Perturbation theory for dark solitons: Inverse scattering transform approach and radiative effects

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(Received 23 September 2004; published 21 December 2004)

A perturbation theory for dark solitons of the nonlinear Schrödinger equation is developed. The theory is based on the inverse scattering transform method. Equations describing dynamics discrete (solitonic) and continuous (radiative) scattering data in the presence of perturbations are derived for *N*-soliton case. Adiabatic equations for soliton parameters and the perturbation-induced radiative field are obtained. The problem of the absence of a threshold for the creation of dark solitons under the action of a perturbation is discussed. A temporal one-soliton pulse with random initial perturbation and a spatial soliton with linear gain and two-photon absorption are considered as examples of application of the developed theory.

DOI: 10.1103/PhysRevE.70.066620 PACS number(s): 42.65.Tg, 42.81.Dp

I. INTRODUCTION

The possibility of propagation of optical bright and dark solitons in lossless fibers was theoretically predicted first by Hasegava and Tappert in 1973 [1,2]. Since then, light solitons in time (temporal pulses in the optical fibers) and space (spatial beams in the waveguides), both bright and dark, have been the object of intensive theoretical and experimental studies [3–5].

The classical, mathematical model for the nonlinear pulse (beam) propagation is the famous nonlinear Schrödinger equation (NLSE). As is well known, the NLSE is a completely integrable Hamiltonian system for both vanishing and nonvanishing at infinite boundary conditions [6]. Dark solitons correspond to nonvanishing boundary conditions, and negative (positive) sign of the dispersion term with positive (negative) sign of the nonlinearity. They appear as an intensity dip in the constant background. Many of their properties have been reviewed in Ref. [7].

In physical applications, additional terms are often present in the NLSE. These terms violate the integrability, but, being small, they can be taken into account by perturbation theory. The most powerful perturbative technique, which fully uses the natural separation of the discrete and continuous (i.e., solitonic and radiative) degrees of freedom of the unperturbed NLSE, is based on the inverse scattering transform (IST). While the IST-based perturbation theory for bright NLSE solitons was developed long ago [8–10], and the corresponding perturbation-induced dynamics of the solitons, including radiative and nontrivial many-soliton effects, was well understood [11,12], the analogous theory for dark solitons was absent. Partly, that can be explained by the fact that the IST formalism for the NLSE with nonvanishing boundary conditions is much more complicated than the one for vanishing boundary conditions. Instead, the simplest techniques based on modified conservative laws or the Hamiltonian formalism have been applied successfully to various problems in the theory of perturbed dark solitons [13,14]. However, these methods are suitable for deriving the corresponding evolution equations only in the lowest approximation, when an unperturbed instantaneous shape of one soliton with slowly varying parameters is assumed. They become irrelevant when considering the *N*-soliton solution or when one wishes to take into account the effects that arise in higher orders of perturbation theory. These effects include, in particular, perturbation-induced emission of radiation by solitons and long-range corrections to the soliton's shape. Note that an effort to derive the adiabatic equations for the dark soliton parameters with the aid of the IST was attempted in Ref. [15]. The authors of Ref. [15] used the so-called direct perturbation theory and based their approach on the assumption that the phase of a dark soliton in the presence of perturbations is fixed by the boundary conditions and it does not change. As was pointed out in Ref. [7], this assumption is, generally speaking, wrong. As a result, the equations derived in Ref. [15] have very narrow applicability limits. In particular, the theory presented in Ref. [15] cannot reproduce the adiabatic equations obtained earlier [13,14] from a simple, but reliable, approach based on the renormalized integrals of motion. Besides that, the radiative part of the field was not considered in Ref. [15].

The aim of this paper is to develop a perturbation theory based on the IST to investigate dark soliton propagation in the presence of a perturbation. For concreteness, we will consider optical dark solitons, although all results can be applied to an arbitrary physical model described by the NLSE with nonvanishing boundary conditions and different signs of the dispersion term and the nonlinearity.

The propagation of dark solitons is described by the equation

$$i\partial_t u + \partial_x^2 u - 2|u|^2 u + p[u,u^*] = 0 \tag{1}$$

with $|u(x,0)| \to \rho_0$ at $|x| \to \infty$, which is often referred to as the defocusing NLSE [note, in this connection, that in the case of temporal dark solitons the Kerr nonlinearity is always focusing, but the group-velocity dispersion is positive, so that the resulting equation has the form of Eq. (1)]. We use classical mathematical notation for the independent variables *t*

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and x. In Eq. (1), u is the complex field envelope, and t is the propagation distance along the optical waveguide (spatial problem) or fiber (temporal problem). In the case of temporal solitons, x is a retarded time measured in a frame of reference moving at the group velocity, while for spatial solitons the variable x stands for the transverse coordinate. The perturbation is represented by the term $p[u,u^*]$. All variables are written in normalized form.

As is well known, the unperturbed defocusing NLSE, i.e., Eq. (1) with $p=0$, has an exact solution in the form of the continuous-wave background

$$u=\rho_0\exp(-2i\rho_0^2 t), \tag{2}$$

which is modulationally stable. The dark soliton can be regarded as a localized nonlinear excitation of the background wave. The corresponding solution is

$$u=\rho_0\frac{1+\exp[i\theta+\nu(x-vt-x_0)]}{1+\exp[\nu(x-vt-x_0)]}e^{-2i\rho_0^2 t}, \tag{3}$$

where $\nu=2\rho_0\sin(\theta/2)$ and $v=-2\rho_0\cos(\theta/2)$. As was pointed out in Ref. [14], when considering Eq. (1) with $p\neq 0$, it is necessary to distinguish the cases of perturbations vanishing and nonvanishing at $|x|\to\infty$. These cases correspond to the constant and varying (in t) backgrounds. In the first case, i.e., when $p[u,u^*]\to 0$ at $|x|\to\infty$, the perturbation does not change the continuous-wave background. Then, $\rho_0=$const (constant background) and introducing the new function $\psi(x,t)$ through the relation

$$u(x,t)=e^{-2i\rho_0^2 t}\psi(x,t), \tag{4}$$

one can transform Eq. (1) into

$$i\partial_t\psi+\partial_x^2\psi-2(|\psi|^2-\rho_0^2)\psi+p[\psi,\psi^*]=0 \tag{5}$$

with nonvanishing boundary conditions $|\psi|^2\to\rho_0^2$ at $x\to\pm\infty$. Without loss of generality one can set

$$\psi(x,0)=\begin{cases}\rho_0 & \text{as}\quad x\to-\infty,\\ \rho_0 e^{i\theta} & \text{as}\quad x\to+\infty.\end{cases} \tag{6}$$

In the second case the perturbation p does not vanish at $|x|\to\infty$ and it will affect the background wave. The background amplitude ρ_0 is no longer constant. In this case the substitution $u(x,t)=\psi(x,t)\exp[-2i\int_0^t\rho_0(\tau)]d\tau$ transforms Eq. (1) into Eq. (5) with ρ_0 being dependent on t. However, as was shown in [14], in many important practical cases of varying background Eq. (1) may be transformed into Eq. (5) with some effective $\rho_0=$const after appropriate change of variables. So we will consider Eqs. (5) and (6) as our starting point.

