Fluid limit of nonintegrable continuous-time random walks in terms of fractional differential equations

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The fluid limit of a recently introduced family of nonintegrable (nonlinear) continuous-time random walks is derived in terms of fractional differential equations. In this limit, it is shown that the formalism allows for the modeling of the interaction between multiple transport mechanisms with not only disparate spatial scales but also different temporal scales. For this reason, the resulting fluid equations may find application in the study of a large number of nonlinear multiscale transport problems, ranging from the study of self-organized criticality to the modeling of turbulent transport in fluids and plasmas.

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I. INTRODUCTION

Continuous-time random walks (CTRWs) have found a wide range of applications in physics since their introduction almost 40 years ago [1]. These generalizations of the standard (discrete) random walk are defined in terms of two probability density functions (pdf's): $p(x-x')$, that gives the probability of the walker moving from x' to x at time t, and $\psi(t-t')$, that gives the probability of having waited at x' and amount of time $t-t'$ before moving to x. Such CTRW is readily "integrated." That is, it is possible to derive a formal expression for the probability of the walker being at x at time t, $n(x,t)$. This quantity is also referred to as the *walker density*. The derivation exploits the spatial invariance of $p(x-x')$ and the temporal invariance of $\psi(t-t')$, to solve formally for the Laplace-Fourier transform of the walker density (in what follows, any quantity will be represented by the same symbol that its Fourier and/or Laplace transforms, but they can still be distinguished by their arguments. s and k are used, respectively, as Laplace and Fourier variables), which happens to be [1–3]

$$n(k,s) = n_0(k)[1-\psi(s)]\{s[1-\psi(s)p(k)]\}^{-1}, \tag{1}$$

where $n_0(k)$ is the Fourier transform of the initial walker density. Laplace-Fourier inversion of the Montroll-Weiss equation [Eq. (1)] completes the integration, providing with the walker density for all t and x. It is also straightforward to prove that this CTRW can be mapped to the following generalized master equation (GME) [4–6]:

$$\frac{\partial n(x,t)}{\partial t} = \int_0^t dt'\left(\int_{-\infty}^{+\infty} dx' K(x-x',t-t')n(x',t') - n(x,t')\int_{-\infty}^{+\infty} dx' K(x-x',t-t')\right), \tag{2}$$

where spatial and temporal invariance are again exploited to show that the GME transition kernel K must be chosen as

$$K(k,s) = s\psi(s)p(k)[1-\psi(s)]^{-1}. \tag{3}$$

This CTRW/GME formalism can be used to model the transport properties of many systems, once appropriate forms for p and ψ are chosen that capture the relevant physics of the mechanism that governs transport in them. Thanks to the central limit theorem [7], it is in many cases sufficient that ψ be an exponential and p a Gaussian. The "fluid limit" of such CTRWs [in which only those details pertinent to the long-time, long-distance system dynamics are kept in Eqs. (1) or (3)] corresponds then to the usual diffusive equation [8,9]:

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2}, \tag{4}$$

for a value of the diffusion coefficient D that is determined uniquely by the chosen distributions.

But the applicability of integrable CTRWs surpasses diffusive systems. Experiments have shown that, in many systems of physical, chemical, and biological interest, the variance of the walker displacement from some initial point increases with time as [3,8–15]

$$\langle x-x_0\rangle \propto t^{\nu/2}, \quad \nu \neq 1, \tag{5}$$

in contrast to what Eq. (4) predicts ($\nu=1$). Transport is termed either "superdiffusive" ($\nu>1$) or "subdiffusive" ($\nu<1$). Integrable CTRWs can still be used to describe transport in many of these cases, but ψ and p must be chosen instead (with certain restrictions that will be made precise later) from within the family of stable Levy distributions (see Appendix A). This family of pdf's, usually denoted by $P_{[\alpha,\beta,\sigma]}(y)$, satisfies a generalized version of the central limit theorem that does not require that the pdf's decay exponentially at large values of the argument [16,17]. It contains, as special cases, the Gaussian distribution for $\alpha=2$, $\beta=0$, and the exponential distribution for $\alpha=1$, $\beta=1$ (the latter is not strictly contained, but exists as a weak limit when $\alpha\to 1$ for $\beta=1$ [18]). A fluid limit also exists for these CTRWs, but it must be expressed instead in terms of fractional differential operators [8,9]. In spite of the somewhat esoteric nature of these operators (see Appendix B), efficient algorithms exist

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that can be used to integrate them numerically for most applications [19–23].

Regretfully, there are many other transport problems of interest that take place in systems where spatial invariance is absent. "Integrable" CTRWs then become useless since the Montroll-Weiss equation [Eq. (1)] ceases to be valid. An example of this situation is provided by the motion of tracer particles in a moving background, as is the case of fluids in which bulk rotation or shear velocity fields are present [24,25]. The problem then becomes strongly inhomogeneous. Still more dramatic is the case in which several transport mechanisms or channels exist that can be switched on/off by some threshold condition. The system is then not only inhomogeneous, but strongly *nonlinear* as well. Many examples of this situation can be found in fluids and plasmas when turbulence is still not fully developed but instead, instability bursts appear whenever some critical local threshold is overcome [26–30]. For instance, in a hot plasma confined in a tokamak, the confined plasma pressure is known to be limited by pressure-gradient-driven instabilities [31]. These instabilities are excited when the pressure gradient becomes larger than a certain critical value set by the local conditions and magnetic field, giving rise to strong, apparently nondiffusive, transport [32–34]. Other examples can be found in the realm of self-organized-criticality (SOC) [35–37], where a superdiffusive transport mechanism (i.e., the avalanches) becomes active only when some threshold condition is met. This appears to be the case, for instance, in earthquake dynamics [38] (where the threshold condition is given by the maximum stress that a given tectonic plate can bear without displacing) or in the transport of magnetic flux quanta in superconductors [39] (where the threshold condition is given by the depth of the local pinning potential).

