Strongly asymmetric clustering in systems of phase oscillators

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In this paper, we look at clustering in systems of globally coupled identical phase oscillators. In particular, we extend and apply techniques developed earlier to study stable clustering behavior involving clusters of greatly differing size. We discuss the bifurcations in which these asymmetric cluster states are created, and how these relate to bifurcations of the synchronized state. Because of the simplicity of systems of phase oscillators, it is possible to say a significant amount about asymmetric clustering analytically. We apply some of the theory developed to one particular system, and illustrate how the techniques can be used to find behavior which might otherwise be missed.

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I. INTRODUCTION

Systems of globally coupled identical oscillators, while simple to describe, can display a surprising variety of interesting behavior [1–5]. These systems have been used as models of a number of physical and biological phenomena (see the references in [1,6]). Among the collective behaviors which can be observed in these systems are synchronization, clustering, "splay-phase" states, and chaotic states (even where the dynamics of individual oscillators do not admit chaotic solutions [7,8]).

The defining feature of globally coupled systems is their full permutation symmetry, which gives rise to a large number of invariant subspaces. Solutions on these subspaces are characterized by the fact that oscillators divide into groups or "clusters" (synchronized solutions are the extreme case) and hence we call these subspaces *cluster subspaces*. These subspaces are central to the system's behavior, and even chaotic behavior is often related in some way to them (e.g., the chaotic attractor contains within its closure unstable cluster states [9]).

An obvious technique for the study of clustering is to restrict the system to its invariant subspaces, and analyze the reduced systems so obtained. But such restrictions give one no idea of behavior transverse to the invariant subspaces. In [10] we developed some fairly general theory describing clustering in globally coupled systems, which included consideration of behavior transverse to the invariant subspaces. Of particular interest were stable, but highly "asymmetric" cluster states which can sometimes be predicted to exist in a domain where the synchronized state is also stable, and are often associated with its loss of stability. By "asymmetric" we will mean cluster states with the great majority of oscillators in one cluster. In this we follow the usage of [1], where, however, the focus was primarily on the symmetric case. Although here we restrict our attention to systems of ordinary differential equations, recently asymmetric clustering has been studied and found to be important in systems of globally coupled maps [11,12].

Asymmetric cluster (AC) states have a certain intuitive appeal, because while the majority of oscillators perform some motion in unison, a few are allowed different dynamics. These dynamics need not be close to those of the dominant cluster. In fact, numerical experiments indicate that for certain systems, the minority of oscillators in the smaller cluster may even be allowed to behave chaotically, while the rest behave approximately periodically. In real-world situations it would be easy to attribute the existence of a few individuals which did not synchronize with the pack to differences in the individuals themselves or aysmmetries in the coupling, but the behavior we describe occurs in systems of symmetrically coupled identical oscillators.

We start by commenting on what can be said about asymmetric clustering in general systems of globally coupled oscillators. Attention is then focused on the easier case of phase oscillators whose simplicity allows for stronger results. These results are illustrated via the study of a particular system which has appeared in the literature before. Finally, the effects of breaking the symmetry of the system are briefly explored.

II. THEORY OF AC STATES

Here we briefly summarize the relevant result from [10] which will be needed. Of interest are systems of N oscillators of the form

$$\dot{x}_i = f(x_i) + g\left(\frac{1}{N}\sum_{j=1}^{N} h(x_i - x_j)\right), \quad i = 1,\ldots,N. \tag{1}$$

x_i may belong to a variety of spaces, for example Euclidian space, a circle, or a torus. Assuming that there exists some asymptotic synchronized state defined by the solution $x_i(t) = \phi_0(t)$, $i=1,\ldots,N$, it is possible to construct a "reduced model,"

$$\dot{x} = f(x) + g(h(x - \phi_0(t))). \tag{2}$$

Formally, what is done to construct this second nonautonomous system is simply to replace each x_j except x_i with $\phi_0(t)$. $\phi_0(t)$ is of course only unique up to a shift in the origin of time. Trivially, Eq. (2) always has the solution $x = \phi_0(t)$. But in addition it can be shown that if (i) the solution $x_i(t) = \phi_0(t)$, $i=1,\ldots,N$ in the full system [Eq. (1)] is

T-periodic and stable, and (ii) the reduced model [Eq. (2)] possesses a stable T-periodic solution $x=\psi_0(t)\neq\phi_0(t)$, then for sufficiently large systems of oscillators, Eq. (1) possesses stable periodic cluster solutions with a small proportion of the oscillators in one cluster—i.e., highly asymmetric cluster states. This is a specialization and amalgamation of theorems 3 and 4 in [10].

Note that if the conditions above are met, N can always be chosen large enough to ensure that the asymmetric cluster states exist. From now on, if we say that asymmetric clusters exist in a system at a given parameter value, we will mean that they exist for sufficiently large N. Moreover, when they do exist, they can usually be found numerically, because the proof tells us where to look: the larger cluster performs a motion close to $\phi_0(t)$ while the smaller cluster does something close to $\psi_0(t)$.

Qualitatively, the argument goes as follows: As long as it is a small proportion of oscillators which perform the unorthodox motion, causing only a minor disturbance to the mean, these oscillators may be tolerated, and need not be pulled back into the pack. Perturbation arguments are used to prove the existence and hyperbolicity of these states. We stress that there is no reason to suppose that the motion of the second cluster is close to the motion of the first cluster.