The paper is organized as follows. Section II begins with a review of the theory of the scattering transform for the corresponding linear eigenvalue problem. Then, N-soliton Jost solutions are calculated. In Sec. III the dynamics of the scattering data in the presence of a perturbation is considered and corresponding equations for the N-soliton case are derived. One-soliton perturbation theory is formulated in Sec. IV. Adiabatic equations for the soliton parameters, an equation for continuous scattering data, which describes radiative effects, and an expression for the radiative field with the use of a specific form of the one-soliton Jost solutions are presented. Some applications of the developed theory, namely, a temporal one-soliton pulse with random initial perturbation and a spatial soliton with linear gain and two-photon absorption are considered in Sec. V. The conclusion is made in Sec. VI.

Regarding notation, we will use asterisks for complex conjugation, and 2×2 matrices will be written with bold letters, except for the Pauli matrices

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

II. INVERSE SCATTERING THEORY FOR THE DEFOCUSING NLSE

A. Scattering data

In this subsection we review the theory of the scattering transform for the Zakharov-Shabat eigenvalue problem corresponding to a defocusing NLSE with nonvanishing boundary conditions. Equation (5) with $p[\psi,\psi^*]=0$ can be represented as the compatibility condition

$$\partial_t\mathbf{U}-\partial_x\mathbf{V}+[\mathbf{U},\mathbf{V}]=0 \tag{7}$$

of two linear matrix equations [6] (the Zakharov-Shabat system):

$$\partial_x\mathbf{M}=\mathbf{U}\mathbf{M}, \tag{8}$$

$$\partial_t\mathbf{M}=\mathbf{V}\mathbf{M}, \tag{9}$$

where λ is a spectral parameter, $\mathbf{V}=-\lambda\mathbf{U}+i\mathbf{L}$,

$$\mathbf{U}=\begin{pmatrix}-i\lambda/2 & \psi^*\\ \psi & i\lambda/2\end{pmatrix},\quad \mathbf{L}=\begin{pmatrix}|\psi|^2-\rho_0^2 & -\partial_x\psi^*\\ \partial_x\psi & \rho_0^2-|\psi|^2\end{pmatrix}. \tag{10}$$

Consider the linear problem (8) for some fixed t. In terms of the matrix $\mathbf{U}$ the boundary conditions (6) can be rewritten as $\lim_{x\to\pm\infty}\mathbf{U}(x,\lambda)=\mathbf{U}_\pm(\lambda)$, where

$$\mathbf{U}_-=\frac{1}{2}\begin{pmatrix}-i\lambda & a\\ a & i\lambda\end{pmatrix},\quad \mathbf{U}_+=e^{-i\theta\sigma_3/2}\mathbf{U}_-e^{i\theta\sigma_3/2}, \tag{11}$$

and we have introduced the notation $a=2\rho_0$. The continuous spectrum $\mathbb{R}_a$ of the problem (8) consists of real λ satisfying $\lambda^2\geqslant a^2$. For $\lambda\in\mathbb{R}_a$ denote by $\mathbf{M}^\pm(x,\lambda)$ the 2×2 matrix Jost solutions of Eq. (8), satisfying the boundary conditions $\mathbf{M}^\pm\to\mathbf{E}^\pm(x,\lambda)$ as $x\to\pm\infty$. It follows from Eq. (8) that

$$\partial_x\mathbf{E}^\pm=\mathbf{U}_\pm\mathbf{E}^\pm. \tag{12}$$

The matrix $\mathbf{E}^-(x,\lambda)$ is taken in the form

$$\mathbf{E}^-(x,\lambda)=\begin{pmatrix}1 & i(k-\lambda)/a\\ i(\lambda-k)/a & 1\end{pmatrix}e^{-ikx\sigma_3/2}, \tag{13}$$

where $k(\lambda)=\sqrt{\lambda^2-a^2}$ with $\operatorname{sgn}k(\lambda)=\operatorname{sgn}\lambda$ and $\mathbf{E}^+=\exp(-i\theta\sigma_3/2)\mathbf{E}^-$. Analytical properties of the Jost solutions are formulated on the Riemann surface determined by the function $k(\lambda)$. The Riemann surface $\mathcal{S}$ consists of two sheets $\mathcal{S}^+$ and $\mathcal{S}^-$ of the complex λ plane with branch cuts on the real axis from $-\infty$ to a and from a to ∞. It is convenient to introduce a change of variables

$$\lambda(\zeta) = \frac{1}{2}\left(\zeta + \frac{a^2}{\zeta}\right), \quad k(\zeta) = \frac{1}{2}\left(\zeta - \frac{a^2}{\zeta}\right), \tag{14}$$

which maps the sheets $\mathcal{S}^\pm$ onto $\mathrm{Im}\,\zeta > 0$ and $\mathrm{Im}\,\zeta < 0$, respectively, and the continuous spectrum $\mathbb{R}_a$ onto the real axis $\mathbb{R}$ on the complex ζ plane. Under this,

$$\mathbf{E}^-(x,\zeta) = \begin{pmatrix} 1 & -ia/\zeta \\ ia/\zeta & 1 \end{pmatrix} e^{-ik(\zeta)\sigma_3 x/2}. \tag{15}$$

The matrix Jost solutions $\mathbf{M}^\pm(x,\zeta)$ can be represented in the integral form

$$\mathbf{M}^\pm(x,\zeta) = \mathbf{E}^\pm(x,\zeta) \pm \int_x^{\pm\infty} \mathbf{\Gamma}^\pm(x,y)\mathbf{E}^\pm(y,\zeta)dy. \tag{16}$$

The potential $\psi(x)$ is expressed through the element of the kernel $\mathbf{\Gamma}^-$ as

$$\psi(x) = \rho_0 + 2\Gamma^-_{21}(x,x). \tag{17}$$

The fundamental solutions $\mathbf{M}^+(x,\zeta)$ and $\mathbf{M}^-(x,\zeta)$ with real ζ are linearly dependent and connected with each other through the monodromy matrix $\mathbf{S}(\zeta)$,

$$\mathbf{M}^-(x,\zeta) = \mathbf{M}^+(x,\zeta)\mathbf{S}(\zeta), \tag{18}$$

with the symmetry properties

$$S_{11}(\zeta) = S^*_{22}(\zeta), \quad S_{12}(\zeta) = S^*_{21}(\zeta), \tag{19}$$

$$M^\pm_{11}(\zeta) = M^{\pm *}_{22}(\zeta), \quad M^\pm_{12}(\zeta) = M^{\pm *}_{21}(\zeta) \tag{20}$$

and normalization condition $|S_{11}|^2 - |S_{21}|^2 = 1$. In addition, since the scattering problem (8) possesses symmetry with respect to the inversion $\zeta \to a^2/\zeta$, the following important involution properties are valid:

$$\mathbf{M}^\pm(x,a^2/\zeta) = (\zeta/a)\mathbf{M}^\pm(x,\zeta)\sigma_2, \tag{21}$$

$$\mathbf{S}(a^2/\zeta) = \sigma_2\mathbf{S}(\zeta)\sigma_2. \tag{22}$$

It follows from Eq. (18) that

$$S_{11}(\zeta) = \Delta^{-1}(\zeta)\det(M^-_1(x,\zeta), M^+_2(x,\zeta)), \tag{23}$$

where $M^\pm_j$ means the jth column of $\mathbf{M}^\pm$, and we have introduced the notation $\Delta(\zeta) = 1 - a^2/\zeta^2$. The columns $M^-_1(x,\zeta)$, $M^+_2(x,\zeta)$ turn out to be analytically continuable to $\mathrm{Im}\,\zeta > 0$, while M^-_2, M^+_1 are analytically continuable to $\mathrm{Im}\,\zeta < 0$. Then, the coefficient $S_{11}(\zeta)$ is analytically continuable to $\mathrm{Im}\,\zeta > 0$, except for the points $\zeta = \pm a$. In addition to $\zeta = 0$, the analytic function $S_{11}(\zeta)$ may have zeros $\zeta_1, \ldots, \zeta_N$ in the region of its analyticity $\mathrm{Im}\,\zeta > 0$. Equation (23) then shows that the columns M^+_2 and M^-_1 are linearly dependent and there exist complex numbers $\gamma_1, \ldots, \gamma_N$ such that

$$M^-_1(x,\zeta_j) = -i\gamma_j M^+_2(x,\zeta_j). \tag{24}$$

Denote $r(\zeta) = S_{21}(\zeta)/S_{11}(\zeta)$ (reflection coefficient) and $S'_{11}(\zeta_j) = \partial_\zeta S_{11}(\zeta)|_{\zeta=\zeta_j}$. One can show [16] the following.