In a recent series of papers [40,41], we proposed an extension of the standard integrable CTRW framework that can accommodate not only inhomogeneous cases (in Ref. [24], a less general CTRW than the one presented here was derived to treat this case) but also many nonlinear ones. The extension is based on the observation that the class of CTRWs that can be mapped to a GME is larger than that of the integrable CTRWs. It also includes all nonintegrable CTRWs with a step-size pdf of the form $p(x-x',\ h(x',t))$, where h is any (nonlinear) arbitrary function of the form

$$h(x',t) = f\left(x',t;n(x',t),\frac{dn}{dx}(x',t),\frac{d^2n}{dx^2}(x',t)\cdots\right). \qquad (6)$$

Therefore the step-size pdf may now contain arbitrary nonlinearities and/or inhomogeneities through the function h, which means that the next step size that the walker chooses at time t may depend (even nonlinearly) in a Markovian way on any local (i.e., defined at x' at time t) quantity.

This extended CTRW can easily address the study of the afore-mentioned nonlinear interaction between multiple transport mechanisms. For instance, assuming for simplicity that only two channels are present, it would be sufficient to choose the following step-size pdf:

$$p(x-x',x',t) = \lambda_1(x',t)P_{[\alpha,\beta,\sigma]}(x-x') + \lambda_2(x',t)P_{2,0,\sigma'}(x-x'), \qquad (7)$$

where we have arbitrarily assumed that the first process is superdiffusive (i.e., characterized by a Levy pdf with $\alpha<2$) while the second is diffusive (characterized by a Gaussian). The two "projectors" λ_1,λ_2 can be defined arbitrarily, as long as they satisfy (at all locations and times) positiveness and

$$\lambda_1(x,t) + \lambda_2(x,t) = 1. \qquad (8)$$

As an illustration of application, consider the case of a SOC system with added diffusion [30]. In it, the superdiffusive mechanism (i.e., the avalanches) takes over the local control of transport above some prescribed critical gradient value. Then, λ_1 and λ_2 should be given by

$$\lambda_1(x,t) = H\left(\left|\frac{dn}{dx}(x,t)\right| - Z_c(x)\right), \quad \lambda_2 = 1-\lambda_1, \qquad (9)$$

with $H(x)$ the usual Heaviside step function. On the other hand, in the case in which the diffusive channel remains active (but subdominant) when the profiles values overcome the local critical gradient [42], we should use instead

$$\tilde{\lambda}_1(x,t) = \lambda_1(x,t) + \epsilon\lambda_2(x,t),$$

$$\tilde{\lambda}_2(x,t) = (1-\epsilon)\lambda_2(x,t), \qquad (10)$$

where λ_1,λ_2 are the same functions defined in Eq. (9) and where the arbitrary parameter $0<\epsilon<1$ sets the relative strength of the diffusive channel with respect to the superdiffusive one when the system is locally supercritical. Many other cases could be also addressed by choosing the appropriate form for the projectors.

The interesting feature of all the CTRWs defined by Eq. (7) is that they can keep active some sort of system memory [through the nonlinearity hidden in $\lambda_j(x',t)$], which makes that the shape of instantaneous slope profile that has been carved by past events may affect the system later evolution. And this happens even if the CTRW is constructed to be Markovian in time by choosing ψ exponential. For this reason, the choices provided by Eqs. (7) and (9) have already proved extremely useful in the investigation of several aspects of transport in systems governed by SOC dynamics [41], also, in the study of particle turbulent transport in plasmas confined in a tokamak or a stellarator [40,43].

However, the present formulation of the extended CTRW/GME is based on an important assumption that is not always justified in practice: that all transport mechanisms share the same temporal dynamics and characteristic scales in the sense that the same waiting-time pdf ψ is used at all times independent of which transport channel is active. For instance, coming back to the example of a magnetically confined plasma, it is well known that the two transport channels that set the dynamics of particle and energy transport in these plasmas—collisional diffusion and turbulence—have different associated time scales. In particular, both time scales can change very differently when external parameters such as the plasma temperature or the strength of the magnetic field are varied [44]. Similar examples are also common in many other fields of physics and chemistry. In this paper we will show that this shortcoming can be easily removed by moving to the fluid limit of the extended CTRW/GME framework, a fact that widens importantly its range of application to real systems.

The paper is then organized as follows. In Sec. II, we review the derivation of the GME associated to the nonintegrable CTRW. Then, after reviewing briefly how the fluid limit is taken for the case of integrable CTRWs in Sec. III, we proceed to calculate the fluid limit of the nonintegrable case in Sec. IV. Next, in Sec. V, we show how the limitations of the extended CTRW/GME mentioned in this section disappear in this limit. Finally, some conclusions are drawn in Sec. VI.

II. NONINTEGRABLE CTRWS: DERIVATION OF THE GENERALIZED MASTER EQUATION

In this section, we will show that the family of CTRWs that have an associated GME is not limited to those which are integrable, but also contains the class of CTRWs defined by the joint step-size, waiting-time pdf given by $p=p(x'-x',h(x',t))$ with h given by Eq. (6) [40,41]. The existence of such a GME is essential to take any kind of fluid limit, since a closed expression for $n(s,k)$ like that provided by Eq. (1) is no longer available in this case.

