Generalized synchronization in time-delayed systems

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We investigate the generalized synchronization between two unidirectionally linearly and nonlinearly coupled chaotic nonidentical Ikeda models and find existence conditions of the generalized synchronization. Also we study the chaos synchronization between nonidentical Ikeda models with variable feedback-delay times and find the existence and sufficient stability conditions for the retarded synchronization manifold with the coupling-delay lag time. Generalization of the approach to the wide class of nonlinear chaotic systems is also presented.

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I. INTRODUCTION

Chaos synchronization [1] is of fundamental importance in a variety of complex physical, chemical, and biological systems [2]. Possible application areas of chaos synchronization are in secure communications, optimization of nonlinear system performance, modeling brain activity, and pattern recognition phenomena [2].

Different types of synchronization have been found in interacting chaotic systems. Complete, generalized, phase, lag, and anticipating synchronizations of chaotic oscillators have been described theoretically and observed experimentally. Complete synchronization is characterized by the convergence of two chaotic trajectories, $y(t)=x(t)$ [1]. Generalized synchronization is defined as the presence of some functional relation between the states of response and drive—i.e., $y(t)=F(x(t))$ [3]. Phase synchronization means entrainment of phases of chaotic oscillators, $n\Phi_x-m\Phi_y=\text{const}$ (n and m are integers), while their amplitudes remain chaotic and uncorrelated [4]. Lag synchronization *for the first time* was introduced by Rosenblum *et al.* [5] under certain approximations in studying synchronization between *bidirectionally* coupled systems described by the ordinary differential equations (no intrinsic delay terms) with *parameter mismatches*: $y(t)\approx x_\tau(t)\equiv x(t-\tau)$ with positive τ. Anticipating synchronization [6–8] also appears as a coincidence of shifted-in-time states of two coupled systems, but in this case the driven system anticipates the driver, $y(t)=x(t+\tau)$ or $x=y_\tau$, $\tau>0$. An experimental observation of anticipating synchronization in external cavity laser diodes [9] has been reported recently; see also [10] for a theoretical interpretation of the experimental results. The concept of inverse anticipating synchronization $x=-y_\tau$ was introduced in [11].

Due to finite signal transmission times, switching speeds, and memory effects, time-delayed systems are ubiquitous in nature, technology, and society [12]. Therefore the study of synchronization phenomena in such systems is of high practical importance. Time-delayed systems are also interesting because the dimension of their chaotic dynamics can be made arbitrarily large by increasing their delay time. From this point of view these systems are especially appealing for secure communication schemes [13].

The role of parameter mismatches in synchronization phenomena is quite versatile. In certain cases parameter mismatches are detrimental to the synchronization quality: in the case of small parameter mismatches the synchronization error does not decay to zero with time, but can show small fluctuations about zero or even a nonzero mean value; larger values of parameter mismatches can result in the loss of synchronization [8,14]. In some cases parameter mismatches change the time shift between the synchronized systems [15]. In certain cases their presence is necessary for synchronization. We emphasize that the crucial role of parameter mismatches for lag synchronization between *bidirectionally* coupled systems was first studied in [5] by Rosenblum *et al.* As such, lag synchronization cannot be observed if two oscillators are completely identical; see, e.g., [16] and references therein.

In a recent paper [17] the complete and generalized synchronizations of one-way, linearly coupled the Mackey-Glass systems is studied. The authors of [17] numerically investigated the largest conditional Lyapunov exponent of the complete synchronization manifold to find the necessary existence conditions of the mode of synchronizations in the Mackey-Glass system. In this paper we analytically investigate the generalized synchronization between two unidirectionally both linearly and nonlinearly coupled chaotic Ikeda models. We use a powerful Krasovskii-Lyapunov functional approach to derive existence conditions of the generalized synchronization in time-delayed systems. Numerical simulations fully support the analytical approach. Generalization of the approach to the wide class of nonlinear systems is also presented.