(i) Zeros of $S_{11}(\zeta)$ are simple and lie on the circle $|\zeta| = a$ in the region $\mathrm{Im}\,\zeta > 0$ [the latter follows from Eq. (22)]. In addition, the quantities $m_j = -i\gamma_j/(S'_{11}(\zeta_j)\zeta_j)$ are real negative.

(ii) The function $r(\zeta)$ possesses the following properties:

$$r(0) = 0, \tag{25}$$

$$|r(\zeta)| \leqslant 1, \tag{26}$$

$$r(a^2/\zeta) = -r^*(\zeta). \tag{27}$$

Equation (25) is also valid for all derivatives of $r(\zeta)$. The equality in Eq. (26) occurs only at $\zeta = \pm a$ with $r(\pm a) = \mp i$.

(iii) In the case of nonreflectionless (i.e., nonsolitonic) potentials, the coefficients $S_{21}(\zeta)$ and $S_{11}(\zeta)$ are singular at the vicinity $\zeta = \pm a$, so that

$$S_{11}(\zeta) \sim \frac{s_\pm}{\zeta^2 - a^2} \text{ and } S_{21}(\zeta) = \mp i S_{11}(\zeta) \tag{28}$$

at $\zeta \to \pm a$.

(iv) There is a condition

$$e^{i\theta} = \prod_{j=1}^N \frac{\zeta_j^*}{\zeta_j} \exp\left\{\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\ln(1 - |r(\zeta)|^2)}{\zeta} d\zeta\right\}. \tag{29}$$

(v) The coefficient $S_{11}(\zeta)$ can be expressed in terms of its zeros and the values of $|r(\zeta)|$ on the real axis:

$$S_{11}(\zeta) = e^{i\theta/2}\prod_{j=1}^N \frac{\zeta - \zeta_j}{\zeta - \zeta_j^*} \exp\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\ln(1 - |r(\mu)|^2)}{\zeta - \mu + i0} d\mu\right\}. \tag{30}$$

The matrix function $\mathbf{\Gamma}^-(x,y)$ satisfies the Gelfand-Levitan-Marchenko equation

$$\mathbf{\Gamma}^-(x,y) + \mathbf{F}(x+y) + \int_{-\infty}^{x} \mathbf{\Gamma}^-(x,y')\mathbf{F}(y'+y)dy' = 0, \tag{31}$$

where $y \leqslant x$, and the matrix kernel $\mathbf{F}(x)$ is

$$\mathbf{F}(x) = \begin{pmatrix} A^*(x) & B(x) \\ B^*(x) & A(x) \end{pmatrix} \tag{32}$$

with

$$A = \frac{ia}{8\pi}\int_{-\infty}^{\infty} \frac{\tilde{r}(\zeta)}{\zeta} e^{-ik(\zeta)x/2} d\zeta + \frac{a}{4}\sum_{j=1}^N \frac{\tilde{c}_j}{\zeta_j} e^{-ik(\zeta_j)x/2}, \tag{33}$$

$$B = \frac{1}{8\pi}\int_{-\infty}^{\infty} \tilde{r}(\zeta) e^{-ik(\zeta)x/2} d\zeta + \frac{1}{4i}\sum_{j=1}^N \tilde{c}_j e^{-ik(\zeta_j)x/2}, \tag{34}$$

where the notations $\tilde{r}(\zeta) = -S_{12}(\zeta)/S_{11}(\zeta)$ and $\tilde{c}_j = i/[\gamma_j S'_{11}(\zeta_j)]$ have been introduced. Here, unlike the case with vanishing boundary conditions, the matrix $\mathbf{F}$ contains both off-diagonal and diagonal parts. Note that the integrand in Eq. (33) is regular at $\zeta=0$ due to Eq. (25). After solving Eq. (31), the potential $\psi(x)$ can be found from Eq. (17).

In conclusion of this subsection we note that the elements $M_{ij}(x,t,\lambda)$ of the matrix $\mathbf{M}$, where $\mathbf{M}$ is an arbitrary solution of Eq. (8), and the corresponding potential $\psi(x,t)$ satisfy the important relations

$$\psi M_{11}M_{12} + \psi^* M_{21}M_{22} = \partial_x(M_{12}M_{21}), \tag{35}$$

$$\psi M_{11}^2 + \psi^* M_{21}^2 = \partial_x(M_{11}M_{21}), \tag{36}$$

which we will use below. These relations can be easily verified by taking the derivative in Eqs. (35) and (36) and using Eq. (8).

B. The Jost solutions and the potential in the reflectionless case

An important particular case is that of the reflectionless (solitonic) potentials $\psi(x)$ when $S_{21}(t,\xi)\equiv 0$ as a function of ζ for some fixed t. It then follows from Eq. (30) that

$$S_{11}(\zeta) = e^{i\theta/2}\prod_{j=1}^{N}\frac{\zeta-\zeta_j}{\zeta-\zeta_j^*}. \tag{37}$$

The kernel $\mathbf{\Gamma}^-(x,y)$ in this case is [16]

$$\mathbf{\Gamma}^-(x,y) = \sum_{j=1}^{N} f_j(x) g_j^T e^{\nu_j y/2}, \tag{38}$$

where $\nu_j=\mathrm{Im}\,\zeta_j$ and g_j^T means the transpose of the column g_j, the columns $f_j(x)$ are determined from the system of N linear equations

$$f_j(x) + \sum_{p=1}^{N} B_{jp}(x) f_p(x) = -h_j e^{\nu_j x/2}, \tag{39}$$

where the columns g_j and h_j are

$$g_j = \frac{\sqrt{c_j}}{2}\begin{pmatrix}1\\ \zeta_j/ia\end{pmatrix}, \quad h_j = \frac{\sqrt{c_j}}{2}\begin{pmatrix}a\\ i\zeta_j^*\end{pmatrix}, \tag{40}$$

and the matrix $B_{jp}(x)$ is

$$B_{jp}(x) = \frac{ia\sqrt{b_j b_p}}{\zeta_j-\zeta_p^*}e^{(\nu_j+\nu_p)x/2}, \tag{41}$$

with $b_j=i/[\zeta_j S_{11}'(\zeta_j)\gamma_j]$. The N-soliton potential $\psi(x)$ is given by Eq. (17). Substituting Eq. (38) into Eq. (16) yields the N-soliton matrix Jost solution $\mathbf{M}^-$:

$$\mathbf{M}^-(x,\zeta) = \mathbf{E}^-(x,\zeta) + 2\sum_{j=1}^{N}\frac{f_j(x)g_j^T\sigma_3\mathbf{E}^-(x,\zeta)e^{\nu_j x/2}}{\nu_j - ik(\zeta)}. \tag{42}$$

The matrix function $\mathbf{M}^+$ can then be found from Eq. (18).

