properties of the system, we compute the ensemble-averaged real-space current-current correlation function defined as

$$C(r) = \left\langle \frac{1}{\sum_{(i,j)} \vec{\mathcal{J}}_{i,j}^{2} N_r} \sum_{i,j} \sum_{i',j'} [\vec{\mathcal{J}}(\mathbf{r}_{i,j}) - \vec{\mathcal{J}}(\mathbf{r}_{i',j'})]^2 \right\rangle. \qquad (5)$$

Here summation over all lattice sites (i,j) is implied, while summation over (i',j') indexes is performed only for lattice site pairs such that $|\mathbf{r}_{i,j} - \mathbf{r}_{i',j'}| = r$, N_r being the number of such pairs. The average $\langle \cdots \rangle$ was taken for various different realizations of the disorder $\{\eta_{i,j}^{\gamma}\}$. The correlation function defined this way is bounded from below by 0 (perfect correlation), from above by 4 (perfect anticorrelation), and takes the value of 2 for vanishing correlation.

In Fig. 3 we show the normalized correlation function, $C(r)/C(1)$, computed for different system sizes and fixed external field amplitude, $F=0.01$. We find that the correlation function follows closely a logarithmic dependence on r for small radial distances, and saturates at a value that depends on the system size, L. This allows us to introduce the corre-

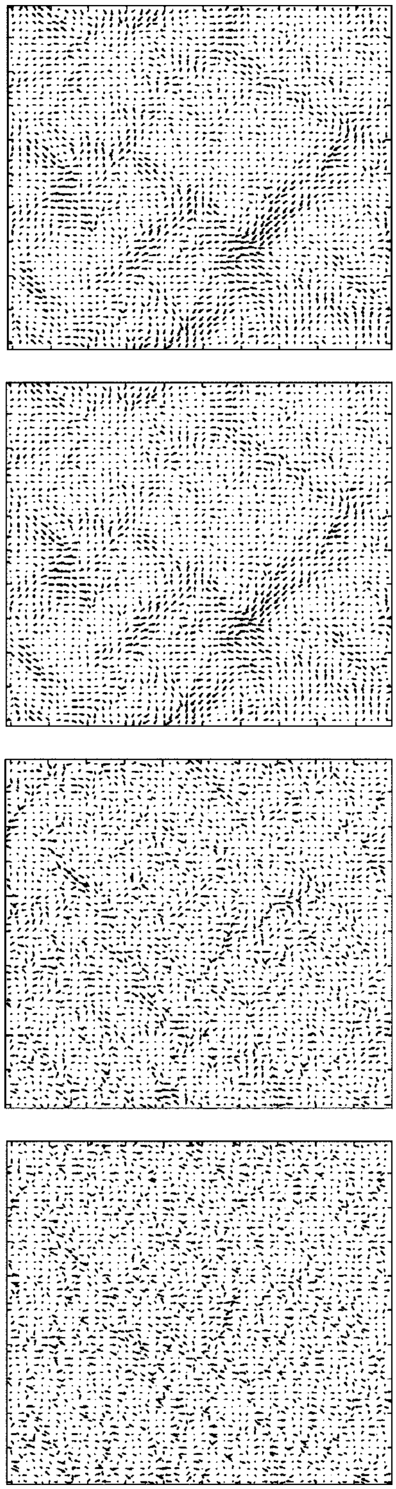

FIG. 2. Snapshots of the steady-state net current fields obtained for systems with $L=50$, $U=0.5$, and different external field amplitudes: (a) $F=0.01$; (b) $F=0.05$; (c) $F=0.10$; (d) $F=0.90$. The direction and the size of each arrow indicates the direction and the magnitude of the net ratchet current in the system.

lation length ξ, defined as a characteristic length at which the correlation vanishes: $C(r)=2$, that is, where $1-0.5C(r) \sim \langle \vec{\mathcal{J}}(\mathbf{r}_{i,j}) \vec{\mathcal{J}}(\mathbf{r}_{i',j'}) \rangle$ becomes zero.