Also we investigate chaos synchronization between nonidentical Ikeda models with variable feedback-delay times and find both existence and sufficient stability conditions for

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the retarded synchronization manifold with the coupling-delay lag time. It is of immense interest to study chaos and synchronization in time-delayed systems with variable feedback-delay times. The basic interest is driven by the fact that there is little research on this important subject in the literature [18,19]. The practical interest is motivated by the appreciation that time-delayed systems with variable delay times are more realistic and would have important implications for more secure communication schemes [19].

II. GENERALIZED SYNCHRONIZATION BETWEEN THE LINEARLY COUPLED IKEDA SYSTEMS

Consider synchronization between the linearly coupled Ikeda systems,

$$\frac{dx}{dt} = -\alpha_1 x + m_1 \sin x_{\tau_1}, \tag{1}$$

$$\frac{dy}{dt} = -\alpha_2 y + m_2 \sin y_{\tau_2} + K(x-y). \tag{2}$$

The Ikeda model was introduced to describe the dynamics of an optical bistable resonator and is well known for delay-induced chaotic behavior [6,11,20]. It also plays an important role in electronics and physiological studies [6]. Physically x is the phase lag of the electric field across the resonator, $\alpha_{1,2}$ are the relaxation coefficients for the driving x and driven y dynamical variables (without loss of generality we assume $\alpha_1=\alpha-\delta$, $\alpha_2=\alpha+\delta$), and $m_{1,2}$ are the laser intensities injected into the systems. τ_1 and τ_2 are the round-trip times of the light in the resonators or feedback delay times in the coupled systems; K is the coupling rate between the driver x and the response system y.

In this section we will investigate kind of synchronizations depending on τ_1, τ_2 and α_1, α_2. First consider the case of synchronization depending on τ_1 and τ_2 ($\alpha_1=\alpha_2=\alpha$). We begin with the case of complete synchronization for $\tau_1=\tau_2$:

$$x = y. \tag{3}$$

We denote the error signal by $\Delta=x-y$. Then from systems (1) and (2) we find the error dynamics $d\Delta/dt=-(\alpha+K)\Delta+m_1 \sin x_{\tau_1}-m_2 \sin y_{\tau_1}$. Thus, under the condition

$$m_1 = m_2, \tag{4}$$

the error dynamics can be written as

$$\frac{d\Delta}{dt} = -(\alpha+K)\Delta - \Delta_{\tau_1} m_1 \cos x_{\tau_1}. \tag{5}$$

It is obvious that $\Delta=0$ is a solution of system (5). To study the stability of the complete synchronization manifold $y=x$ one can use a Krasovskii-Lyapunov functional approach. According to [12,6,21], the sufficient stability condition for the trivial solution $\Delta=0$ of the time-delayed equation $d\Delta/dt=-r(t)\Delta+s(t)\Delta_\tau$ is $r(t)>|s(t)|$. This can be found by investigating the positively defined Krasovskii-Lyapunov fuctional

$$V(t) = \frac{1}{2}\Delta^2 + \mu\int_{-\tau}^{0}\Delta^2(t+t_1)dt_1, \tag{6}$$

where $\mu>0$ is an arbitrary positive parameter. As shown in [12,6,21], the solution $\Delta=0$ is stable if the derivative of the functional (6) along the trajectory of the equation $d\Delta/dt=-r(t)\Delta-s(t)\Delta_\tau$ is negative. In general this negativity condition is of the form $4(r-\mu)\mu>s^2$ and $r>\mu>0$. As the value of μ that will allow s^2 as large as possible is $\mu=r/2$, the asymptotic stability condition for $\Delta=0$ can be written as

$$r^2 > s^2, \tag{7}$$

which is equivalent to $r>|s|$. This result is valid for both constant and time-dependent coefficients r and s [in the latter case $r(t)$ and $s(t)$ should be bounded continuous functions [12]].