In general, there will not be an analytic form for the synchronized solution $\phi_0(t)$ to use in the reduced model. Although this might at first seem to be an impediment to exploring the model in the search for AC behavior, this is only apparent. To see why, we simply note that $\phi_0(t)$ is by definition a solution of the system $\dot{x}=f(x)+g(h(0))$. The reduced model is then a projection of the simple skew product system,

$$\dot{y}=f(y)+g(h(0)), \tag{3}$$

$$\dot{x}=f(x)+g(h(x-y)). \tag{4}$$

The analysis above has so far been about the existence of two-cluster states. It can, however, be extended to the case of more than two clusters [10]. Basically, given the existence of k stable solutions in the reduced model, we can prove the existence of stable asymmetric k-cluster states in the full system.

III. PHASE SYSTEMS

Note that up to this point, the results have applied to arbitrary systems of identical globally coupled oscillators. Sometimes, however, under the assumption of sufficiently weak coupling, systems of oscillators can be described using a single phase variable for each oscillator [13]. In this case, the one-dimensionality of the oscillators allows us to say more about asymmetric clustering than in the general case. Although the assumptions needed to justify the phase approximation cannot always be justified, phase oscillators have been used to model a number of biological phenomena, including neurons [14].

Consider a system of phase oscillators where the coupling includes Fourier modes up to the mth mode,

$$\dot{\phi}_i=\omega+\frac{1}{N}\sum_{j=1}^{N}\sum_{k=1}^{m}[a_k\sin k(\phi_i-\phi_j)+b_k\cos k(\phi_i-\phi_j)],$$
$$i=1,\ldots,N. \tag{5}$$

This form includes as a special case the system studied in [15]. We can make three general statements about system (5):

(i) If there is only one Fourier mode in the coupling, i.e., $m=1$, there can be no AC states.

(ii) For $m=2$, given a set of parameter values a_1,a_2,b_1,b_2, we can calculate analytically whether the system possesses asymmetric two-cluster states at these values.

(iii) For general m, there can be AC states with up to m clusters.

These statements will allow us to explore in greater detail the behavior of the particular system we study below. They arise via the construction and analysis of a reduced model as detailed above. In order to see this, we first note that Eq. (5) has a synchronized solution of the form

$$\phi_i(t)=Rt,\quad i=1,\ldots,N, \tag{6}$$

where $R\equiv\omega+\sum_{k=1}^{m}b_k$. This solution is stable as long as $\sum_{k=1}^{m}ka_k<0$. At $\sum_{k=1}^{m}ka_k=0$, it undergoes a degenerate transcritical bifurcation to be described in Sec. IV. Substituting the solution into Eq. (5) gives

$$\dot{\phi}=\omega+\sum_{k=1}^{m}[a_k\sin k(\phi-Rt)+b_k\cos k(\phi-Rt)]. \tag{7}$$

The coordinate change $\theta=\phi-Rt$ gives

$$\dot{\theta}=\sum_{k=1}^{m}[a_k\sin k\theta+b_k(\cos k\theta-1)]\equiv f_{\{a_k\},\{b_k\}}(\theta), \tag{8}$$

which is the reduced model for this system. This is a one-dimensional autonomous system, whose only solutions can be fixed points. Since it is periodic, these fixed points will generically come in stable and unstable pairs, except at bifurcation values [20]. From the point of view of asymmetric clustering, we are interested in the zeros of $f_{\{a_k\},\{b_k\}}(\theta)$. The next stage is to use complex representations of sin and cos to get

$$f_{\{a_k\},\{b_k\}}(\theta)=\sum_{k=1}^{m}\left[a_k\frac{(e^{ik\theta}-e^{-ik\theta})}{2i}+b_k\left(\frac{(e^{ik\theta}+e^{-ik\theta})}{2}-1\right)\right], \tag{9}$$

which has zeros at the same values as

$$\tilde{f}_{\{a_k\},\{b_k\}}(\theta)\equiv\sum_{k=1}^{m}\left[a_k\frac{(e^{i(m+k)\theta}-e^{i(m-k)\theta})}{2i}+b_k\left(\frac{(e^{i(m+k)\theta}+e^{i(m-k)\theta})}{2}-e^{im\theta}\right)\right]. \tag{10}$$

Setting $z\equiv e^{i\theta}$, we see that $\tilde{f}_{\{a_k\},\{b_k\}}(\theta)$ can be written as a polynomial in z of the form

$$P(z) = \sum_{k=0}^{2m} c_k z^k \tag{11}$$

for complex c_k. Moreover, it has one solution at $z=1$, so that it can be written

$$P(z) = (z-1)\sum_{k=0}^{2m-1} d_k z^k. \tag{12}$$

The polynomial $\Sigma_{k=0}^{2m-1} d_k z^k$ has $2m-1$ roots, $\{z_i\}_{i=1,\ldots,2m-1}$, in general distinct. Let $\mathbf{P}\subset\mathbb{C}$ be defined as any vertical strip of the form $\{z\in\mathbb{C}:\mathrm{Re}(z)\in[n\pi,(n+2)\pi)\}$. Then the map $z\mapsto e^{iz}:\mathbf{P}\to\mathbb{C}$ is invertible, and hence to each of the roots z_i there is a unique $\theta_i\in\mathbf{P}$ such that $e^{i\theta_i}=z_i$.

Returning to the equation $\tilde{f}_{\{a_k\},\{b_k\}}(\theta)=0$, we know from the geometry used to define it that its solutions generically come in pairs which are either both real or both complex. These correspond to pairs of z_i either on or off the unit circle. Thus there are always two real roots to $\tilde{f}_{\{a_k\},\{b_k\}}(\theta)$ with $m-1$ further pairs of roots, in general complex.

If $m=1$, this means that there can be no stable zeros apart from the trivial one. Hence we can never predict AC states when there is only one Fourier mode in the coupling.