On the basis of the above observations, we conclude that

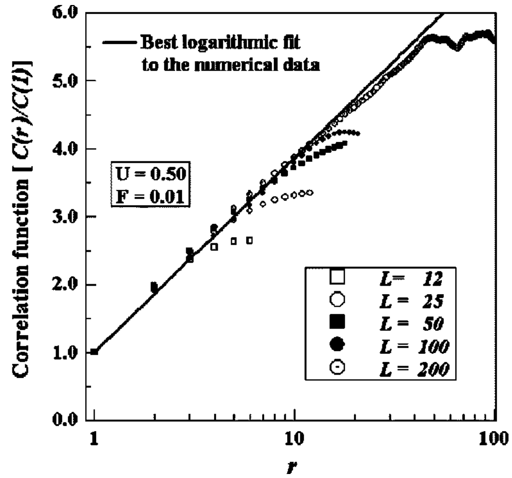

FIG. 3. Linear-logarithmic plot of the current-current correlation function, $C(r)/C(1)$, computed for $U=0.5$, $F=0.01$, and for different system sizes L.

the current-current correlation function follows the scaling relation

$$C(r) \sim \log(r) f\left(\frac{r}{\xi}\right), \tag{6}$$

where $f(x \leqslant 1) \sim \text{const}$. Note that the correlation length in Eq. (6) depends both on the amplitude of the external field and on the system size; i.e., $\xi=\xi(F,L)$. Moreover, we find that the correlation length, ξ, scales with both of these quantities following power laws. That is, for small external field amplitudes, the correlation length depends on the system size as $\xi(L) \sim L^{\alpha}$, where $\alpha \simeq 0.98 \pm 0.02$ (i.e., ξ grows approximately linearly with L). On the other hand, for intermediate external field amplitudes, ξ is found to be weakly dependent on L and to monotonically decrease with F, following an inverse power law, $\xi \sim F^{-\beta}$, where $\beta \simeq 0.78 \pm 0.02$. The transition between the two regimes occurs at the L-dependent field amplitude $F_1^c(L)$. These scaling laws suggest the scaling relation

$$\xi \sim F^{-\beta} g(FL^{\alpha/\beta}), \tag{7}$$

where $g(x) \sim x^B$ for $F \leqslant F_1^c(L)$ and $g(x) \sim \text{const}$ for $F_1^c(L) \leqslant F \leqslant F_2^c(L)$. The significance of $F_2^c(L)$ will be discussed later. In Fig. 4 we show the data collapse, performed according to Eq. (7). As one can observe, the collapse is excellent for small and intermediate $FL^{\alpha/\beta}$ values, systematic deviations being observed only in the large saturated regime. In general, three scaling regimes can be distinguished. For small F [i.e., $F \leqslant F_1^c(L)$] the behavior of ξ is determined by the system size dependent scaling. In the second regime, corresponding to the intermediate values $F_1^c(L) \leqslant F \leqslant F_2^c(L)$, scaling is dominated by the external field amplitude. Deviations from the scaling behavior predicted by Eq. (7) are observed only for large F [i.e., for $F \geqslant F_2^c(L)$]. This behavior is an artifact of the model's discrete nature since, as the particles are on a lattice, we cannot measure ξ smaller than 1. As a result, in the large F limit the correlation length does not

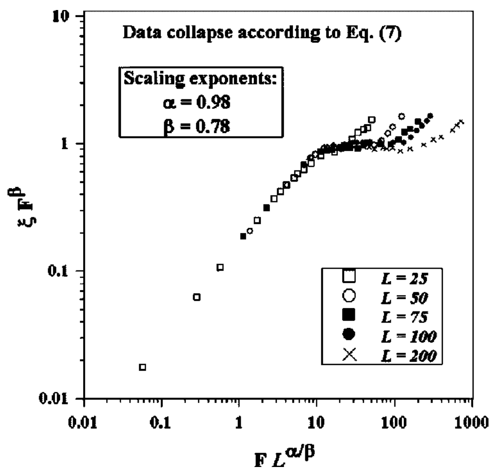

FIG. 4. The size dependence of the correlation length shown for different system sizes and rescaled according to Eq. (7).

follow Eq. (7) down to arbitrary small distances, but rather saturates at $\xi \simeq 1$.