The reflectionless scattering data with the single ($N=1$) zero $\zeta_1=\upsilon+i\nu$ of the function $S_{11}(\zeta)$ in Eq. (30) correspond to the one-soliton solution. The one-soliton kernel $\mathbf{\Gamma}^-(x,y)$ is

$$\mathbf{\Gamma}_s^-(x,y) = \frac{i\nu e^{\nu(x+y)/2}}{2a(\gamma_1+e^{\nu x})}\begin{pmatrix} ia & \zeta_1\\ -\zeta_1^* & ia\end{pmatrix}, \tag{43}$$

with $\zeta_1=-ae^{-i\theta/2}$ and $\nu=a\sin(\theta/2)$. It then follows from Eq. (17) that the one-soliton potential is

$$\psi_s = \rho_0\frac{1+\exp[i\theta+\nu(x-z)]}{1+\exp[\nu(x-z)]}, \tag{44}$$

where we have introduced the notation $z=\ln(\gamma_1)/\nu$. The one-soliton Jost solutions and the corresponding scattering data are given in Appendix A.

III. DYNAMICS OF THE SCATTERING DATA

Equation (5) can be cast in the matrix form

$$\partial_t\mathbf{U} - \partial_x\mathbf{V} + [\mathbf{U},\mathbf{V}] + \mathbf{P} = 0, \tag{45}$$

where

$$\mathbf{P} = \begin{pmatrix}0 & ip^*\\ -ip & 0\end{pmatrix}. \tag{46}$$

From Eq. (45) and the fact that $\mathbf{M}^\pm$ satisfies Eq. (8) one can get

$$(\partial_x - \mathbf{U})(\partial_t - \mathbf{V})\mathbf{M}^\pm + \mathbf{P}\mathbf{M}^\pm = 0. \tag{47}$$

Introducing a new unknown $\mathbf{J}^\pm(x,t,\zeta)$ defined through the relation

$$(\partial_t - \mathbf{V})\mathbf{M}^\pm = \mathbf{M}^\pm\mathbf{J}^\pm, \tag{48}$$

one can obtain that $\mathbf{J}^\pm$ satisfies $\partial_x\mathbf{J}^\pm=-\mathbf{M}^{\pm-1}\mathbf{P}\mathbf{M}^\pm$, and therefore $\mathbf{J}^\pm=\mathbf{C}^\pm+\int_x^{\pm\infty}\mathbf{M}^{\pm-1}\mathbf{P}\mathbf{M}^\pm dx'$, where the constant matrices $\mathbf{C}^\pm$ are determined from the boundary conditions at $x\to\pm\infty$. Since $\mathbf{V}=-\lambda\mathbf{U}_-$, $\mathbf{M}^-=\mathbf{E}^-$ as $x\to-\infty$, Eq. (48) for $\mathbf{M}^-$ at $x\to-\infty$ becomes $\lambda\mathbf{U}_-\mathbf{E}^-=\mathbf{E}^-\mathbf{J}^-$ or, taking into account Eq. (12), $\mathbf{J}^-=\lambda(\mathbf{E}^-)^{-1}\partial_x\mathbf{E}^-$, and after using Eq. (15) one obtains $\mathbf{C}^-=\mathbf{J}^-(-\infty)=-i\Omega(\zeta)\sigma_3/2$, where

$$\Omega(\zeta) = \frac{1}{4}\left(\zeta^2 - \frac{a^4}{\zeta^2}\right). \tag{49}$$

Similarly, one can show that $\mathbf{C}^+=\mathbf{C}^-$. Then we have

$$\mathbf{J}^\pm = -\frac{i}{2}\Omega(\zeta)\sigma_3 + \int_x^{\pm\infty}\mathbf{M}^{\pm-1}\mathbf{P}\mathbf{M}^\pm dx' \tag{50}$$

and, hence, the following equations of motion for $\mathbf{M}^\pm$:

$$(\partial_t - \mathbf{V})\mathbf{M}^\pm = \mathbf{M}^\pm\left[-\frac{i}{2}\Omega(\zeta)\sigma_3 + \int_x^{\pm\infty}\mathbf{M}^{\pm-1}\mathbf{P}\mathbf{M}^\pm dx'\right]. \tag{51}$$

Equation (51) is valid only for $\mathrm{Im}\,\zeta=0$. Introducing the matrix $\mathbf{M}(x,t,\lambda)=(M_1^-,M_2^+)$, the columns of which admit analytical continuation to $\mathrm{Im}\,\zeta>0$, and as before defining the new unknown matrix $\mathbf{J}(x,t,\lambda)=(J_1,J_2)$ through the relation $(\partial_t-\mathbf{V})\mathbf{M}=\mathbf{M}\mathbf{J}$, one can similarly obtain

$$J_1 = \begin{pmatrix}-i\Omega(\zeta)/2\\ 0\end{pmatrix} - \int_{-\infty}^{x}\mathbf{M}^{-1}\mathbf{P}M_1^- dx', \tag{52}$$

$$J_2 = \begin{pmatrix} 0 \\ i\Omega(\zeta)/2 \end{pmatrix} + \int_x^\infty \mathbf{M}^{-1}\mathbf{P}M_2^+ dx'. \tag{53}$$

Thus, we have the equations of motion

$$(\partial_t - \mathbf{V})M_1^- = \mathbf{M}J_1, \tag{54}$$

$$(\partial_t - \mathbf{V})M_2^+ = \mathbf{M}J_2, \tag{55}$$

valid for Im $\zeta > 0$ except at ζ_j and $\zeta = \pm a$, where $\mathbf{M}$ fails to be invertible. Making the natural assumption that the zeros $\zeta = \zeta_j$ are simple, one can show (see below) that each singularity is removable since det $\mathbf{M} = S_{11}\Delta$.

The equations of motion for $\mathbf{M}^\pm$ and $\mathbf{M}$ determine the evolution of the scattering data. Differentiating Eq. (18) with respect to t and using Eq. (51) yields

$$\partial_t \mathbf{S}(t,\zeta) - \frac{i}{2}\Omega(\zeta)[\sigma_3, \mathbf{S}(t,\zeta)] = -\int_{-\infty}^{\infty} (\mathbf{M}^+)^{-1}(x,t,\zeta)\mathbf{P}\mathbf{M}^-(x,t,\zeta)dx. \tag{56}$$

The equations of motion for the coefficients $S_{11}(t,\zeta)$ and $S_{21}(t,\zeta)$ are contained in Eq. (56). Taking into account that det $\mathbf{M}^\pm = \Delta$, we have

$$\frac{\partial S_{11}}{\partial t} = -i\Delta^{-1}\int_{-\infty}^{\infty} (pM_{12}^+ M_{11}^- + p^* M_{22}^+ M_{21}^-)dx, \tag{57}$$

$$\frac{\partial S_{21}}{\partial t} + i\Omega(\zeta)S_{21} = i\Delta^{-1}\int_{-\infty}^{\infty} (pM_{11}^+ M_{11}^- + p^* M_{21}^+ M_{21}^-)dx. \tag{58}$$

The expression defining the zeros $\zeta_j(t)$ of $S_{11}(t,\zeta)$ is $S_{11}(t,\zeta_j(t)) = 0$. Differentiating with respect to t gives

$$\partial_t S_{11}(t,\zeta_j(t)) + \frac{\partial \zeta_j}{\partial t} S_{11}'(\zeta_j) = 0. \tag{59}$$