If $m=2$, $\Sigma_{k=0}^{2m-1} d_k z^k$ is a cubic polynomial, and can be solved analytically to give the roots z_1, z_2, and z_3, from which we can calculate θ_1, θ_2, and θ_3. One θ_i must be real, but if the other pair are also real, then there are further stable zeros. If the base solution at $\theta=0$ is also stable, then there will be stable asymmetric two-cluster states in the full system.

The third statement follows from the fact that $\Sigma_{k=0}^{2m-1} d_k z^k$ is a polynomial with one root on the unit circle (corresponding to a real value of θ), and another $m-1$ pairs of roots which may or may not be on the unit circle. If all of these are on the unit circle, and hence there are $m-1$ stable solutions to $\tilde{f}_{\{a_k\},\{b_k\}}(\theta)=0$, then the arguments of Sec. 5 of [10] can be used to prove the existence of m-cluster solutions in the full system.

We mention that a relationship between Fourier modes and clustering has been pointed out previously in [1], although in a different context.

IV. LOSS OF STABILITY OF THE SYNCHRONIZED STATE

In globally coupled systems, barring the uninteresting case that a synchronized state loses stability on its own subspace, generically, the synchronized state loses stability in a degenerate transcritical bifurcation. In this bifurcation, a large number of cluster states collide and the synchronized state loses stability. The bifurcation is discussed in some detail in [6,10]. It is shown that at the bifurcation, states from every two-cluster subspace collide with the synchronized state, but theory in [16] implies that with the possible exception of totally symmetric clusters, these are all unstable both before and after the bifurcation. The situation is shown schematically in Fig. 1. We will call this bifurcation the DT bifurcation.

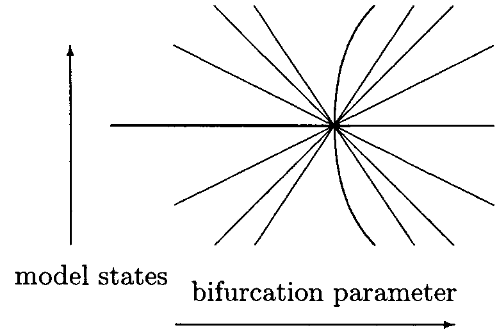

FIG. 1. A schematic representation of a degenerate transcritical (DT) bifurcation. The horizontal line represents the synchronized solution which loses stability. The other straight lines represent unstable solution branches. A possibly stable symmetric branch is also shown.

The DT bifurcation in the full system manifests itself as a transcritical bifurcation of the base solution in the reduced model. For phase systems this is particularly simple to characterize: basically an unstable solution collides with the base solution and exchanges stability with it (the reader can glance ahead to Fig. 5 for example).

It is an interesting observation that frequently AC states are born shortly before the DT bifurcation, and may attract the majority of phase space in the region near the bifurcation. We now calculate explicitly when AC states states can be expected to exist near the DT bifurcation point for a phase system with two Fourier modes.

The reduced model can be written

$$\dot{\theta} = a_1\sin\theta + b_1(\cos\theta-1) + a_2\sin\theta + b_2(\cos\theta-1) \equiv F(\theta). \tag{13}$$

A transcritical bifurcation of the base solution occurs when $F'(0)=0$, i.e., at $a_1+2a_2=0$. (See any textbook on bifurcation theory such as [17] for the nondegeneracy conditions on this bifurcation.) As in Sec. III, we rewrite the system in complex form and find that $F(\theta)$ has zeros when $P(z)=\Sigma_{j=0}^{4} c_j z^j$ has zeros where $z=\underline{e^{i\theta}}$, $c_4=b_2/\underline{2}+a_2/(2i)$, $c_3=b_1/2+a_1/(2i)$, $c_2=-(b_1+b_2)$, $c_1=\overline{c_3}$, and $c_0=\overline{c_4}$. Because of the root at unity, we can always write

$$P(z) = (z-1)(c_4z^3 + c_5z^2 + c_6z - c_0), \tag{14}$$

where $c_5=c_3+c_4$ and $c_6=\overline{c_5}$. It takes a little algebra to check that at bifurcation (i.e., when $a_1+2a_2=0$),

$$P_{\mathrm{bif}}(z) = (z-1)^2(c_4z^2 + c_7z + \overline{c_4}), \tag{15}$$

where $c_7=b_1/2+b_2$. Again following Sec. III, we are interested in when the quadratic polynomial $c_4z^2+c_7z+\overline{c_4}$ has two zeros on the unit circle. Indeed, the two solutions to this polynomial are

$$\frac{(-c_7 \pm \sqrt{c_7^2 - 4|c_4|^2})\overline{c_4}}{2|c_4|^2}. \tag{16}$$

The reader can check that there are two distinct solutions on the unit circle when $c_7^2-4|c_4|^2<0$, which translates to

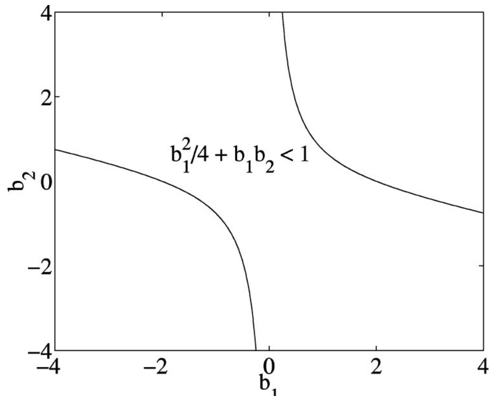

FIG. 2. A plot showing the region $b_1^2/4+b_1b_2<1$ in which we can expect asymmetric clustering.

$$a_2^2 > b_1^2/4 + b_1b_2. \tag{17}$$

Thus for these parameter values there will be stable AC states present when the synchronized state undergoes the DT bifurcation [21]. A sketch of the shape of this region in b_1-b_2 space for fixed a_2 is shown in Fig. 2.