The difficulties of proving that a GME can be associated to this CTRW become clear when trying to "integrate" it along the lines outlined in Sec. I. First, we express the probability of finding the walker as [2]

$$n(x,t)=\int_0^t \eta(x;t-t')Q(x;t')dt', \tag{11}$$

where $\eta(x;t-t')$ represents the probability that the walker, located at x' at time t', still remains in the same position at time t:

$$\eta(x,\tau)=\int_0^\tau d\tau' \psi(\tau',x). \tag{12}$$

$Q(x;t)$ represents the total probability of the walker arriving at position x at time t by any possible route. Next, we Laplace transform Eq. (11) to get

$$n(x,s)=\eta(x;s)Q(x;s). \tag{13}$$

The Laplace transform of $\eta(x,t-t')$ is trivially obtained in terms of $\psi(s,x)$ by Laplace transforming Eq. (12):

$$s\eta(x,s)=1-\psi(s,x). \tag{14}$$

Regarding $Q(x;t)$, it satisfies the recursive equation as shown in Ref. [2],

$$Q(x;t)-\delta(x)\delta(t)=\int_{-\infty}^{\infty}dx'\,p(x-x',h(x';t))\times\int_0^t dt'\,\psi(x';t-t')Q(x';t'), \tag{15}$$

that only assumes that the walker is initially located at x. From Eq. (15) it is clear that, if p and ψ would not depend explicitly on h, $Q(q,s)$ would be readily available via a Fourier-Laplace transformation. However, in the case of our CTRW, $Q(x,t)$ may depend on $n(x',t)$ (through the function h), and the standard approach from Ref. [2] is no longer applicable. The CTRW under consideration is therefore "nonintegrable" due to the presence of the nonlinearity.

To derive the GME associated with this CTRW we must establish the link to the CTRW the link through Eq. (15) instead. We start by introducing an auxiliary function in Laplace space,

$$\phi(x;s)=\psi(x;s)/\eta(x;s), \tag{16}$$

that allows us to rewrite Eq. (15) as

$$Q(x;t)-\delta(x)\delta(t)=\int_{-\infty}^{\infty}dx'\,p(x-x',\ h(x';t))\times\int_0^t dt'\,\phi(x';t-t')n(x',t'), \tag{17}$$

after transforming the temporal convolution in the right-hand side (rhs) of Eq. (15) with the help of the Laplace transform $\mathsf{L}[\cdot]$:

$$\mathsf{L}\left(\int_0^t dt'\,\psi(x';t-t')Q(x';t')\right)=\psi(x',s)Q(x',s)=\phi(x',s)n(x',s)=\mathsf{L}\left(\int_0^t dt'\,\phi(x';t-t')n(x';t')\right), \tag{18}$$

where we have also used Eq. (13). Next, we Laplace transform Eq. (17), multiply the result by $s\eta(x;s)$, and use Eq. (13) (after adding and subtracting $\delta(x)$] to obtain

$$[sn(x,s)-\delta(x)]-\delta(x)[s\eta(x;s)-1]=s\eta(x;s)g(x;s), \tag{19}$$

where $g(x,s)$ stands for the Laplace transform of the rhs of Eq. (17). $g(x,s)$ is eliminated by combining the Laplace transform of Eq. (17) with Eqs. (14) and (16) to get

$$g(x;s)=\frac{\delta(x)\psi(x;s)-\phi(x;s)n(x,s)}{[s\eta(x;s)-1]}. \tag{20}$$

After inserting this expression for $g(x,s)$, Eq. (19) is Laplace inverted to yield the final GME we sought:

$$\frac{\partial n(x,t)}{\partial t}=-\int_0^t dt'\,\phi(x;t-t')n(x,t')+\int_0^t dt'\times\int_{-\infty}^{\infty}dx'\,\phi(x';t-t')p(x-x',\ h(x';t))n(x',t'). \tag{21}$$

The resulting GME transition kernel in Eq. (21) is thus

$$K(x,x',t,t') = \phi(x';t-t')p(x-x',\ h(x';t)), \quad (22)$$

that reduces to the usual transition kernel given by Eq. (3) if spatial invariance is again assumed by disregarding any possible dependence on $h(x',t)$. The function $\phi(t-t')$ is usually known as the *memory function* since it becomes a delta function only when the CTRW is Markovian [45].

III. FLUID LIMIT OF INTEGRABLE CTRWS

As we mentioned in Sec. I, fluid limit means that all details of the CTRW that are irrelevant at very large temporal and spatial scales are neglected [6,46–50]. Mathematically, this limit can be taken either on the Montroll-Weiss equation [Eq. (1)] or on the associated GME [Eq. (2)]. In this section, we collect some well known results regarding this calculation that will be useful when addressing the calculation of the same limit for the nonintegrable case in Sec. IV.

Before proceeding, a few comments are appropriate about the adequate choices for waiting-time and step-size pdf's. As we said in Sec. I, the generalized central limit suggests that both should be chosen from within the family of stable Levy distributions [16]. The only details about these distributions that are of concern at this stage (see Appendix A for more details) are that they can be defined in terms of their Fourier transform [17]:

$$P_{\alpha,\beta,\sigma}(k) = \exp\left\{-\sigma^\alpha|k|^\alpha\left[1 - i\beta\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right\}, \quad (23)$$

with $\alpha \in (0,2]$, $|\beta| \leqslant 1$ and $0 < \sigma < \infty$ (the meaning of each label is discussed in Appendix A). Usually, one can choose any stable Levy pdf as step-size pdf (note that the choice $\alpha = 2$, $\beta = 0$ is the Gaussian pdf), but waiting-time pdf's can only be defined for positive lapses of time (i.e., for $t-t' \geqslant 0$). For this reason, they must be chosen within the subfamily of Levy pdf's known as *positive extremal distributions* ($\alpha < 1$, $\beta = 1$) [17], that are only defined for positive values of y (see Appendix A). Also, the exponential can be used, since it can be shown that it is the limiting pdf when the limit $\alpha \to 1$ for $\beta = 1$ is taken [18].