Using Eqs. (24) and (57), one therefore finds

$$\frac{\partial \zeta_j}{\partial t} = \frac{\zeta_j^2}{(a^2 - \zeta_j^2)S_{11}'(\zeta_j)\gamma_j}\int_{-\infty}^{\infty} [p(M_{11}^-)^2 + p^*(M_{21}^-)^2]dx, \tag{60}$$

where the integrand is evaluated at x, t, and $\zeta = \zeta_j$. To obtain the evolution equation for γ_j, we differentiate Eq. (24) with respect to t, use Eqs. (54) and (55), and take the limit $\zeta \to \zeta_j$. As a result, one obtains

$$\frac{1}{\gamma_j}\frac{\partial \gamma_j}{\partial t}M_1^-(x,\zeta_j) + i\Omega(\zeta_j)M_1^-(x,\zeta_j) = -\lim_{\zeta\to\zeta_j}\int_{-\infty}^{\infty} \mathbf{M}(x,\zeta)\mathbf{M}^{-1}(x',\zeta)\mathbf{P}(x')M_1^-(x',\zeta_j)dx'. \tag{61}$$

Assuming that singularities at $\zeta = \zeta_j$ in $\mathbf{M}^{-1}(x,\zeta)$ are simple poles, multiplying the matrices and applying the l'Hôpital rule, we arrive at

$$\frac{\partial \gamma_j}{\partial t} + i\Omega(\zeta_j)\gamma_j = \frac{\zeta_j^2}{(a^2 - \zeta_j^2)S_{11}'(\zeta_j)}\int_{-\infty}^{\infty} \{pM_{11}^-\partial_\zeta(i\gamma_j M_{12}^+ + M_{11}^-) + p^* M_{21}^-\partial_\zeta(i\gamma_j M_{22}^+ + M_{21}^-)\}dx, \tag{62}$$

with $\zeta = \zeta_j$. Equations (58), (60), and (62) describe the evolution of the scattering data. It is necessary to stress that no assumptions about the perturbation term $\mathbf{P}$ have been made yet, and these equations are valid for arbitrary $p[\psi,\psi^*]$ in Eq. (5). However, Eqs. (58), (60), and (62) are coupled to the equations for unknown $\mathbf{M}$ and $\mathbf{M}^\pm$ and, in this sense, are practically useless. As is well known, the coupling disappears for $\mathbf{P} = \mathbf{0}$ and the dynamics of the scattering data in this case turns out to be trivial:

$$S_{21}(t) = S_{21}(0)\exp[-i\Omega(\zeta)t], \tag{63}$$

$$\zeta_j(t) = \zeta_j(0), \tag{64}$$

$$\gamma_j(t) = \gamma_j(0)\exp[-i\Omega(\zeta_j)t]. \tag{65}$$

In particular, substituting Eqs. (64) and (65) with $j=1$ into Eq. (44) yields the one-soliton solution (3) with $\nu x_0 = \ln[\gamma_1(0)]$.

One can also immediately write equations for variations of the scattering data under the variations of the potentials $\delta\psi(x,t)$, $\delta\psi^*(x,t)$ for some fixed t. We make use of the formula

$$\delta\mathbf{S}(\zeta) = \int_{-\infty}^{\infty} (\mathbf{M}^+)^{-1}(x,\zeta)\delta\mathbf{Q}(x)\mathbf{M}^-(x,\zeta)dx, \tag{66}$$

where

$$\delta\mathbf{Q}(x) = \begin{pmatrix} 0 & \delta\psi^* \\ \delta\psi & 0 \end{pmatrix}. \tag{67}$$

Comparing this expression with Eq. (56), we get

$$\delta S_{11}(\zeta) = \frac{i}{\Delta}\int_{-\infty}^{\infty} (\delta\psi M_{12}^+ M_{11}^- + \delta\psi^* M_{22}^+ M_{21}^-)dx, \tag{68}$$

$$\delta S_{21}(\zeta) = \frac{1}{i\Delta}\int_{-\infty}^{\infty} (\delta\psi M_{11}^+ M_{11}^- + \delta\psi^* M_{21}^+ M_{21}^-)dx. \tag{69}$$

If $p[u,u^*]$ is a small perturbation, one can substitute the unperturbed N-soliton solutions ψ, ψ^* determined by Eqs. (17) and (38) and N-soliton Jost solutions $\mathbf{M}^\pm$ determined by Eq. (42) into the right-hand side of Eqs. (58), (60), and (62) [or into Eqs. (68) and (69) for small variations of the potential]. This yields evolution equations for the scattering data in the lowest approximation of perturbation theory. This procedure can be iterated to yield higher orders of perturbation theory. The appearing hierarchy of equations (58), (60), and (62) is applied to an arbitrary number of solitons and, in particular, describes nontrivial many-soliton effects in the presence of perturbations. In this paper we restrict ourselves to the case of a one-soliton pulse.

So far, we implied that a is a constant. As was said in Sec. I, this corresponds to a perturbation that vanishes at $|x|\to\infty$. However, it is not difficult to get a generalization for the case of varying background (i.e., when p does not vanish at $|x|\to\infty$). Since all results presented in Sec. II are valid for arbitrary fixed t, and a enters expressions of Sec. III only as a parameter, we can formally put $a=a(t)$ in Eqs. (58), (60), and (62). If $p=0$, it immediately follows (since $|\zeta_j|=a$) from Eq. (60) that $\partial a/\partial t=0$. To obtain the equation for a, we follow the idea suggested in Refs. [7,14]. Considering the nonpropagating (i.e., that which does not depend on x) background ψ_b, and taking the limit $|x|\to\infty$ in Eq. (5), we get the evolution equation

$$i\frac{\partial \psi_b}{\partial t}+p[\psi_b,\psi_b^*]=0. \qquad (70)$$

Writing the background field as $\psi_b=(a/2)\exp(i\alpha)$, and splitting real and imaginary parts in Eq. (70), one can obtain equations for the background intensity and phase,

$$\frac{\partial a}{\partial t}=-2(\cos\alpha\,\mathrm{Re}\,p+\sin\alpha\,\mathrm{Im}\,p), \qquad (71)$$

$$\frac{\partial \alpha}{\partial t}=\frac{2}{a}(\sin\alpha\,\mathrm{Re}\,p-\cos\alpha\,\mathrm{Im}\,p), \qquad (72)$$

where $p[\psi,\psi^*]$ is evaluated at ψ_b. Equations (71) and (72) complete Eqs. (58), (60), and (62) for the case of varying background.

IV. ONE-SOLITON PERTURBATION THEORY

In this section we consider the simplest, but important, case of a one-soliton initial pulse. Taking $N=1$ in Eq. (60), we have

$$\frac{\partial \theta}{\partial t}+\frac{2i}{a}\frac{\partial a}{\partial t}=\frac{4\sin(\theta/2)}{(e^{i\theta}-1)\gamma_1}\int_{-\infty}^{\infty}[p(M_{11}^-)^2+p^*(M_{21}^-)^2]dx, \qquad (73)$$

where M_{11}^- and M_{21}^- are defined by Eqs. (A9) and (A10). We further assume that a is a constant, since in many practical cases the perturbation term which does not vanish at $|x|\to\infty$ can be transformed into a vanishing one after an appropriate change of variables [14]. One can easily check that

$$[M_{11}^-(\zeta_1)]^2=\frac{4\gamma_1}{a^3\sin\theta(e^{i\theta}-1)}\frac{\partial \psi_s}{\partial t}. \qquad (74)$$