Clearly there is a large region of parameter space including the area around $b_1=b_2=0$ (this point corresponds to antisymmetric functions) where there are stable AC states. Thus it is a frequent occurrence to get stable two-cluster states in the system at the moment of the DT bifurcation. Moreover, the parameter set where this is so can be calculated explicitly for a phase system coupled by two Fourier modes.

V. BIFURCATIONS CREATING AC STATES

AC states exist whenever there is a nontrivial stable solution in the reduced model. For phase systems, this means that such states are created when $P(z)$ in Eq. (12) has a double root on the unit circle distinct from the root at unity. For the two Fourier mode system, $P(z)$ can then be written

$$P(z) = (z-1)(z-e^{it})^2(pz-q) \tag{18}$$

with the four conditions

$$p = c_4, \tag{19}$$

$$q = c_0e^{-2it}, \tag{20}$$

$$-(2pe^{it}+q) = c_3 + c_4, \tag{21}$$

$$pe^{2it} + 2qe^{it} = c_2 + c_3 + c_4. \tag{22}$$

First, we can confirm, as expected, that the fourth root, q/p, is also on the unit circle since $c_0=\overline{c_4}$. We can also check that the last two equations actually correspond to the same condition. The third equation can be rewritten as

$$-(2c_4e^{it} + \overline{c_4}e^{-2it}) = c_3 + c_4. \tag{23}$$

In this complex equation, t parametrizes a codimension-1 surface through parameter space (i.e., a_1,a_2,b_1,b_2 space), and portions of this surface lying in the region of stability of the synchronized solution (i.e., where $a_2^2>b_1^2/4+b_1b_2$) represent bifurcations creating stable asymmetric cluster solutions in the full system.

We will refer to the bifurcation in which AC states are created as the AC bifurcation to distinguish it from the DT bifurcation. Equation (23) will be used to characterize the AC bifurcation curve for the system to be studied below.

VI. A PARTICULAR SYSTEM

A. Description of the system

We now apply the theory developed above to a particular system of phase oscillators studied in [15]. In that reference, the authors make the correct and important point that numerical simulations of systems of globally coupled identical oscillators can converge to linearly *unstable* states in the absence of noise. This phenomenon of "slow switching" has been further explored in [4,18] and the existence of structurally stable heteroclinic cycles in systems with symmetry shown. Here we find that apart from the slow-switching states, the system studied in [15] also contains large parameter sets where asymmetric clustering behavior occurs, in regions where the synchronized state is also stable.

The system in question is

$$\dot{\phi}_i = \omega + g\frac{1}{N}\sum_{j=1}^{N}\{-\sin(\phi_i-\phi_j+\alpha) + r\sin[2(\phi_i-\phi_j)]\}, \quad i=1,\ldots,N, \tag{24}$$

where $\phi_i\in[0,2\pi)$, N is the number of oscillators, and ω,r,g, and α are parameters. The reader can check that this is of the general form (5) with two Fourier modes. ω can be eliminated by shifting to rotating coordinates via the coordinate change $\phi_i\to\phi_i-\omega t$ for each i, and so from the outset we set $\omega=0$. Subsequently, if time is appropriately rescaled ($t\to gt$), then g is eliminated, so we set $g=1$, to get the simplified form

$$\dot{\phi}_i = \frac{1}{N}\sum_{j=1}^{N}\{-\sin(\phi_i-\phi_j+\alpha) + r\sin[2(\phi_i-\phi_j)]\} \quad i=1,\ldots,N. \tag{25}$$

This system has full permutation symmetry, and possesses a stable synchronized state while $\alpha<\alpha_c\equiv\arccos(2r)$. At $\alpha=\alpha_c$, a DT bifurcation occurs.

B. Construction of the reduced model

We simply follow the schema laid out in Sec. III. Equation (25) has the synchronized solution $\phi_i(t)=-\sin(\alpha)t$, $i=1,\ldots,N$, which can be substituted back to give

$$\dot{\phi} = -\sin[\phi+\sin(\alpha)t+\alpha] + r\sin\{2[\phi+\sin(\alpha)t]\}. \tag{26}$$

Rotating out time dependence by substituting $\theta=\phi+\sin(\alpha)t$ gives us the autonomous equation

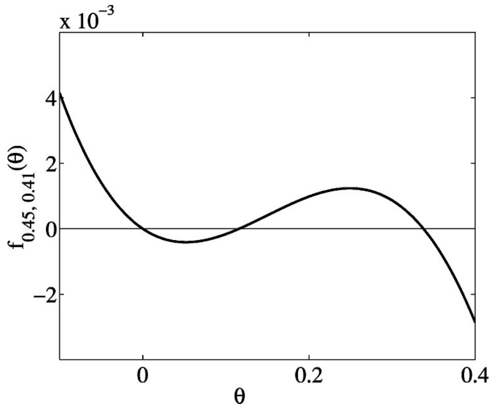

FIG. 3. A plot of $f_{0.45,0.41}(\theta)$ in the region near zero, showing the existence of the base solution at $\theta=0$, and another stable solution between $\theta=0.3$ and $\theta=0.4$. The value of α is such that we are in the region shortly before the synchronized solution loses stability (which happens at $\alpha \simeq 0.451$ in this case).

$$\dot{\theta} = -\sin(\theta+\alpha) + r\sin(2\theta) + \sin\alpha \quad [\equiv f_{r,\alpha}(\theta)]. \quad (27)$$

This equation always has the fixed point $\theta=0$, which we call the base solution. As the system is periodic, generically all its solutions come in pairs (except at bifurcations) and so it also has an unstable solution. From Sec. II, the question is whether it has any further pairs of solutions.