Next, it is useful to introduce some notation. In what follows, the labels α,β,σ will always refer to step-size pdf's. Regarding the waiting-time pdf's, only α and σ are free parameters, since $\beta = 1$. To avoid confusion with the step-size labels, we will use instead γ (for α) and τ (for σ) to refer to waiting-time labels. Therefore we will assume that the integrable CTRW is defined by a waiting-time step size:

$$\psi(t-t') = P_{[\gamma,1,\tau]}(t-t'), \quad \gamma \leqslant 1, \quad 0 < \tau < \infty, \quad (24)$$

and step-size pdf:

$$p(x-x') = P_{[\alpha,\beta,\sigma]}(x-x'), \quad \alpha \leqslant 2, \quad |\beta| \leqslant 1, \quad 0 < \sigma < \infty. \quad (25)$$

The fluid limit can now be taken, for instance, on the Montroll-Weiss equation [Eq. (1)]. We only need to take the limit of long distances (in Fourier space, $k \to 0$) in $p(k)$ and of long times (in Laplace space, $s \to 0$) in $\psi(s)$. This reduces to approximating Eq. (23) as

$$p(k) \simeq 1 - \sigma^\alpha|k|^\alpha\left[1 - i\beta\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right], \quad (26)$$

and approximating the Laplace transform of positive extremal Levy pdf's, given by Eq. (A6), by

$$\psi(s) \simeq 1 - A_\gamma^{-1}\tau^\gamma s^\gamma. \quad (27)$$

where we have also included the exponential law if $\gamma = 1$ and defined the constant

$$A_\gamma = \begin{cases} \cos\left(\frac{\pi\gamma}{2}\right), & \gamma < 1 \\ 1, & \gamma = 1 \end{cases}. \quad (28)$$

After inserting Eqs. (27) and (26) in Eq. (1), the fluid limit of the Montroll-Weiss equation becomes

$$n(s,k) \simeq n_0(k)\left\{s + C(\alpha,\gamma)s^{1-\gamma}|k|^\alpha \times\left[1 - i\beta\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right]\right\}^{-1}, \quad (29)$$

where the coefficient $C(\alpha,\gamma) = A_\gamma\sigma^\alpha/\tau^\gamma$ has been defined. Equation (29) can be rewritten as

$$sn(s,k) - n_0(k) = -C(\alpha,\gamma)s^{1-\gamma}|k|^\alpha \times\left[1 - i\beta\,\mathrm{sgn}(k)\tan\left(\frac{\pi\alpha}{2}\right)\right]n(s,k). \quad (30)$$

After using the identity Eq. (B7), Eq. (30) can be Fourier inverted by introducing the two Riemann-Liouville fractional differential operators defined by Eq. (B2), which satisfy [19,21]

$$\mathsf{F}\left[\frac{\partial^\alpha n}{\partial(\pm x)^\alpha}\right] \equiv (\mp ik)^\alpha n(k). \quad (31)$$

$\mathsf{F}[\cdot]$ represents the Fourier transform. The resulting equation becomes thus a fractional differential equation (FDE) in space:

$$sn(s,x) - n_0(x) = -\frac{C(\alpha,\gamma)s^{1-\gamma}}{2\cos\left(\frac{\pi\alpha}{2}\right)} \times\left((1+\beta)\frac{\partial^\alpha n}{\partial x^\alpha} + (1-\beta)\frac{\partial^\alpha n}{\partial(-x)^\alpha}\right). \quad (32)$$

In order to carry out next the Laplace inversion of Eq. (32), two choices are possible. The first one is to multiply both sides by $s^{\gamma-1}$ and introduce the Caputo fractional differential operator [51] [Eq. (B8)], which verifies [19,21] (for $\gamma < 1$)

$$\mathsf{L}\left[\frac{\partial_c^{\gamma} n}{\partial_c t^{\gamma}}\right] \equiv s^{\gamma} n(s,x) - s^{\gamma-1} n_0(x), \tag{33}$$

where $\mathsf{L}[\cdot]$ represents the Laplace transform. The result is the FDE in space and time:

$$\frac{\partial_c^{\gamma} n}{\partial t_c^{\gamma}} = -\frac{C(\alpha,\gamma)}{2\cos\left(\frac{\pi\alpha}{2}\right)}\left((1+\beta)\frac{\partial^{\alpha} n}{\partial x^{\alpha}} + (1-\beta)\frac{\partial^{\alpha} n}{\partial(-x)^{\alpha}}\right). \tag{34}$$

A second possibility is to Laplace invert Eq. (32) directly. This can be done by introducing the Riemann-Liouville differential operator with start point at $t=0$ [Eq. (B1)] [48]:

$$\frac{\partial n}{\partial t} = -{}_0D_t^{1-\gamma}\left[\frac{C(\alpha,\gamma)}{2\cos\left(\frac{\pi\alpha}{2}\right)}\left((1+\beta)\frac{\partial^{\alpha} n}{\partial x^{\alpha}} + (1-\beta)\frac{\partial^{\alpha} n}{\partial(-x)^{\alpha}}\right)\right]. \tag{35}$$

The interpretation and applications of Eqs. (34) and (35) have been discussed in detail in the literature for different choices of α and γ [6,8,9,48,50,52]. We will only remark here that the exponent that determines the superdiffusive or subdiffusive character of transport is equal to $\nu=2\gamma/\alpha$ [see Eq. (5)]. Thus superdiffusive behavior is observed if $2\gamma > \alpha$, diffusive if $2\gamma=\alpha$, and subdiffusive if $2\gamma<\alpha$ [8,9]. For the choices $\alpha=2$, $\gamma=1$, Eqs. (34) and (35) reduce to the standard diffusive equation [see Eq. (4)] with diffusive coefficient $D=C(2,1)=\sigma^2/\tau$.