We notice that one can still use the functional (6) to estimate the sufficient stability condition for the trivial solution $\Delta=0$ of the time-delayed equation $d\Delta/dt=-r(t)\Delta+s(t)\Delta_\tau$, which is $r(t)>|s(t)|$ for time-dependent τ [12]. Namely, as presented in [12], when $\tau=\tau(t)$ is continuously differentiable and bounded, the solution $\Delta=0$ to $d\Delta/dt=-r(t)\Delta-s(t)\Delta_{\tau(t)}$ is uniformly asymptotically stable if $a(t)>\mu>0$ and $[2r(t)-\mu](1-d\tau/dt)\mu>s^2(t)$ uniformly in t. Applying the same procedure as in the case of constant feedback-delay time, we can find the value of μ that will allow s^2 to be as large as possible: $\mu=r$. Thus we find that the sufficent stability condition for the $\Delta=0$ solution of the time delay equation with time-dependent coefficients $d\Delta/dt=-r(t)\Delta-s(t)\Delta_{\tau(t)}$ is

$$r^2(t)\left(1-\frac{d\tau(t)}{dt}\right) > s^2(t). \tag{8}$$

Note that for the constant-delay-time cases the inequality (8) is reduced to the well-known sufficient stability condition $r>|s|$ [21,6,11].

Thus we obtain that the sufficient stability condition for the complete synchronization manifold $y=x$ for Eqs. (1) and (2) can be written as

$$\alpha + K > |m_1|. \tag{9}$$

As Eq. (5) is derived for small Δ, condition (9) it is valid only locally. Condition (4) is the existence condition for complete synchronization between the unidirectionally coupled systems (1) and (2). Condition (9) can also be used for the estimation of the critical coupling strength K needed for the high-quality synchronization [21].

Next we consider the case of $\tau_1\neq\tau_2$. From the analysis above it is clear that complete synchronization is not the synchronization manifold for systems with different values of feedback delay times. Then for such a case we study the possibility of generalized synchronization between the driver and driven systems (1) and (2). For this purpose we use the auxiliary system method to detect generalized synchronization [3,17]: that is, given another identical driven auxiliary system $z(t)$, generalized synchronization between $x(t)$ and $y(t)$ is established with the achievement of complete synchronization between $y(t)$ and $z(t)$. In fact, the auxiliary method allows us to find the local stability condition of the generalized synchronization [17].

Thus consider complete synchronization between the following Ikeda systems:

$$\frac{dy}{dt} = -\alpha y + m_2 \sin y_{\tau_2} + K(x-y), \quad (10)$$

$$\frac{dz}{dt} = -\alpha z + m_2 \sin z_{\tau_2} + K(x-z). \quad (11)$$

Applying the error dynamics approach described above for the case of complete synchronization we find complete synchronization between the $y(t)$ and $z(t)$; therefore, generalized synchronization between systems (1) and (2) exists if

$$\alpha + K > |m_2|. \quad (12)$$

Equation (12) in fact is the local sufficient stability condition for the generalized synchronization between systems (1) and (2). We note that using the error dynamics approach above it is straightforward to study generalized synchronization between systems (1) and (2) for $m_1 \neq m_2$.

Finally in this section we present an example of the linearly coupled Ikeda model when parameter mismatches for the relaxation coefficients is the only way to achieve synchronization. Consider synchronization between the Ikeda systems

$$\frac{dx}{dt} = -\alpha_1 x + m \sin x_{\tau_1},$$

$$\frac{dy}{dt} = -\alpha_2 y + m \sin y_{\tau_1} + K x_{\tau_2}. \quad (13)$$

First we consider the case of constant feedback-delay time and show that $y=x_{\tau_2}$ is the retarded synchronization manifold if the parameter mismatch $\alpha_2-\alpha_1=2\delta$ is equal to the coupling rate K [22]. This can be seen by the dynamics of the error $\Delta=x_{\tau_2}-y$:

$$\frac{d\Delta}{dt} = -(\alpha+\delta)\Delta + (2\delta - K)x_{\tau_2} + m \cos x_{\tau_1+\tau_2}\Delta_{\tau_1}. \quad (14)$$