C. Analysis of the model and the full system

An example of a plot of $f_{r,\alpha}(\theta)$ is shown in Fig. 3. We have chosen parameter values where the reduced model does indeed display a second stable solution, and even without any analysis, a quick exploration of r-α parameter space suggests that this is not uncommon.

Our next task is to characterize completely the region of parameter space (i.e., r-α space) where AC states exist in this system. Following [15], we explore $0<r<0.5$, $0<\alpha<\alpha_c$. We first use techniques from Sec. V to calculate that the AC bifurcation takes place on the line parametrized by

$$\alpha = -\cot\left(\frac{2\sin t + \sin(2t)}{1 + 2\cos t - \cos(2t)}\right), \quad (28)$$

$$r = -\frac{\sin\alpha}{2\sin t + \sin(2t)}. \quad (29)$$

For thoroughness we then pick values of r and α in the relevant region of phase space and use the methods illustrated in Sec. III to confirm that $f_{r,\alpha}$ has nontrivial stable zeros for these parameter values. The details are sketched in Appendix A. In Fig. 4, we have plotted the AC bifurcation line, and parameter values where the reduced model has a second stable state, and there are hence AC solutions in the full system.

To get a flavor for what this diagram means, we explore a few regions of parameter space in the reduced model. First, we look at what happens when we cross the DT bifurcation line in a region where there are no asymmetric clusters. At

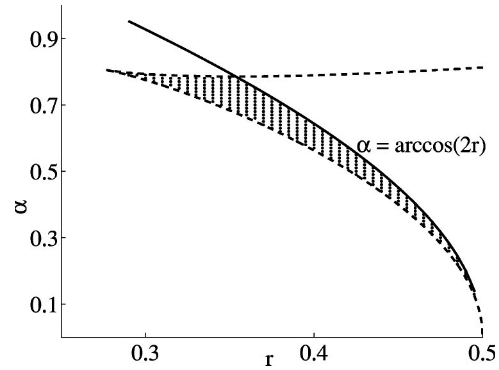

FIG. 4. A scatter plot of the parameter values in r-α space where the reduced model has a second stable solution and where there must be stable AC states in the full system. Also plotted is the DT bifurcation line $\alpha=\arccos(2r)$ (thick line), and the relevant portion of AC bifurcation line (dashed line). See Appendix A for details of the calculations. The second-order bifurcation point where the AC and the DT bifurcation lines meet can be calculated using Eq. (17) to be at $r=\sqrt{1/8}\simeq 0.354$, $\alpha=\pi/4$.

$r=0.25$, the DT bifurcation occurs at $\alpha=\pi/3\simeq 1.047$. In Fig. 5, we see what happens locally to the base solution in the reduced model at the DT bifurcation: An unstable state exchanges stability with the base solution in a transcritical bifurcation.

In this particular instance, there is no more global behavior to comment on—the unstable solution which exchanges stability with the base solution is the only other real zero of $f_{r,\alpha}$.

Although the local picture remains the same, the global picture becomes more interesting if we fix $r=0.45$ and cross through the AC and DT bifurcation lines in succession. Figure 6 shows what happens. First a new pair of solutions is born in the AC bifurcation, and then the unstable solution collides with the base solution at the DT bifurcation. Translating to the full model, first pairs of AC states are born. Then the newly born unstable clusters collide with the synchronized state at the DT bifurcation. We believe that this scenario is very common in systems of globally coupled oscillators and have observed it on other occasions [10].

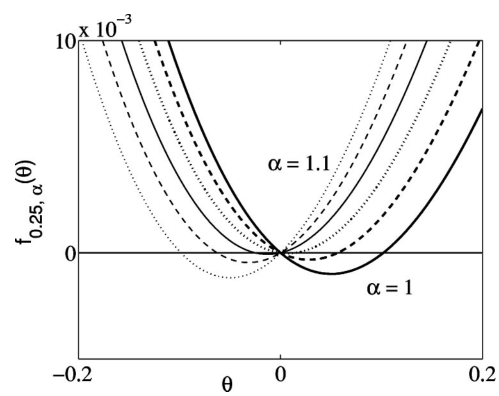

FIG. 5. Plots of $f_{0.25,\alpha}(\theta)$ for $\alpha=1.00$, 1.02, 1.04, 1.06, 1.08, and 1.10 in a region near $\theta=0$. As we move through the sequence, the synchronized state loses stability. This manifests in the reduced model as a transcritical bifurcation of the base solution.

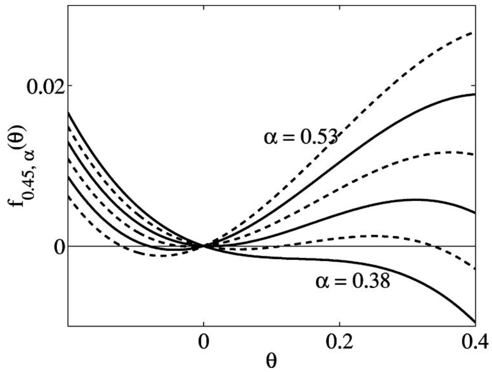

FIG. 6. Plots of $f_{0.45,\alpha}(\theta)$ for $\alpha=0.38$, 0.41, 0.44, 0.47, 0.50, and 0.53 in a region near $\theta=0$. As we move through the sequence, first a stable and unstable pair of solutions is born, and then the newly formed unstable solution collides with the synchronized state and exchanges stability with it.