Noting also that $\partial\psi_s/\partial t=-\exp(i\theta)(\partial\psi_s^*/\partial t)$ and $[M_{21}^-(\zeta_1)]^2=-\exp(i\theta)[M_{11}^-(\zeta_1)]^2$, we get from Eq. (73) the following equation for the phase θ:

$$\frac{\partial \theta}{\partial t}=\frac{4}{a^3\cos(\theta/2)\sin^2(\theta/2)}\mathrm{Re}\int_{-\infty}^{\infty}p\frac{\partial \psi_s^*}{\partial t}dx, \qquad (75)$$

which coincides (up to notations $\theta=\pi-2\varphi$, $p\to-p$ and scaling) with the equation obtained by Kivshar and Yang [14]. The equation for the second soliton parameter γ_1 follows from Eq. (62) and has the form

$$\frac{\partial \gamma_1}{\partial t}+i\Omega(\zeta_1)\gamma_1=\frac{ia}{4\nu^2}\mathrm{Re}\int_{-\infty}^{\infty}pe^{\nu x}\frac{(\nu-w)^2}{w}\left\{1-\frac{we^{-i\theta}}{\nu-w}-xw(1+e^{-i\theta})\right\}dx, \qquad (76)$$

where w is defined by Eq. (A8). Equations for $S_{11}(\zeta)$ and $S_{21}(\zeta)$ with $\mathrm{Im}\,\zeta=0$ follow from Eqs. (57) and (58):

$$\frac{\partial S_{11}}{\partial t}=\frac{e^{i\theta/2}}{i\Delta(\zeta)}\frac{(\zeta-\zeta_1)}{(\zeta-\zeta_1^*)}\int_{-\infty}^{\infty}\{pM_{11}^-M_{12}^-+p^*M_{21}^-M_{22}^-\}dx, \qquad (77)$$

$$\frac{\partial S_{21}}{\partial t}+i\Omega(\zeta)S_{21}=\frac{ie^{-i\theta/2}}{\Delta(\zeta)}\frac{(\zeta-\zeta_1^*)}{(\zeta-\zeta_1)}\int_{-\infty}^{\infty}\{p(M_{11}^-)^2+p^*(M_{21}^-)^2\}dx, \qquad (78)$$

where the functions $M_{ij}^-(x,\zeta)$ are defined by Eqs. (A4)–(A7). Equations (77) and (78) are completed by initial conditions $S_{11}(0,\zeta)=S_{11}(\zeta)$ and $S_{21}(0,\zeta)=0$, where $S_{11}(\zeta)$ is defined by Eq. (A1). Note that due to the property (28) the coefficients $S_{11}(\zeta)$ and $S_{21}(\zeta)$ are singular functions at the points $\zeta=\pm a$ and $t>0$, so that the singular factor $1/(\zeta^2-a^2)$ does not enter in the expansion parameter.

If the perturbation term has the form $p[\psi,\psi^*]=f(|\psi|^2)\psi$, where f is some arbitrary real function, then Eqs. (77) and (78) can be simplified with the aid of Eqs. (35) and (36).

The action of the perturbation p on the soliton generates a radiation field ψ_c so that at any t the total field is

$$\psi(x,t)=\psi_s(x,t)+\psi_c(x,t) \qquad (79)$$

with $\psi_c(x,0)=0$. The reflection coefficient $\tilde{r}(\zeta)=-S_{12}/S_{11}$ is a measure of the radiation field present in the pulse; for a pure soliton, $\tilde{r}(\zeta)=0$. The perturbation changes $S_{11}(\zeta)$ and $S_{12}(\zeta)=S_{21}^*(\zeta)$ in accordance with Eqs. (77) and (78), respectively. To obtain ψ_c we represent the matrix functions $\mathbf{\Gamma}^-$ and $\mathbf{F}$ in Eq. (31) in the form $\mathbf{\Gamma}^-=\mathbf{\Gamma}_s^-+\delta\mathbf{\Gamma}^-$, $\mathbf{F}=\mathbf{F}_s+\delta\mathbf{F}$, where $\mathbf{\Gamma}_s^-$ and $\mathbf{F}_s$ correspond to the one-soliton solution. The function $\mathbf{\Gamma}_s^-$ is given by Eq. (43), and from Eqs. (32)–(34) we have

$$\mathbf{F}_s=\frac{\nu e^{\nu x/2}}{2\gamma_1}\begin{pmatrix}1 & -i\zeta_1/a\\ i\zeta_1^*/a & 1\end{pmatrix}, \qquad (80)$$

$$\delta\mathbf{F}(x)=\frac{1}{8\pi}\int_{-\infty}^{\infty}\mathbf{E}^-(x,\zeta)\begin{pmatrix}0 & \tilde{r}(\zeta)\\ \tilde{r}^*(\zeta) & 0\end{pmatrix}d\zeta. \qquad (81)$$

Substituting these expressions into Eq. (31), and assuming $\delta\mathbf{\Gamma}^-\ll\mathbf{\Gamma}_s^-$, $\delta\mathbf{F}\ll\mathbf{F}_s$, one can obtain the integral equation for $\delta\mathbf{\Gamma}^-(x,y)$,

$$\delta\mathbf{\Gamma}^-(x,y)+\int_{\infty}^{x}\delta\mathbf{\Gamma}^-(x,y')\mathbf{F}_s(y+y')dy'=\mathbf{\Phi}, \qquad (82)$$

where

$$\mathbf{\Phi}(x,y) = -\delta\mathbf{F}(x+y) - \int_{-\infty}^{x} \mathbf{\Gamma}_s^-(x,y')\delta\mathbf{F}(y+y')dy' \tag{83}$$

is a known function. Equation (82) is an integral equation with degenerate kernel $\mathbf{F}_s(y+y')$ and can be easily solved. In fact, we need only $\delta\Gamma_{21}^-(x,x,t)$. Details are given in Appendix B, and the result is

$$\delta\Gamma_{21}^-(x,x,t) = -\frac{1}{8\pi}\int_{-\infty}^{\infty}\left\{\frac{\tilde{r}(\zeta)}{(1+\zeta_1/\zeta)}(M_{21}^-)^2(x,t,\zeta) + \frac{\tilde{r}^*(\zeta)}{(1+\zeta_1^*/\zeta)}(M_{22}^-)^2(x,t,\zeta)\right\}d\zeta, \tag{84}$$

where $M_{11}^-(x,t,\zeta)$, $M_{21}^-(x,t,\zeta)$ are defined by Eqs. (A4), (A5), and (A7). Then, as follows from Eq. (17), the radiation field ψ_c is given by

$$\psi_c(x,t) = 2\delta\Gamma_{21}^-(x,x,t). \tag{85}$$

Equation (75) loses its validity as $\theta\to 0$, i.e., at small amplitudes of the original soliton. This can be understood in the following way. It is known that in the case of dark solitons the presence of certain perturbations in the initial soliton pulse result in the creation of new solitons with small amplitudes and large velocities without a threshold [21,22]. This is connected to the fact that the continuous spectrum of the linear problem (8) has edges at the branch points $\lambda=\pm a$ which correspond to the so-called virtual levels. An analogous situation takes place for external perturbations too. Indeed, formal exact solution of Eq. (77) can be represented as

$$S_{11}(\zeta,t) = S_{11}(\zeta,0) + \frac{i\zeta^2}{a^2-\zeta^2}\epsilon G(\zeta,t), \tag{86}$$

where $\epsilon\ll 1$, the complex function $G(\zeta,t)$ is regular at the vicinity $\zeta=\pm a$, and $S_{11}(\zeta,0)$ is defined by Eq. (A1). As follows from Eq. (86), an equation defining zeros of $S_{11}(\zeta,t)$ is