The sufficient stability condition for the retarded synchronization manifold $y=x_{\tau_2}$ can be written as $\alpha+\delta=\alpha_2>|m|$. Thus, we find that the retarded chaos synchronization manifold $y=x_{\tau_2}$ occurs *only* under parameter mismatch—i.e., $\alpha_1 \neq \alpha_2$. By analyzing the corresponding error dynamics one can also establish that without the parameter mismatch—i.e., $\alpha_1=\alpha_2=\alpha$—neither $y=x_{\tau_2-\tau_1}$ nor $y=x_{\tau_1-\tau_2}$ is the synchronization manifold. We also emphasize that for both $\alpha_1=\alpha_2$ and $\alpha_1 \neq \alpha_2$ system (13) admits neither complete (we notice that for special case of $\tau_2=0$ $y=x_{\tau_2}$ is the complete synchronization manifold, which exists if $\alpha_1 \neq \alpha_2$) nor anticipating *chaos* synchronization. We emphasize that this result is due to the linear coupling between the synchronized systems. The importance of the role of the form of coupling between the synchronized systems is underlined in [6]. In the case of nonlinear (sinusoidal) coupling for identical drive and response Ikeda systems, depending on the relation between the feedback delay time and the coupling delay time retarded, complete or anticipating synchronization can occur; see, e.g., [23] and references therein.

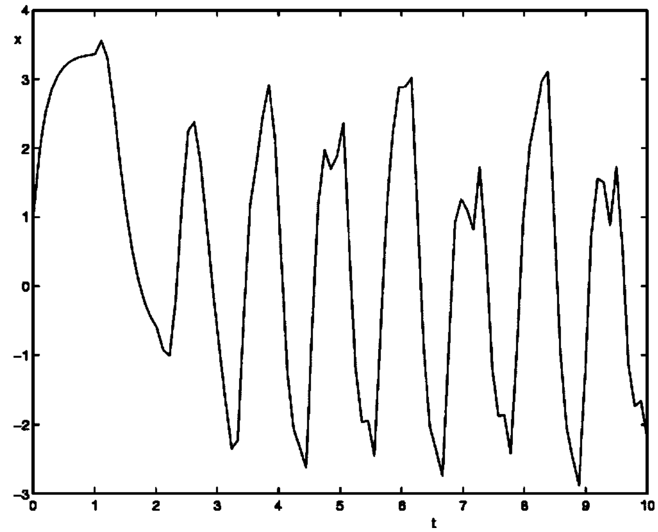

FIG. 1. Numerical simulation of the Ikeda model [Eq. (1)]: time series of the driver $x(t)$.The parameters of the Ikeda model are $\tau_1=1$, $\alpha=5$, and $m_1=20$. Dimensionless units.

Next we consider the case of time-dependent delay time $\tau_1(t)$. First we notice that as in the case of time-independent delay times $2\delta=K$ is the condition of existence for the $y=x_{\tau_2}$ synchronization manifold. Next applying the general formula (8) derived earlier in the paper we write the sufficient stability condition for the synchronization manifold $y=x_{\tau_2}$ in the following form:

$$\alpha_2^2\left(1-\frac{d\tau_1(t)}{dt}\right) > m^2. \quad (15)$$

As an example consider the following sinusoidal form of the variable delay time:

$$\tau_1(t) = \tau_0 + \tau_a \sin(\omega t), \quad (16)$$

where τ_0 is the zero-frequency component, τ_a is the amplitude, and $\omega/2\pi$ is the frequency of the modulation. Then for the concrete form of variable delay time (16) the sufficient stability condition (15) can be written as

$$\alpha_2^2[1-\tau_a\omega\cos(\omega t)] > m^2. \quad (17)$$

Numerical simulations fully confirm the analytical results. Eqs. (1), (2), (10), and (11) were simulated using the DDE23 program [24] in MATLAB 6 for $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $m_1=m_2=20$, and $K=30$. Figure 1 shows the time series of the driver $x(t)$. Figure 2 shows generalized synchronization between x and y.