In summary, we studied the diffusive motion of a single particle under the influence of an alternating field on 2D Euclidean lattices with quenched random potential. In contrast with the intuitive picture, that would indicate no systematic difference between an alternating driven diffusion and random diffusion at time scales larger than the period of the alternating field, we offer evidence that the interplay between the quenched randomness of the potential landscape and the external alternating bias leads to the emergence of large-scale vorticity patterns in the net steady-state currents. These patterns represent a unique form of self-organized pattern formation. The emerging stationary currents are found to be strongly correlated, with the two-point correlation function following a logarithmic distance dependence. The current fields are largely independent of the particular realization of the disorder. The observed correlation length scales linearly with the system size and displays an inverse power-law dependence on the external field amplitude.

An intuitive explanation for the observed phenomenon is based on the particle's ability to "choose" globally different paths when driven by the alternating field in two opposing directions. The superposition of these globally different flow fields results in large-scale flow patterns and vorticity. This picture is consistent with a slow (logarithmic) decay of the correlations and the finding that the correlation length is of the order of the system size. Note that this phenomenon cannot occur in one dimension, where all the energy barriers are in series, thus alternative trajectories are not possible. Yet, it will likely emerge in three dimensions as well.

These results could find application in a number of areas. Driven diffusion in the presence of randomness is present in a number of systems, ranging from the diffusion of vortices in thin disordered superconductors [11] to electron diffusion in disordered conductors or the diffusion of colloidal particles in colloidal suspensions [12]. Each of these systems can be driven by external alternating fields, potentially reproducing the conditions discussed in the present paper. Therefore, with rapidly developing visualization techniques and high sensitivity instrumentation the predicted spatial organizing patterns could be observed experimentally as well.

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[1] T. Nattermann and J. Villain, Phase Transitions **11**, 5 (1988).
[2] J. P. Hirth and J. Lothe, *The Theory of Dislocations* (McGraw-Hill, New York, 1968).
[3] J. Nittmann, G. Daccord, and H. E. Stanley, Nature (London) **314**, 141 (1985).
[4] L. F. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926).
[5] S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, Rev. Mod. Phys. **53**, 175 (1981); S. Havlin and D. Ben-Avraham, Adv. Phys. **36**, 695 (1987); J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990); F. Cecconi, M. Marsili, J. R. Banavar, and A. Maritan, Phys. Rev. Lett. **89**, 088102 (2002); E. Lopez, S. V. Buldyrev, N. V. Dokholyan, L. Goldmakher, S. Havlin, P. R. King, and H. E. Stanley, Phys. Rev. E **67**, 056314 (2003); J. S. Andrade, A. D. Araujo, S. V. Buldyrev, S. Havlin, H. E. Stanley, Phys. Rev. E **63**, 051403 (2001); S. Ciliberti, G. Caldarelli, P. De los Rios, L. Pietronero, and Y. C. Zhang, Phys. Rev. Lett. **85**, 4848 (2000).
[6] A. Ajdari and J. Prost, C.R. Acad. Sci., Ser. IIc: Chim **315**, 1635 (1992); M. O. Magnasco, Phys. Rev. Lett. **71**, 1477 (1993); R. D. Astumian and M. Bier, *ibid.* **72**, 1766 (1994); R. D. Astumian, Science **276**, 917 (1997); F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69**, 1269 (1997).
[7] B. Robertson and R. D. Astumian, J. Chem. Phys. **94**, 7414 (1991).
[8] K. W. Kehr, K. Mussawisade, T. Wichmann, and W. Dieterich, Phys. Rev. E **56**, R2351 (1997).
[9] It must be emphasized that the numerical solution of Eqs. (1)–(3) for arbitrarily small values of *F* and/or for arbitrarily large system sizes is not possible. This is due to the slow convergence of the conjugate gradient method in these limits and, most importantly, due to the growing impact of the randomization, induced by the numerical errors [M. A. Makeev, I. Derényi, and A.-L. Barabási (unpublished)].
[10] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes*, 2nd ed. (Cambridge University Press, Cambridge, England 1992).
[11] C.-S. Lee, B. Jankó, I. Derényi, and A.-L. Barabási, Nature (London) **400**, 337 (1999); J. E. Villegas, S. Savelév, F. Nori, E. M. Gonzalez, J. V. Anguita, R. Garcia, and J. L. Vicent, Science **302**, 1188 (2003); M. B. Hastings, C. J. Olson Reichhardt, and C. Reichhardt, Phys. Rev. Lett. **90**, 247004 (2003).
[12] K.-Q. Zhang and X. Y. Liu, Nature (London) **429**, 739 (2004).