IV. NONINTEGRABLE CTRWS: FLUID LIMIT

We will now derive the fluid limit of the GME Eq. (21) for the choices of waiting-time and step-size pdf's suggested by the generalized central limit and our discussion in Sec. I. Regarding the same waiting-time pdf, the same choice already made in the integrable case [see Eq. (24)] will be used. However, we consider instead as step-size pdf the combination of two arbitrary stable Levy pdf's:

$$p(x-x',x',t) = \lambda_1(x',t)P_{[\alpha_1,\beta_1,\sigma_1]}(x-x') + \lambda_2(x',t)P_{[\alpha_2,\beta_2,\sigma_2]}(x-x'). \tag{36}$$

Keep in mind that the projectors λ_1,λ_2 are completely arbitrary, as long as they satisfy the conditions given by Eq. (8). Extension to the case with N transport mechanisms is straightforward.

A. Fluid limit in terms of FDEs

The fluid limit must now be taken directly on the GME Eq. (21), since an equation analogous to the Montroll-Weiss equation that we used in the integrable case [see Eq. (29)] is not available for nonintegrable CTRWs. We proceed by first taking the temporal part of the fluid limit $(s\to 0)$ of the memory function:

$$\phi(s) = \frac{s\psi(s)}{1-\psi(s)} \sim A_{\gamma}\tau^{-\gamma}s^{1-\gamma}, \tag{37}$$

for which only Eq. (27) is required [the coefficient A_{γ} was introduced in Eq. (28) in Sec. II]. We will use this result to take the temporal part of the fluid limit of the first term on the rhs of GME Eq. (21), that introducing again the Caputo derivative and taking advantage of Eq. (33), becomes

$$\begin{aligned}\int_0^t dt'\,\phi(x;t-t')n(x,t') &= \mathsf{L}^{-1}[\phi(s)n(x,s)] \\ &\simeq \mathsf{L}^{-1}[A_{\gamma}\tau^{-\gamma}s^{1-\gamma}n(x,s)] \\ &= A_{\gamma}\tau^{-\gamma}\left(\frac{\partial_c^{1-\gamma}n}{\partial_c t^{1-\gamma}} + \frac{t^{\gamma-1}n_0(x)}{\Gamma(\gamma)}\right) \\ &= A_{\gamma}\tau^{-\gamma}\left[{}_0D_t^{1-\gamma}n\right].\end{aligned} \tag{38}$$

To derive this expression, use has also been made of [21]

$$L[t^{\gamma}] = \Gamma(\gamma+1)s^{-(\gamma+1)}, \tag{39}$$

and of the relation between the Caputo derivative and the Riemann-Liouville derivative with start point at $t=0$ [Eq. (B10), in Appendix B]. Doing the same with the time convolution appearing inside the second term of the rhs of Eq. (21), we can rewrite the GME as

$$\frac{\partial n(x,t)}{\partial t} = -A_{\gamma}\tau^{-\gamma}\left\{\left[{}_0D_t^{1-\gamma}n\right](x,t) + \int_{-\infty}^{\infty} dx'\,p(x-x',\ h(x';t)) \times\left[{}_0D_t^{1-\gamma}n\right](x,t)\right\}. \tag{40}$$

Next, we take the spatial part of the fluid limit by computing the Fourier transform of Eq. (40) and taking its limit when $k\to 0$:

$$\frac{\partial n(k,t)}{\partial t} = -\sum_{j=1}^{2} C(\alpha_j,\gamma)\Lambda_j^{(\gamma)}(k,t)|k|^{\alpha_j-1} \cdot\left[|k| - ik\beta_j\tan\left(\frac{\pi\alpha_j}{2}\right)\right], \tag{41}$$

where we have defined the quantities

$$\Lambda_j^{(\gamma)}(x,t) \equiv \lambda_j(x,t)\left[{}_0D_t^{1-\gamma}n\right](x,t), \quad j=1,2. \tag{42}$$

The diffusive coefficients $C(\alpha,\gamma)\equiv A_{\gamma}\sigma^{\alpha}/\tau^{\gamma}$ are the same as those defined in Sec. II for the integrable cases.

The Fourier inverse of Eq. (41) can then be written explicitly by introducing again the Riemann-Lioville fractional differential operators [Eq. (B2), Appendix B]:

$$\frac{\partial n(x,t)}{\partial t} = -\sum_{j=1}^{2}\frac{C(\alpha_j,\gamma)}{2\cos\left(\frac{\pi\alpha_j}{2}\right)}\left((1+\beta_j)\frac{\partial^{\alpha_j}}{\partial x^{\alpha_j}} + (1-\beta_j)\frac{\partial^{\alpha_j}}{\partial(-x)^{\alpha_j}}\right)\cdot\Lambda_j^{(\gamma)}(x,t), \tag{43}$$

which should be compared with Eq. (35), that we obtained in Sec. II for integrable CTRWs. A first comment to be made is that, in the extended CTRW case, an equation with a temporal fractional derivative in terms of the Caputo operator is not available due to the presence of the nonlinearity in the projectors, in contrast to what happened with Eq. (34) in the integrable case.

We finish this section by noting that the rather complicated Eq. (43) can be written in a more compact form if both step-size pdf's are symmetric (i.e., if $\beta_j=0$, $\forall_j$). Then, we can introduce the Riesz operator [Eq. (B5), Appendix B] and rewrite Eq. (43) as

$$\frac{\partial n(x,t)}{\partial t} = C(\alpha_1,\gamma)\frac{\partial^{\alpha_1}\Lambda_1^{(\gamma)}}{\partial |x|^{\alpha_1}} + C(\alpha_2,\gamma)\frac{\partial^{\alpha_2}\Lambda_2^{(\gamma)}}{\partial |x|^{\alpha_2}}. \quad (44)$$

B. Interpretation of FDEs Eqs. (43) and (44)

We proceed now to interpret each term in the fluid limit given by Eq. (43) in what follows. To do it, it is convenient to set $\gamma=1$ for the moment, and consider the Markovian version of Eq. (43):

$$\frac{\partial n(x,t)}{\partial t} = -\sum_{j=1}^{2} \frac{C(\alpha_j,1)}{2\cos\left(\frac{\pi\alpha_j}{2}\right)}\left((1+\beta_j)\frac{\partial^{\alpha_j}}{\partial x^{\alpha_j}} + (1-\beta_j)\frac{\partial^{\alpha_j}}{\partial(-x)^{\alpha_j}}\right)\cdot[\lambda_j(x,t)n(x,t)]. \quad (45)$$