The final scenario in Fig. 4 is what happens when we pass through a region where AC bifurcations occur, but do not shortly precede the DT bifurcation. If we fix r at 0.3 and increase α from 0.77 to 0.8, we pass through an AC region, getting two saddle-node bifurcations as shown in Fig. 7. Thus AC states are created and destroyed while the synchronized solution remains stable throughout.

D. Numerical simulations

For completeness, we confirm numerically that stable clusters exist in the full system when there is multistability in the reduced system. At $r=0.3$, $\alpha=0.79$, it is easy to compute that $f_{0.3,0.79}(\theta)$ has a stable zero at $\theta_s \simeq 1.188$, and hence that there are stable AC solutions in the full model with a spacing between clusters of about 1.188. We simulate the full model at these parameter values [22]. In this particular instance, the system almost always converges to AC states from random initial conditions as illustrated in Fig. 8, even though the synchronized solution is stable. This is a physically interesting situation, where the system prefers a few oscillators to perform a motion different from the majority, but cannot support a significant number of oscillators in the smaller cluster.

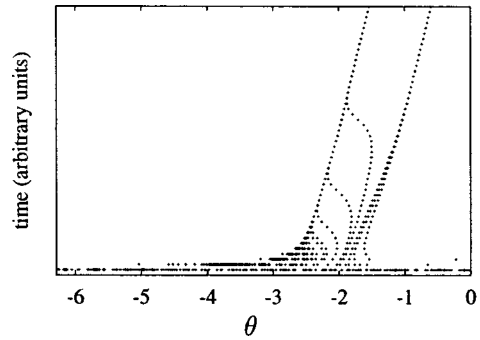

FIG. 8. Oscillators with random initial conditions coalescing to an asymmetric two-cluster state. Parameter values are $r=0.3$, $\alpha=0.79$. Time steps have been chosen to be approximately one rotation (for clarity). In this case, $N=200$ and the smaller cluster contains three oscillators. It is interesting to note how excess oscillators leave the smaller cluster as it forms. We can say that the cluster "supports" a maximum number of oscillators, and any excess oscillators above this number are expelled.

If we step α up, at some point the AC solutions disappear and we find the synchronized solution. The way this happens is depicted in Fig. 9.

It is worth asking how closely the AC-bifurcation line corresponds to the moment of creation of asymmetric clusters in a large but finite system. For this purpose we simulated the system for $N=200$ and looked for stable asymmetric clusters containing a single oscillator in the smaller cluster in regions close to the AC bifurcation. The results of this simulation are shown in Fig. 10. For parameter values close to the AC bifurcation line, the asymmetric cluster states are easily found. It is worth noting that for parameter values actually *on* the bifurcation line, only the synchronized state is found, suggesting that the bifurcation which creates the states with one oscillator separated from the rest happens to the right of the AC bifurcation line.

VII. FURTHER COMMENTS ON THE LOSS OF STABILITY OF THE SYNCHRONIZED STATE

We have made some further observations about the basin of attraction of the synchronized state as we approach the DT

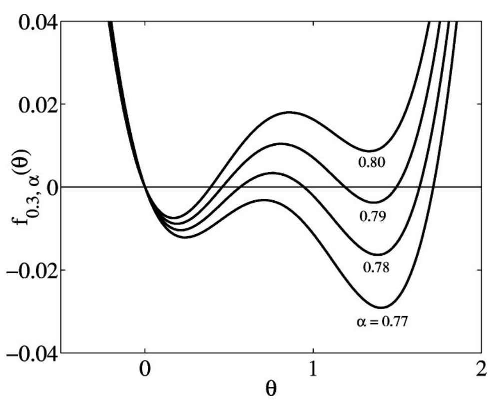

FIG. 7. Plots of $f_{0.3,\alpha}(\theta)$ for $\alpha=0.77$, 0.78, 0.79, and 0.80. As we move through this sequence of parameter values, there are two saddle-node bifurcations during which a pair of solutions (one of which is stable) is created, and then a pair is destroyed.

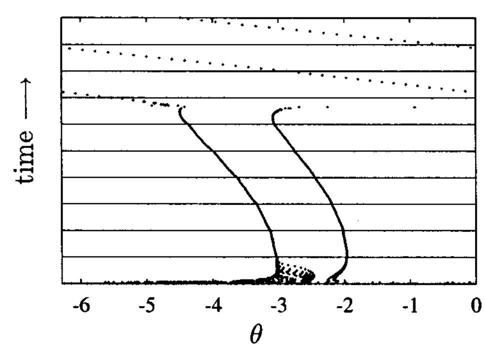

FIG. 9. An AC state losing stability to the synchronized solution. $r=0.3$. α starts at a value of 0.796 and each horizontal line represents an increase in α of 0.002. We see that between 0.806 and 0.808, the cluster state loses stability.

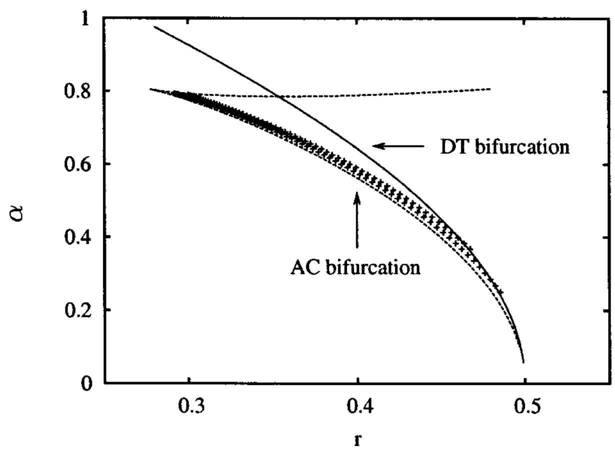

FIG. 10. The crosses indicate parameter values at which simulations of the full system with $N=200$ were carried out and stable asymmetric cluster states containing a single oscillator in the smaller cluster were found. The AC and DT bifurcation lines are shown for reference.

bifurcation. These observations are based on a large number of numerical simulations. There appear to be two distinct scenarios in the lead up to bifurcation. The first is that although the synchronized state becomes more and more weakly attracting, it still appears to attract the entirety of phase space, right up to the moment of bifurcation. In this scenario, after bifurcation it is the attractors which include stable heteroclinic cycles and contain unstable cluster states in their closure, rather than stable cluster states, which tend to be found [4,18]. We speculate that the bifurcation is a transcritical-homoclinic bifurcation as described in [6].