$$\zeta^2 = \frac{a^2}{1+i\epsilon G(\zeta,t)\exp[-i\varphi(\zeta)]}, \tag{87}$$

where $\varphi(\zeta)=\arg(\theta/2+2\zeta-2\zeta_1)$. Then, since $\epsilon\ll 1$, one can easily show that there always exist at least two eigenvalues with $\mathrm{Im}\,\zeta>0$:

$$\mathrm{Im}\,\zeta_\pm = \frac{a}{2}\epsilon|\mathrm{Re}\,G(a)|, \tag{88}$$

$$\mathrm{Re}\,\zeta_\pm = \pm a\left\{1-\mathrm{sgn}(\mathrm{Re}\,G)\frac{\epsilon}{2}\mathrm{Im}\,G(a)\right\}, \tag{89}$$

corresponding to a pair of dark solitons with equal small amplitudes and opposite large velosities. If $G(\zeta)\neq G(-\zeta)$, another such pair can be obtained by replacing $G(a)\to G(-a)$. Thus, if the amplitude $\mathrm{Im}\,\zeta_1$ of the original soliton is small enough so that $\mathrm{Im}\,\zeta_1\leqslant\mathrm{Im}\,\zeta_\pm$, i.e., it is comparable with or less than the amplitudes of the spontaneously emerging solitons, then one-soliton perturbation theory fails. Instead, as the first step, it is necessary to substitute the corresponding multisoliton Jost functions into the right hand side of Eq. (75). The criterion of the validity of the one-soliton perturbation theory can be written as

$$\sin\theta \gg \epsilon|\mathrm{Re}\,G(a)|. \tag{90}$$

Equation (87) is a transcendental equation and, generally speaking, it has an infinite (or large) set of close roots with small $\mathrm{Im}\,\zeta$ and $\mathrm{Re}\,\zeta\sim a$. As is known [16], in the case of the focusing NLS with vanishing boundary conditions, such clustering and condensing of the zeros of S_{11} is equivalent to emerging of a radiative component (i.e., the continuous spectrum can be exactly reproduced by taking the limit). However, this is not true for the defocusing NLS with nonvanishing boundary conditions. In this case, as was pointed out in Ref. [16], dispersion relations [i.e., equations connecting the energy E to the momentum P; see Eqs. (104) and (105) below] for continuous and discrete spectrum modes are essentially different (the solitonic one cannot even be written in an explicit form) and the continuous spectrum cannot be obtained from the solitonic part of the spectrum by such zero condensing (or in any other way).

V. APPLICATIONS

A. One-soliton pulse with random initial perturbation

In this subsection we consider temporal dark solitons (i.e., soliton propagation in optical fibers). Suppose that $p=0$, but the soliton input is randomly perturbed so that a pulse $\psi(x)=\psi_s(x)+\delta\psi(x)$ is injected into the fiber. This case corresponds to an inhomogeneous stochastic perturbation in the terminology of Refs. [12,17,18]. The stochasticity arises from an indeterminacy associated with the input pulse, and not from any agency in the fiber itself. One of the sources of the inhomogeneous stochasticity is the amplified spontaneous emission (ASE) noise. The ASE leads to random jitter in the soliton arrival time (the Gordon-Haus effect) T, which is connected with the soliton velocity v by the relation $T=vx$. Thus, the variance $\langle\delta T^2\rangle$ is proportional to $\langle\delta v^2\rangle$. The theory of the Gordon-Haus effect for dark solitons was given in Refs. [19,20], where the noise $\delta\psi(x)$ was assumed to be a homogeneous random process δ correlated in time (white noise). As long as we consider the influence of the noise on localized structures (the solitons over the background) and calculate adiabatic changes of the soliton parameters, the approximation of noise δ correlated in x is quite justified, if the width of the noise spectrum $\Delta\omega\gg\nu$, where ν is the characteristic localization length of the structure (soliton width). However, when considering a continuous spectrum, i.e., unlocalized objects (radiation), the approximation of δ-correlated noise is no longer valid, since (as will be seen below) it results in infinite total energy of the radiation emitted by the soliton. Moreover, the homogeneous random part $\delta\psi$, i.e., with a correlator depending only on the difference $x-x'$, leads to infinite spectral density of the radiation. So we assume that the noise is concentrated in the region occupied by the soliton, and take $\delta\psi(x)$ in the form $\delta\psi(x)=f(x)\varepsilon(x)$, where $f(x)$ is a real deterministic function that vanishes fast enough at the infinity, and $\varepsilon(x)$ is a zero-mean, homogeneous random process with correlation function

$$\langle \varepsilon(x)\varepsilon^*(x')\rangle = D(x-x'), \quad (91)$$

where $\langle \cdots \rangle$ means statistical averaging. The noise $\delta\psi$ is an inhomogeneous random process and its correlation function depends not only on the difference $x-x'$, but on the observation point x too. A suitable choice for $f(x)$ is $f(x) = \mathrm{sech}(\nu x/2)$ (the noise envelope traces the soliton shape), though, as we will see below, final results are not too sensitive to the specific form of $f(x)$. It is assumed that the intensity of the noise is small compared to the square of the soliton amplitude, so that $\langle \delta\psi^2\rangle \ll \nu^2$. The presence of $\delta\psi(x)$ will modify the soliton eigenparameter ζ_1 in a random way, and, aside from this, will result in a continuum (radiative) contribution $\delta\psi_c$ accompanying the modified soliton into the fiber. The corresponding variation of the eigenparameter $\delta\zeta_1$ can be written as

$$\delta\zeta_1 = \left(\left. \frac{\partial S_{11}(\zeta)}{\partial \zeta} \right|_{\zeta=\zeta_1} \right)^{-1} \delta S_{11}(\zeta_1), \quad (92)$$

where δS_{11} is the variation of the transmission coefficient $S_{11}(\zeta)$ induced by the given realization of $\delta\psi$. It follows from Eqs. (68) and (A11) that

$$\delta S_{11}(\zeta_1) = \frac{\zeta_1^2}{a^2-\zeta_1^2}\int_{-\infty}^{\infty}\left(\delta\psi - \frac{a^2}{\zeta_1^2}\delta\psi^*\right)(M_{11}^-)^2 dx, \quad (93)$$

where M_{11}^- is determined by Eq. (A9). Using Eqs. (91)–(93) and performing averaging, one can obtain

$$\langle |\delta\zeta_1|^2\rangle = a^2 I(\nu), \quad \langle \delta\zeta_1^2\rangle = -a^2 e^{-i\theta} I(\nu), \quad (94)$$

where we have introduced the function

$$I(\nu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{D(x-x')f(x)f(x')dx\,dx'}{8\cosh^2(\nu x/2)\cosh^2(\nu x'/2)}, \quad (95)$$

which depends on the specific form of the noise correlation function. It then follows from Eq. (94) that the variance of the soliton velocity is

$$\langle \delta v^2\rangle = a^2 I(\nu)\sin^2(\theta/2). \quad (96)$$

If the noise is δ correlated in time (zero correlation time), so that $D(x)=D_0\delta(x)$ and $f(x)=1$ (pure homogeneous noise), from Eqs. (95) and (96) one obtains $\langle \delta v^2\rangle = (a/3)\sin(\theta/2)$, which coincides with the result obtained in Ref. [19]. Choosing $f(x)$ in the form suggested above, we get $\langle \delta v^2\rangle = (4a/15)\sin(\theta/2)$.