III. GENERALIZED SYNCHRONIZATION BETWEEN THE NONLINEARLY COUPLED IKEDA SYSTEMS

In this section we study synchronization between the following nonlinearly coupled Ikeda systems:

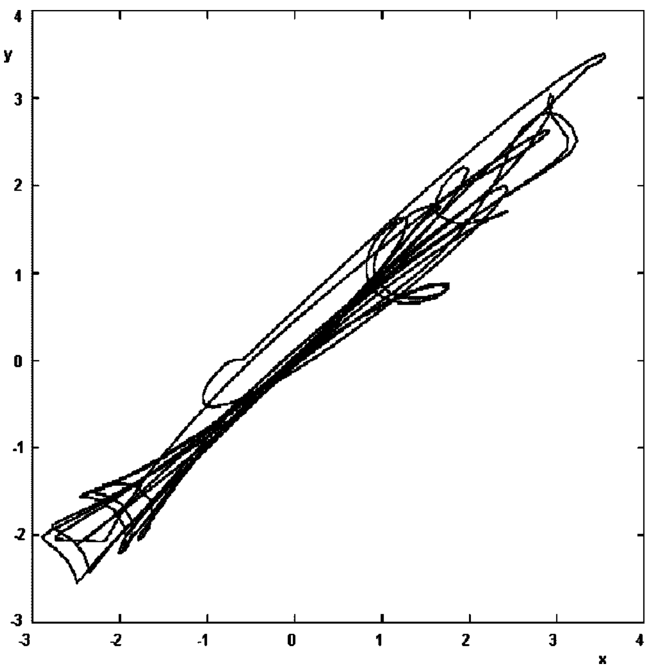

FIG. 2. Numerical simulation of systems (1) and (2): generalized synchronization between x and y. The parameters are $\tau_1=1$, $\tau_2=2$, $\alpha=5$, $m_1=20$, $m_2=20$, and $K=30$. Dimensionless units.

$$\frac{dx}{dt} = -\alpha x + m_1 \sin x_{\tau_1}, \quad (18)$$

$$\frac{dy}{dt} = -\alpha y + m_2 \sin y_{\tau_2} + K \sin x_{\tau_3}. \quad (19)$$

As established in [23], for $\tau_1=\tau_2$ under conditions

$$m_1 = m_2 + K, \quad (20)$$

systems (18) and (19) allow for the synchronization manifold

$$y = x_{\tau_3-\tau_1}. \quad (21)$$

The stability condition of the synchronization manifold (21) was derived using the error dynamics approach and the Krasovskii-Lyapunov functional:

$$\alpha > |m_2|. \quad (22)$$

We note that for $\tau_3>\tau_1$, $\tau_3=\tau_1$, and $\tau_3<\tau_1$, Eq. (21) is the retarded, complete, and anticipating synchronization manifold [11,23], respectively.

For $\tau_1 \neq \tau_2$ obviously the manifold (21) is longer no the synchronization manifold. To investigate generalized synchronization between x and y variables we have to consider complete synchronization between the systems:

$$\frac{dy}{dt} = -\alpha y + m_2 \sin y_{\tau_2} + K \sin x_{\tau_3}, \quad (23)$$