The rhs of Eq. (45) contains the contributions of the two transport channels. Let us focus on just one of them (say, $j=1$), which consists of two terms. The first one, proportional to $1+\beta_1$, is the only one that survives if $\beta_1=1$ [note that, from the "microscopic" level, $\beta=1$ corresponds to having the walker moving under a step-size Levy pdf in which the only steps allowed for the walker take it to larger x's (except for an exponentially vanishing contribution to lower x's]. However, the α-fractional derivative is nothing else but an integral over $(-\infty,x]$ [see Eq. (B1)]:

$$\frac{\partial^{\alpha}\Lambda_1^{(1)}}{\partial x^{\alpha}} \equiv \frac{1}{\Gamma(p-\alpha)}\frac{d^p}{dx^p}\int_{-\infty}^{x}\frac{\lambda_j(x',t)n(x',t)dx'}{(x-x')^{\alpha-p+1}}, \quad (46)$$

where p is the integer part of α. Therefore this fractional derivative collects the contributions of all walkers that end up at x at time t *from* $x'\leqslant x$. But, since the argument in the integral is the "projected" walker density $\lambda_j(x',t)n(x',t)$, only those locations x' for which $\lambda_j(x',t)\neq 0$ can contribute to the density of walkers at x. In the case in which the projector describes some instability threshold [as in Eq. (9)], it follows that the first of the two contributions to Eq. (43) from the first transport mechanism simply states that any change in walker density at point x and time t *can only come from points* $x'\leqslant x$ *that, at that same time, are unstable*!

Analogously, the second contribution to the first transport mechanism [the term proportional to $(1-\beta_1)$ in Eq. (45)] gives the contribution to the change in $n(x,t)$ from points with $x'\geqslant x$ that are unstable at time t. In the general case, a combination of both terms applies [8,9]. For example, Eq. (44) would correspond to the case in which the combination of the two contributions yields a symmetric Levy pdf: each walker has equal probability of moving to larger or smaller x's from any given location.

Before discussing the non-Markovian $(\gamma<1)$ case, it is important to note that Eq. (45), in spite of being Markovian, may contain some sort of system memory. It is the "memory-through-profile" mechanism associated to the first (or second) transport mechanism, which is contained in the projector $\lambda_1(x,t)$ [or $\lambda_2(x,t)$]. As we already discussed in Sec. II, the previous history of the system, that has been carved in the system profile by past transport events, can in this way affect the future system evolution.

Let us look now at the non-Markovian case with $\gamma<1$, that allows us to model memory effects in a probabilistic manner associated with the microscopic waiting-time pdf ψ. In this case, $\Lambda_1^{\gamma}(x',t)$ appearing in Eq. (43) is more complicated than just the projected density $\lambda_j(x,t)n(x,t)$ that we just discussed. Writing $\Lambda_1^{\gamma}(x',t)$ explicitly, it happens that [recall Eq. (B1), Appendix B]

$$\Lambda_1^{\gamma}(x',t) = \frac{1}{\Gamma(m-1+\gamma)}\frac{d^m}{dt^m}\left(\int_0^t \frac{\lambda_1(x',t')n(x',t')}{(t-t')^{2-\gamma-m}}dt'\right), \quad (47)$$

with m the integer part of $1-\gamma$. Note that $\Lambda_1^{\gamma}(x',t)$ may now be nonzero even if $\lambda_1(x',t)=0$ at time t. The reason is that this term collects now contributions from all past times $t'<t$ when $\lambda_1(x',t')\neq 0$. Again, if λ_1 represents some kind of instability threshold, this would mean that $\Lambda_1^{\gamma}(x',t)$ is determined by the values of the density of walkers *at all* $t'<t$ *when that particular site was unstable!*

To finish this section it is interesting to clarify the relationship between Eq. (44) and the so-called distributed-order fractional kinetics (DOFK) introduced by Caputo [53] and very recently reviewed in Ref. [54]. In a sense, Eq. (44) is the simplest nonlinear generalization of DOFK, that substitutes the linear combination of fractional derivatives characteristic of DOFK with a nonlinear combination that is mediated through a nonlinear threshold condition.

V. ACCOMMODATING MULTIPLE CHARACTERISTIC TIME SCALES

As we mentioned in the introduction, one of the limitations of the extended CTRW/GME is that it assumes that all transport channels share the same waiting-time pdf. Such an assumption is central to the proof of the existence of an associated GME, but it is not justified from a physics point of view in many applications. This problem can, however, be satisfactorily dealt with in the fluid limit we just derived in Sec. IV.

To prove it, note first that the choices of step-size and waiting-time pdf given by Eqs. (24) and (36) are equivalent to considering each transport mechanism as an independent CTRW, given by

$$\{P_{[\alpha_j,\beta_j,\sigma_j]}(x-x');\ P_{[\gamma,1,\tau]}(t-t')\}, \quad j=1,2. \quad (48)$$

The obvious way to introduce multiple scales is to assume that also γ and τ can be channel dependent. But this invalidates the derivation of the GME presented in Sec. II. One way to overcome this problem is to prove that any transport channel can be "rescaled," in a sense to be clarified later, so that its *rescaled waiting-time* pdf coincides with that of the other channels. Of course, this means that all the information intrinsic to that mechanism is stored instead in the "rescaled step-size" pdf. As we proceed to show now, this rescaling is only possible in the fluid limit.