The other scenario is that the basin of attraction of the synchronized state shrinks as bifurcation approaches. Shortly before bifurcation, it becomes much more likely that we will find stable cluster states rather than the synchronized state, to the point where, even in a region where we know the synchronized state to be stable, a thousand sets of initial conditions do not find it. In this scenario, after bifurcation it is stable cluster states which tend to be found.

VIII. BREAKING THE SYMMETRY

Real applications will rarely consist of identical oscillators with entirely symmetrical coupling. In some situations, breaking the symmetry or adding noise has fundamental effects on the behavior of globally coupled systems [5,15,19], often destroying various invariant structures in phase space. Hence it is a valid question to ask what happens to the scenarios described above if the symmetry is broken. In reply to this question, we note first that all the states we describe away from the bifurcation points, in particular the synchronized state and asymmetric cluster states, are hyperbolic. We thus expect these states to survive for sufficiently small perturbations of the system. What we mean is that symmetry broken versions of these states (which continue to be periodic, but where there is now some phase spread within each cluster) continue to exist.

Further, although the degenerate transcritical bifurcation of the synchronized state will be destroyed by the symmetry breaking (and replaced with a large number of saddle-node bifurcations), the bifurcations giving rise to asymmetric cluster states (which are saddle-node bifurcations) will survive in one-parameter families. Of course there is the practical question of what "sufficiently small perturbations" means. In order to get some sense of an answer, we have simulated a version of Eq. (24) with nonidentical oscillators,

$$\dot{\phi}_i = \omega + \epsilon_i + g\frac{1}{N}\sum_{j=1}^{N}\{-\sin(\phi_i - \phi_j + \alpha) + r\sin[2(\phi_i - \phi_j)]\}, \quad i = 1,\ldots,N. \tag{30}$$

The quantities ϵ_i—perturbations to the normal frequency of the oscillators—are uniformly distributed on intervals of the form $[-10^{-D}/2, 10^{-D}/2]$. The parameter D thus controls the size of the perturbations. As before, we choose $N=200$. At parameter values $r=0.3$, $\alpha=0.79$, where asymmetric clusters coexist with the synchronized state, and in fact attract the majority of phase space (see Sec. VI D), we find as expected that for sufficiently small perturbations (i.e., large D), the asymmetric cluster states continue to exist and to dominate phase space.

But as we move away from perfect symmetry, various trends become apparent. To understand them, consider the particular bifurcation giving rise to a set of cluster states with two oscillators in the smaller cluster. There are $N(N-1)$ such cluster states, and with full permutation symmetry, the $N(N-1)$ bifurcations creating these clusters occur at the same parameter value. However, with broken symmetry—assuming none of the bifurcations is destroyed—they will occur at slightly different parameter values. This means that at a given parameter value, different states will be more or less stable, in the sense of being closer to or further from bifurcation. Thus, we expect the symmetry breaking to mean that some clusters are favored over others—in other words, certain oscillators prefer to be in the smaller cluster.

Numerically this behavior manifests as follows: As the perturbation increases, the oscillators in the smaller cluster tend to come from one edge of the distribution. Because the smaller cluster lies to the right of the larger one at these parameter values, oscillators whose perturbed frequencies cause them to move in this direction are more likely to end up in the second cluster than those whose perturbations incline them to move in the opposite direction [23].

To make this notion more precise, we define $q_i = 2\epsilon_i/10^{-D}$ for the ith oscillator drawn from a distribution parametrized by D, and $Q(\mathcal{C}) = (1/|C|)\Sigma_{j\in C} q_j$ as the average q for a cluster $\mathcal{C}$, where C is the index set of the oscillators in the cluster. Then a Q value near 0 for oscillators in the smaller cluster would represent unbiased selection of oscillators from the distribution to make up the cluster, while Q values differing from 0 would represent a bias in the choice of oscillators to form the smaller cluster. What happens is that as the symmetry is broken, oscillators which form the smaller cluster are increasingly chosen from one end of the distribution. This trend is illustrated in Fig. 11.

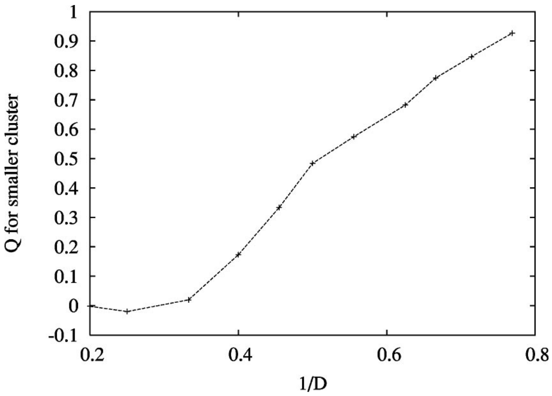

FIG. 11. Variation of the average Q value of the oscillators in the smaller cluster as the symmetry is broken (via the parameter D). Each point is based on 1000 simulations with oscillator initial conditions drawn from a uniform distribution on the circle. For each value of D, there are one or two strongly preferred cluster sizes. For perturbations greater than the values shown, the clusters start to blur together, and there can be some ambiguity about which cluster a particular oscillator belongs to.