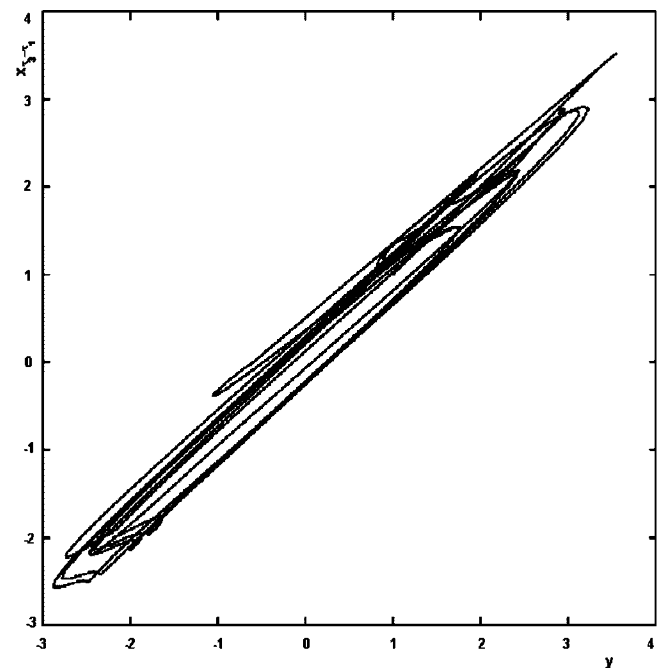

FIG. 3. Numerical simulation of systems (18) and (19): generalized synchronization between $x_{\tau_3-\tau_1}$ and y. The parameters are $\tau_1=1$, $\tau_2=2$, $\tau_3=3$, $\alpha=5$, $m_1=20$, $m_2=3$, and $K=17$. Dimensionless units.

$$\frac{dz}{dt} = -\alpha z + m_2 \sin z_{\tau_2} + K \sin x_{\tau_3}. \quad (24)$$

One finds that $y=z$ is the complete synchronization manifold between systems (23) and (24). The corresponding sufficient synchronization condition reads $\alpha>|m_2|$. We conclude that for $\tau_1=\tau_2$ we find complete synchronization between systems (18) and (19); for $\tau_1 \neq \tau_2$, one observes generalized synchronization between systems (18) and (19). Equations (18), (19), (23), and (24) were simulated for $\alpha=5$, $\tau_1=1$, $\tau_2=2$, $\tau_3=3$, $m_1=20$, $m_2=3$, and $K=17$. Figure 3 shows the generalized synchronization between $x_{\tau_3-\tau_1}$ and y.

IV. GENERAL APPROACH

Consider complete synchronization between the linearly coupled time-delayed systems of general form

$$\frac{dx}{dt} = -\alpha x + m_1 f(x_{\tau_1}), \quad (25)$$

$$\frac{dy}{dt} = -\alpha y + m_2 f(y_{\tau_2}) + K(x-y), \quad (26)$$

where f is differentiable generic nonlinear function.

One finds that for $\tau_1=\tau_2$ under the condition

$$m_1 = m_2, \quad (27)$$

Eqs. (25) and (26) admit the complete synchronization manifold

$$y = x. \tag{28}$$

The sufficient stability condition of Eq. (28) can be derived from the Krasovskii-Lyapunov functional approach:

$$\alpha + K > |m_1 f'(x_{\tau_1})|. \tag{29}$$

Here f' stands for the derivative of f with respect to time. For $\tau_1 \neq \tau_2$, Eq. (28) is no longer the synchronization manifold.

To investigate generalized synchronization between systems (25) and (26) we apply an auxiliary system approach to investigate complete synchronization between systems:

$$\frac{dy}{dt} = -\alpha y + m_2 f(y_{\tau_2}) + K(x - y), \tag{30}$$

$$\frac{dz}{dt} = -\alpha z + m_2 f(z_{\tau_2}) + K(x - z). \tag{31}$$

We find that the complete synchronization manifold $y = z$ between systems (30) and (31) is stable under the condition $\alpha > |m_1 f'(y_{\tau_2})|$. This is also the existence condition of generalized synchronization between systems (25) and (26).

Next consider a situation where a time-delayed chaotic master (driver) system

$$\frac{dx}{dt} = -\alpha_1 x + m_1 f(x_{\tau_1}) \tag{32}$$

drives a nonidentical slave (response) system

$$\frac{dy}{dt} = -\alpha_2 y + m_2 f(y_{\tau_1}) + K x_{\tau_2}. \tag{33}$$

By investigating the error signal $\Delta = x_{\tau_2} - y$ dynamics we find that under the conditions $\alpha_2 - \alpha_1 = K$, $m_1 = m_2$ $y = x_{\tau_2}$ is the synchronization manifold and the manifold is sufficiently stable if $\alpha_2 > |k_1 f'(x_{\tau_1+\tau_2})|$. We note that systems (32) and (33) admit the retarded chaos synchronization manifold $y = x_{\tau_2}$ *only* under parameter mismatch—i.e., $\alpha_1 \neq \alpha_2$. We also notice that without the parameter mismatch—i.e., $\alpha_1 = \alpha_2 = \alpha$—neither $y = x_{\tau_2 - \tau_1}$ nor $y = x_{\tau_1 - \tau_2}$ is the synchronization manifold. Moreover, in general for both $\alpha_1 = \alpha_2$ and $\alpha_1 \neq \alpha_2$ systems (32) and (33) admit neither complete nor anticipating *chaos* synchronization.

