Comment on "Existence and design of trans-vacuum-speed metamaterials"

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It is shown that metamaterials considered by Ziolkowski and Cheng [Phys. Rev. E **68**, 026612 (2003)] as media that support trans-vacuum-speed pulses must exhibit instantaneous response. Thus, they are not physically realizable as linear passive materials.

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Considering linear passive causal materials with the constitutive relations of the form

$$D_x(z,t) = \epsilon_0 E_x(z,t) + \epsilon_0 \int_0^\infty G(\tau) E_x(z,t-\tau) d\tau, \tag{1}$$

or, in the frequency domain,

$$D_x(z,\omega) = \epsilon_0 [1 + \chi(\omega)] E_x(z,\omega), \tag{2}$$

the authors of [1] claim that the susceptibility $\chi(\omega)$ can have a nonzero limit at $\omega \to \infty$. This means the assumption that the permittivity $\epsilon(\omega) = \epsilon_0[1+\chi(\omega)]$ does not tend to ϵ_0 when the frequency tends to infinity. The authors argue that although it is "generally assumed because of causality and, hence, satisfaction of the Kramers-Krönig relations" that $\epsilon(\omega)$ tends to ϵ_0 at infinity, this is in fact not required by causality.

It is correct that this property is not required by causality, but it is incorrect that it is "generally assumed because of causality." Actually, $\epsilon(\omega)$ tends to ϵ_0 at infinity because no material can exhibit instantaneous response. The proof can be found in many textbooks (e.g., [2,3]), and we briefly revise it for the reader's convenience. Physically, this property follows from the fact that if the field varies very quickly, electrons (whose mass is nonzero) cannot react to the field and the material is not polarized at all (e.g., [2,4]). Mathematically, the limit $\chi(\omega)|_{\omega\to\infty} = 0$ follows from simple properties of the integral defining $\chi(\omega)$,

$$\chi(\omega) = \int_0^\infty G(\tau) e^{-j\omega\tau}\, d\tau. \tag{3}$$

Indeed, in all passive media without instantaneous response the kernel function defined in (1) is finite at all τ [and $G(\tau) \to 0$ at $\tau \to \infty$]. Denoting $\max\{G(\tau)\} = M$, we have an estimate,

$$|\chi(\omega)| = \left| \int_0^\infty G(\tau) e^{-j\omega\tau}\, d\tau \right| < M \left| \int_0^\infty e^{-j\omega\tau}\, d\tau \right|. \tag{4}$$

The last integral obviously tends to zero for $\omega \to \infty$.

As is known from the properties of the Laplace transform, the asymptotic behavior of $\chi(\omega)$ for large frequencies is related to the behavior of $G(\tau)$ near zero. Integrating (3) by parts, one arrives to the following asymptotic expansion [Ref. [3], p. 333]:

$$\chi(\omega) \approx -j \frac{G(+0)}{\omega} - \frac{G'(+0)}{\omega^2} - \cdots. \tag{5}$$

Here $G(+0) = \lim_{\tau \to +0} G(\tau)$.

The above derivation is based on two key assumptions. First, $G(\tau) = 0$ at $\tau < 0$ (causality). Second, $G(\tau)$ is everywhere finite and continuous (no instantaneous response). If we allow instantaneous response, $G(\tau)$ can take infinite values and does not have to be a continuous function. In particular, we can assume that $G(\tau) = \delta(\tau)$ (Dirac δ function) and arrive to a model of a causal material with a constant permittivity at all frequencies. This is actually the case assumed in [1]: The inverse Fourier transform of susceptibility $\chi(\omega)$ given by Ref. [1], Eq. (14) can be easily calculated, and it is of the form $G(\tau) = \chi_\gamma \delta(\tau)$ + a finite function of τ (the finite function is a combination of decaying exponents and trigonometric functions specified in [5]). These are valid models at moderate frequencies where one can neglect retardation of the medium response. However, instantaneous models lose their sense at extremely high frequencies.

The reason for the confusion is apparently the quasistatic model of artificial materials (a circuit model built from some inductances, capacitances, and resistances), which is valid only at moderate frequencies. Arbitrarily extending the validity region to infinitely high frequencies, the authors of [1] arrived to a conclusion of possible "trans-vacuum speed" propagation of information. A similar reasoning allowed one of the authors of [1] to conclude in [5] that "superluminal transmission of information" is possible in an artificial material formed by an array of metal wires and an array of metal split-ring resonators modeled by

$$\epsilon = \epsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2 - j\Gamma^e \omega} \right), \quad \mu = \mu_0 \left(1 - \frac{F\omega^2}{\omega^2 - \omega_o^2 - j\Gamma^m \omega} \right), \tag{6}$$

where $F > 0$.

This is an expected conclusion since according to this model at $\omega \to \infty$ the permeability tends to a constant that is smaller than μ_0. But in fact this model is valid only in the quasistatic regime where the split-ring resonators are small compared to the wavelength. At high frequencies, the resonator response to magnetic fields quickly diminishes and,

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moreover, the effective permeability loses its physical meaning [2]. As a result, we arrive to a natural conclusion that $\lim_{\omega\to\infty}\mu(\omega)=\mu_0$. Thus, the pulse front propagates with the speed of light and there is no superluminal transmission of information.

In summary, Ziolkowski and Cheng in [1] assumed that $\lim_{\omega\to\infty}\chi(\omega)=\chi_\gamma\neq 0$. This means that their hypothetical metamaterial produces instantaneous response. "Trans-vacuum-speed" behavior under this assumption is an expected result, because the front of a pulse propagates with the speed [Ref. [1], Sec. IV]

$$v_{\text{front}}=\frac{c}{\sqrt{1+\chi_\gamma}}. \tag{7}$$

It is enough to choose $-1<\chi_\gamma<0$. However, instantaneous response is not possible in passive linear media, which means that $\chi_\gamma=0$.

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