A second observation is that as the perturbation increases, the average size of the smaller cluster increases, as illustrated in Fig. 12.

We can interpret these trends as follows.

(i) As we break the symmetry of the system, cluster states with certain oscillators in them are more stable and attract more of phase space.

(ii) The greater stability of these states allows them to support a larger number of oscillators.

IX. GENERAL REMARKS AND CONCLUSIONS

It is in general hard to characterize completely the behavior of systems of coupled oscillators, due largely to the high dimension of the state space. Due to its simplicity and physical importance, a large emphasis has traditionally been placed on synchronization. However, we have shown that asymmetric clustering is another phenomenon which is simple enough to be amenable to analysis—in fact, for simple systems of oscillators, asymmetric clustering behavior can be completely characterized. It is important both because it is often associated with the loss of stability of the synchronized state, and because such behavior could easily (and wrongly) be attributed to noise or experimental error in a real physical situation.

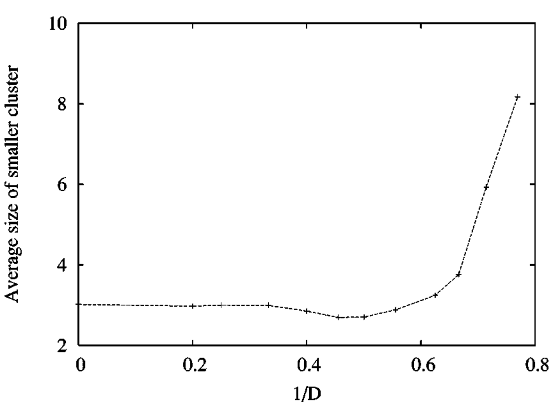

FIG. 12. Variation of the average size of the smaller cluster, based on the same numerical experiments as in Fig. 11.

We have also seen numerically that the phenomenon of asymmetric clustering survives significant destruction of the symmetry of the system, although particular asymmetric clusters become preferred over others.

It is worth noting that although this paper is about asymmetric clustering, numerically we can often construct more symmetric clusters simply by moving oscillators from the larger to the smaller cluster. This process sometimes leads to the breakdown of the cluster state, but not always. In fact, the proportion β of oscillators in a cluster can be treated as a parameter like any other from the point of view of exploring clustering.

ACKNOWLEDGMENT

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APPENDIX A: FINDING THE AC BIFURCATION AND FINDING ZEROS OF $f_{r,\alpha}$

We can quite easily calculate when an AC bifurcation takes place in Eq. (24) using the general technique in Sec. V. First note that for this system, $a_1=-\cos\alpha$, $b_1=-\sin\alpha$, $a_2=r$, and $b_2=0$. So $c_3=-\sin\alpha/2$ and $c_4=-ir/2$.

The AC bifurcation condition [Eq. (23)] can be written as

$$2ir(\cos t + i\sin t) - ir[\cos(2t) - i\sin(2t)] = -\sin\alpha + i(\cos\alpha - r). \tag{A1}$$

Equating real and imaginary parts and a little manipulation leads to

$$\alpha = -\cot\left(\frac{2\sin t + \sin(2t)}{1 + 2\cos t - \cos(2t)}\right), \tag{A2}$$

$$r = -\frac{\sin\alpha}{2\sin t + \sin(2t)}. \tag{A3}$$

These expressions are used to plot the portion of the curve shown in Fig. 4.

We now sketch briefly how we confirm the behavior of $f_{r,\alpha}$ for the values of r and α as plotted in Fig. 4.

1. We use the complex representations of $\sin\theta$ and $\cos\theta$ to get that $f_{r,\alpha}=0$ when

$$0 = -irz^4 + Mz^3 + 2Sz^2 - Pz + ir \tag{A4}$$

$$=(z-1)(-irz^3 + Qz^2 + Rz - ir), \tag{A5}$$

where $z=e^{i\theta}$, $C=(\cos\alpha)/2$, $S=(\sin\alpha)/2$, $M=(iC-S)$, $P=(iC+S)$, $Q=(M-ir)$, $R=(P-ir)$. The root of 1 corresponds to the root of zero in the original equation for $f_{r,\alpha}$.

2. The cubic polynomial that remains on the right-hand side of Eq. (A5) after factoring generically has three distinct roots in the complex plane, which we call z_1, z_2, z_3. We solve for these roots using a symbolic algebra package (giving a long and unwieldy expression for each root in terms of r and α).

3. From arguments in Sec. III to each z_i, there corresponds a unique $\theta_i \in [0, 2\pi)$. The points we plot in Fig. 4 are those values of r and α where all three θ_i are real (i.e., all three z_i lie on the unit circle.)

Note that from Sec. III there can be no more than one nontrivial stable real root of $f_{r,\alpha}$. Hence we can only predict asymmetric two-cluster states and never three-cluster states for this system (which is not to say that three-cluster states cannot in general exist). If more Fourier modes were incorporated into the coupling, this situation would change.

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[20] Throughout the text, we bear in mind that roots can have multiplicity of greater than 1, but avoid repeating this fact everywhere that it is relevant.
[21] Strictly speaking, what we have proved is that AC states are present at parameter values arbitrarily close to the bifurcation point, not actually at the bifurcation point.
[22] Obviously the full system must be simulated with small added noise to avoid the possibility of spurious clustering behavior as mentioned in [15].
[23] More precisely, it is these states which have larger basins of attraction, and to which random initial conditions converge.