In the case of time-dependent feedback-delay time $\tau_1(t)$ analysis of the error dynamics for the retarded synchronization manifold $y = x_{\tau_2}$ shows that the existence conditions $\alpha_2 - \alpha_1 = K$ and $m_1 = m_2$, $y = x_{\tau_2}$ hold also for the variable feedback delay case. The sufficient stability condition for the time-delayed equations (32) and (33) with time-dependent feedback delay $\tau_1(t)$ can be written as

$$\alpha_2^2 \left(1 - \frac{d\tau_1(t)}{dt}\right) > [m_1 f'(x_{\tau_1(t)+\tau_2})]^2. \tag{34}$$

Finally considering complete synchronization between the nonlinearly coupled time-delayed systems,

$$\frac{dx}{dt} = -\alpha x + m_1 f(x_{\tau_1}), \tag{35}$$

$$\frac{dy}{dt} = -\alpha y + m_2 f(y_{\tau_2}) + K f(x_{\tau_3}), \tag{36}$$

it is straightforward to establish that for $\tau_1 = \tau_2$, under the condition

$$m_1 - K = m_2, \tag{37}$$

Eqs. (35) and (36) admit the synchronization manifold

$$y = x_{\tau_3 - \tau_1}. \tag{38}$$

The sufficient stability condition of the synchronization manifold (38) is $\alpha > |m_2 f'(x_{\tau_3})|$. As for $\tau_1 \neq \tau_2$, Eq. (38) is not a synchronization manifold; we investigate the case of generalized synchronization by studying the possibility of complete synchronization between the following systems:

$$\frac{dy}{dt} = -\alpha y + m_2 f(y_{\tau_2}) + K f(x_{\tau_3}), \tag{39}$$

$$\frac{dz}{dt} = -\alpha z + m_2 f(z_{\tau_2}) + K f(x_{\tau_3}). \tag{40}$$

The complete synchronization $y = z$ is stable under the condition $\alpha > |m_2 f'(y_{\tau_2})|$. Under this condition there is generalized synchronization between systems (35) and (36).

V. CONCLUSIONS

We conclude the paper with the following remarks. Usually parameter mismatches are considered to have a detrimental effect on the synchronization quality between coupled identical systems; larger values of parameter mismatches can even result in the loss of synchronization [8,14]. However, it appears that in reality the relation between chaos synchronization in time-delayed systems and parameter mismatches is quite intricate and complex. In [15] it was shown that parameter mismatches can change the time shift between the synchronized states. (Knowledge of the time shift between the synchronized states is of considerable practical importance for the message recovery and information processing using chaos control methods.) In [10] we have demonstrated that perfect anticipating synchronization between two bidirectionally coupled external cavity laser diodes is possible in the presence of parameter mismatches. In this paper we have studied the relation between parameter mismatches and complete and generalized synchronizations using the powerful Krasovskii-Lyapunov functional approach. In the example of two unidirectionally both linearly and nonlinearly coupled chaotic nonidentical Ikeda systems we have shown that the presence of parameter mismatches can change the mode of synchronization from a complete to a generalized one. Most importantly we have derived sufficient existence conditions for the generalized synchronization. We have also investigated chaos synchronization in variable delay time systems and found both existence and sufficient stability conditions for the retarded synchronization manifold with the coupling-delay lag time.

These results are of certain importance in the context of relations between parameter mismatches, coupling types, and mode of synchronizations. Recent studies have disclosed that secure communication schemes based on both low- and high-dimensional chaotic systems using a nongeneralized mode of synchronization and systems without delay-time modulations are vulnerable; see, e.g., [25,19] and references there in. In light of this, the investigation of generalized synchronization in time-delayed systems and a synchronization study between systems with delay-time modulations are also important from the application point of view [26,19].

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