To prove it, note first that each of the individual CTRW defined in Eq. (48) is integrable. Therefore its fluid limit is given by Eq. (29), which is a function of α, β, γ, and the ratio $(\sigma^\alpha/\tau^\gamma)$. Therefore the fluid limit of each individual CTRW is invariant under the scale transformation

$$\{\gamma,\tau,\alpha,\beta,\sigma\} \rightarrow \{\gamma,\tau',\alpha,\beta,\sigma'\}, \tag{49}$$

if it holds that

$$[\sigma'/\sigma]^\alpha = [\tau'/\tau]^\gamma. \tag{50}$$

Therefore, as long as the temporal and spatial *essential* dynamics of each individual CTRW remains unchanged (by essential, we mean α, γ and β), it is always possible to rescale the temporal and spatial scale parameters of all CTRWs (which are associated to the temporal and spatial characteristic scales of each transport mechanism) so that their rescaled temporal scale parameters are all the same. On the other hand, the value of γ cannot be rescaled in this fashion. For this reason, consideration of several transport mechanisms with different temporal *essential* dynamics is not possible in this framework, not even in the fluid limit.

VI. CONCLUSIONS

In the previous sections we have shown that the fluid limit of the extended CTRW/GME framework defined by Eq. (7) (and its generalizations to a larger number of transport channels) can indeed overcome some of the limitations of the microscopic GME. In particular, we have shown that it can account adequately for the interaction between multiple transport mechanisms with disparate characteristic time scales as long as they share the same essential temporal dynamics.

Also, we have shown that the resulting fluid equations [Eqs. (43) and (44)] are capable of implementing the memory-through-profile mechanism into the dynamics in an appropriate way, that remains active even when the individual transport channels are Markovian. For this reason alone, these equations suggest themselves as a valuable generalization of previous studies of SOC dynamics based on some nonlinear versions of the standard diffusive equation [41,55,56]. Otherwise, any approach that attempts to study this problem by relying on linear FDEs must consider temporal fractional derivatives to account for that memory effect. The problem thus becomes strongly non-Markovian [15,34].

Finally, note that Eq. (43) is useful to model particle transport in systems in which critical thresholds exist that can excite/damp instability-driven transport. In particular, we would like to mention its application to turbulent transport in plasmas magnetically confined in a tokamak or stellarator [40,43]. In these works, it was shown that the combination of nonlinearity and superdiffusive transport channels may provide us with explanations for the observation of anomalous scalings in the global confinement time and nondiffusive propagation of perturbations observed in the experiments for many years [57–60]. It remains, however, to be seen how this formalism can be extended to account also for energy (heat) and momentum transport. Anomalous heat transport is a topic that has seen renewed interest in recent times [61,62]. But its accommodation in this framework still requires a much better understanding of fractional generalization of Boltzmann equilibrium concepts than currently available. It thus remains a very active field of work [63,64].

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APPENDIX A: LEVY DISTRIBUTIONS

The Levy-Gnedenko family of pdf's comprises all the possible limit distributions that are strictly stable with respect to the *sum of N independent and identically distributed (i.i.d.) random variables* [16,17]. The family is defined in terms of three parameters, and its members are denoted by $P_{\alpha,\beta,\sigma}(y)$. They can be defined in terms of their Fourier transform or characteristic function as $(0 < \alpha \leqslant 2,\ |\beta| \leqslant 1)$ [17]

$$P_{\alpha,\beta,\sigma}(k) = \exp\left\{ - \sigma^\alpha |k|^\alpha \left[1 - i\beta\,\mathrm{sgn}(k) \tan\left(\frac{\pi\alpha}{2}\right)\right]\right\}. \tag{A1}$$

The three labels define the properties of each distribution. First, β measures the *asymmetry* of the distribution. This comes from the fact that

$$P_{\alpha,\beta,\sigma}(y) = P_{\alpha,-\beta,\sigma}(-y). \tag{A2}$$

It can very within $-1 \leqslant \beta \leqslant 1$ for all $\alpha \neq 1$, 2, for which only $\beta=0$ is possible. Second, α gives the asymptotic behavior of the distribution at large y. Thus for $0 < \alpha < 2$ all Levy distributions exhibit heavy tails. Certainly, for $\alpha \neq 1$, it holds that

$$P_{\alpha,\beta,\sigma}(y) \sim \begin{cases} C_\alpha\left(\frac{1-\beta}{2}\right)\sigma^\alpha |y|^{-(1+\alpha)}, & y \rightarrow -\infty \\ C_\alpha\left(\frac{1+\beta}{2}\right)\sigma^\alpha |y|^{-(1+\alpha)}, & y \rightarrow +\infty \end{cases}, \tag{A3}$$

where the constant is given by

$$C_\alpha = \frac{(\alpha-1)\alpha}{\Gamma(2-\alpha)\cos(\pi\alpha/2)}; \tag{A4}$$

$\Gamma(x)$ is Euler gamma function. In the special case $\alpha=1$, the PDF decays as $P_{1,0,\sigma}(y) \sim (\sigma/\pi)|y|^{-2}$. Finally σ is called a *scale parameter* because

$$P_{\alpha,\beta,\sigma}(ay) = P_{\alpha,\,\mathrm{sgn}(a)\beta,|a|\sigma}(y). \tag{A5}$$

1. Extremal Levy distributions

A Levy distribution is called *extremal* if its skewness value is maximum: $\beta = \pm 1$ for $\alpha \neq 1$, 2. It is important to notice that, according to the previous equations, the power-law decay is only observed in one tail in the case of all extremal distributions ($\beta=\pm1$), the other decaying instead exponentially. In the case of $1<\alpha<2$, $\beta=+1$ implies that the exponential tail exists for $y\to-\infty$, while $\beta=-1$ has a right exponential tail for $y\to\infty$. For $0<\alpha<1$ the extremal distributions are *one sided* [52]: they are defined only for $y>0$ if $\beta=-1$ and for $y<0$ if $\beta=1$. In that case, the exponential tail is found in the limit $y\to0+$ for $\beta=-1$, and for $y\to0-$ for $\beta=1$. Their Laplace transform is given by

$$P_{\alpha,1,\sigma}(s) = \exp\left(-\frac{\sigma^\alpha}{\cos(\pi\alpha/2)}s^\alpha\right). \tag{A6}$$

2. Moments of Levy distributions

Another important property of the Levy distributions is that all moments higher than α are infinite. That is, the momenta of $P_{\alpha,\beta,\sigma}$ verify

$$\langle|x|^p\rangle = \begin{cases}\infty, p\geqslant\alpha\\ [c_{\alpha,\beta}(p)]^p\sigma^p, p<\alpha\end{cases}, \tag{A7}$$

where the coefficient is not relevant for our discussion (it can be found in Ref. [17]). Thus only the Gaussian distribution ($\alpha=2$) has a finite variance. Furthermore, all distributions with $\alpha\leqslant1$ have also infinite first moments.

3. Explicit expressions of Levy distributions

There are only three Levy distributions for which an analytical expression exists [17]: The *Cauchy distribution*. Its real space representation is

$$P_{1,0,\sigma}(y) = \frac{\sigma}{\pi(y^2+\sigma^2)}; \tag{A8}$$

the *Gauss distribution*,

$$P_{2,0,\sigma}(y) = \frac{1}{2\sigma\sqrt{\pi}}e^{-y^2/4\sigma^2} \tag{A9}$$

(note that the relation of σ with the usual width w of the Gaussian is thus $2\sigma^2=w^2$); and the *Levy distribution*,

$$P_{1/2,1,\sigma}(y) = \left(\frac{\sigma}{2\pi}\right)^{1/2}\frac{1}{y^{3/2}}e^{-\sigma/2y}. \tag{A10}$$

APPENDIX B: FRACTIONAL DIFFERENTIAL OPERATORS

The *Riemann-Liouville fractional derivative operators* can be defined explicitly by means of the integral operators [19,21]:

$${}_aD^\alpha_x f(x) \equiv \frac{1}{\Gamma(p-\alpha)}\frac{d^p}{dx^p}\left[\int_a^x \frac{f(x')dx'}{(x-x')^{\alpha-p+1}}\right],$$

$${}^bD^\alpha_x f(x) \equiv \frac{-1}{\Gamma(p-\alpha)}\frac{d^p}{d(-x)^p}\left[\int_x^b \frac{f(x')dx'}{(x'-x)^{\alpha-p+1}}\right]. \tag{B1}$$

In these expressions, $\Gamma(x)$ is the usual Euler Gamma function, and p represents the integer part of α. a (or b) is called the start (end) point of the operator.

In the cases in which the start point a or the end point b extend all the way to infinity, we will use the notation

$$\frac{d^\alpha f}{dx^\alpha} \equiv {}_{-\infty}D^\alpha_x f(x);$$

$$\frac{d^\alpha f}{d(-x)^\alpha} \equiv {}^{+\infty}D^\alpha_x f(x). \tag{B2}$$

These operators are particularly interesting since they satisfy, under Fourier transformations, that [19,21]

$$F\left[\frac{d^\alpha f}{dx^\alpha}\right] = (-iq)^\alpha f(q), \tag{B3}$$

$$F\left[\frac{d^\alpha f}{d(-x)^\alpha}\right] = (iq)^\alpha f(q). \tag{B4}$$

As a matter of fact, it is also possible to define them via Eqs. (B3) and (B4).

Another useful fractional operator is the so-called *Riesz fractional derivative operator* [19,21]. It is defined as the symmetrization:

$$\frac{d^\alpha}{d|x|^\alpha} \equiv -\frac{1}{2\cos(\pi\alpha/2)}\left[\frac{d^\alpha}{dx^\alpha}+\frac{d^\alpha}{d(-x)^\alpha}\right]. \tag{B5}$$

Thus, the Riesz operator verifies under Fourier transform that

$$F\left[\frac{d^\alpha f}{d|x|^\alpha}\right] = -|q|^\alpha f(q), \tag{B6}$$

which follows from Eqs. (B3) and (B4) thanks to the complex identity

$$(-iq)^\alpha + (iq)^\alpha = 2|q|^\alpha\cos\left(\frac{\pi\alpha}{2}\right), \tag{B7}$$

where $i=\sqrt{-1}$, the usual imaginary unit.

Finally, the *Caputo fractional derivative operator* is defined as [51]

$$\frac{d_c^\gamma f}{d_cx^\gamma}(x) \equiv \frac{1}{\Gamma(\gamma-p)}\int_0^x \frac{d^pf}{dx^p}(x')\frac{d\tau}{(x-x')^{\gamma+1-p}}, \tag{B8}$$

where p is the integer part of γ. The Caputo fractional derivative is usually associated to derivatives in time. The need for defining a different fractional derivative when time is involved (instead of using the Riemann-Lioville operator with start point at $t=0$) has to do with the fact that the Laplace transform of the Caputo derivative verifies [19,21]

$$L\left[\frac{d_c^\gamma f}{d_c t^\gamma}(t)\right] = s^\gamma f(s) - \sum_{k=0}^{p-1} s^{\gamma-k-1}\frac{d^k f}{dt^k}(0), \tag{B9}$$

which depends only on the initial values of $f(t)$ and its integer derivatives. The Laplace transform of ${}_0D^\gamma_t f(t)$ depends instead on $f(t)$ and the initial values of fractional derivatives of lower order that γ, which do not have a clear physical meaning in the case of real applications [19,21]. The relation between Riemann-Lioville and Caputo derivatives is given by [21]

$$_0D_t^\gamma f(t) = \frac{d_c^\gamma f}{d_c t^\gamma} + \frac{t^{-\gamma} f(0)}{\Gamma(1-\gamma)}. \tag{B10}$$

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