Introducing the Fourier transform of $D(x)$ in the form $D(x)=\int_{-\infty}^{\infty} C(\omega)\exp(-i\omega x)d\omega$ and performing integration over x in Eq. (95), we have

$$I(\nu) = \frac{\pi^2}{8\nu^6}\int_{-\infty}^{\infty} C(\omega)\frac{(\nu^2+4\omega^2)^2}{\cosh^2(\pi\omega/\nu)}d\omega. \quad (97)$$

Equations (96) and (97) determine the variance of the soliton velocity for arbitrary form $C(\omega)$ of the noise spectrum. Consider, for example, the case when the random function $\varepsilon(x)$ has the form $\varepsilon(x)=\varepsilon_0\exp(i\omega_0 x + i\varphi)$, where the random amplitude ε_0 is a zero-mean, normally distributed value with variance σ^2, and the random phase φ is uniformly distributed between 0 and 2π. The correlation function of such a process in the frequency domain is

$$C(\omega) = (\sigma^2/2)\delta(\omega-\omega_0). \quad (98)$$

In this case the noise has an infinite correlation time and is concentrated at the frequency ω_0. The variance of the soliton velocity is

$$\langle \delta v^2\rangle = \frac{\pi^2\sigma^2(\nu^2+4\omega_0^2)^2}{16\nu^4\cosh^2(\pi\omega_0/\nu)}. \quad (99)$$

To take into account a finite correlation time we consider an important particular case, when the ε-noise spectrum has a Lorenzian shape

$$C(\omega) = \frac{D_0}{\pi\tau_c[(\omega-\omega_0)^2+(1/\tau_c)^2]}, \quad (100)$$

where D_0 is the integral intensity of the noise. In the time domain this corresponds to the correlation function $D(x) = D_0\exp(-|x|/\tau_c)\cos(\omega_0 x)$, where τ_c is a correlation time. It follows from Eqs. (96) and (97) that

$$\langle \delta v^2\rangle = \frac{D_0\mu}{8\pi^3}\int_{-\infty}^{\infty}\frac{(\pi^2+4\xi^2)^2 d\xi}{[\mu^2+(\xi-\xi_0)^2]\cosh^2\xi}, \quad (101)$$

where $\mu = \pi/(\nu\tau_c)$, $\xi_0 = \pi\omega_0/(\nu)$. In Fig. 1 the dependence of the variance of the soliton velocity v on the parameter μ is shown for different values of ξ_0 at $D_0=0.1$ and $\nu=1$.

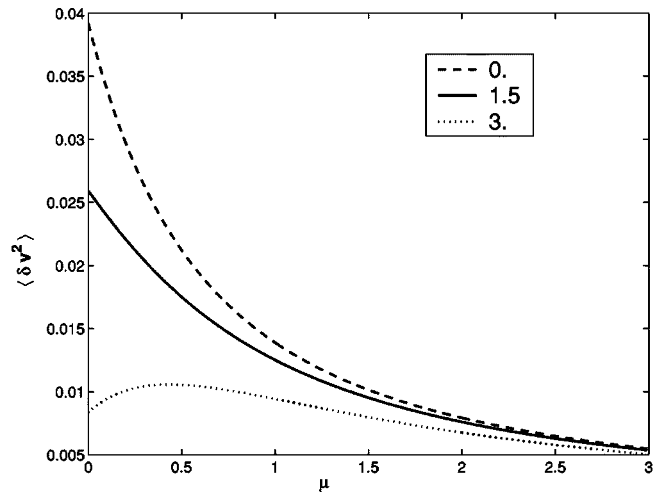

FIG. 1. The dependence of the variance of v on the parameter μ for different values of ξ_0.

Consider now the radiative contribution. Equation (5) with $p=0$ conserves the field momentum P and the energy E,

$$P = \frac{1}{2i}\int_{-\infty}^{\infty}\left(\frac{\partial\psi}{\partial x}\psi^* - \frac{\partial\psi^*}{\partial x}\psi\right)dx - \rho_0^2\theta, \quad (102)$$

$$E = \int_{-\infty}^{\infty}\left\{\left|\frac{\partial\psi}{\partial x}\right|^2 + (|\psi|^2-\rho_0^2)^2\right\}dx. \quad (103)$$

These quantities are written in the regularized form [14,16], so that the corresponding contributions of the background are extracted and, in particular, the integrals (102) and (103) are finite on the soliton solution (44). The integrals of motion (102) and (103) can be explicitly expressed [16] in terms of the continuum ($\zeta \in \mathbb{R}$) and discrete (solitonic) scattering data:

$$P = \int_{-\infty}^{\infty} P_{rad}(\zeta)d\zeta + 2i\sum_{j=1}^{N}\left(\frac{\lambda_j k_j}{4} - \rho_0^2 \arccos\frac{\lambda_j}{a}\right), \tag{104}$$

$$E = \int_{-\infty}^{\infty} E_{rad}(\zeta)d\zeta + \frac{i}{3}\sum_{j=1}^{N} k_j^3, \tag{105}$$

where $\lambda_j = \lambda(\zeta_j)$, $k_j = k(\zeta_j)$, and the spectral densities of the momentum and the energy are

$$P_{rad}(\zeta) = \frac{\zeta}{8\pi}\Delta^2(\zeta)\ln[1 + |S_{21}(\zeta)|^2], \tag{106}$$

$$E_{rad}(\zeta) = \frac{\zeta}{2}\left(1 + \frac{a^2}{\zeta^2}\right)P_{rad}(\zeta). \tag{107}$$

In Eqs. (104) and (105) the soliton contribution is separated from that of the radiative component ($\int d\zeta$) of the wave field described by the continuous-spectrum scattering data. The dispersion relation [taking $\sim\exp(iqx - iKt)$] corresponding linearized version of Eq. (5) is $K(q) = q^2/2$, which means the t dependence $\sim\exp(-iq^2t/2)$. On the other hand, as follows from Eq. (63), in the nonlinear case the t dependence for the continuous spectrum data is $\sim\exp[-i\Omega(\zeta)t]$. Then, considering the radiative component as a superposition of free waves governed by the linear Schrödinger equation, one can conclude that the spectral parameter ζ is connected to the frequency of the emitted quasilinear waves q by the relation

$$q^2 = \frac{1}{4}\left(\zeta^2 - \frac{a^4}{\zeta^2}\right). \tag{108}$$

Note that $q^2 > 0$, since $\operatorname{sgn} k(\lambda) = \operatorname{sgn} \lambda(\zeta)$. The quantities $P_{rad}(q)$ and $E_{rad}(q)$ can be regarded as spectral densities (in the frequency domain) of the momentum and the energy carried by the radiation.

The coefficient $S_{21}(\zeta)$ is no longer zero and, for a given realization of $\varepsilon(x)$, we have from Eqs. (69) and (A4)–(A6)

$$S_{21}(\zeta) = \frac{e^{-i\theta/2}}{i\Delta(\zeta)}\frac{(\zeta - \zeta_1^*)}{(\zeta - \zeta_1)}\int_{-\infty}^{\infty}\{\delta\psi(M_{11}^-)^2 + \delta\psi^*(M_{21}^-)^2\}dx. \tag{109}$$

Note that $\langle S_{21}(\zeta)\rangle = 0$. Writing down the expression for $|S_{21}(\zeta)|^2$, performing averaging over $\varepsilon(x)$, introducing the frequency noise correlator $C(\omega)$, and calculating integrals over x and x', one can obtain

$$\langle|S_{21}(\zeta)|^2\rangle = \frac{\pi^2}{\nu^2\Delta^2}\int_{-\infty}^{\infty}\frac{C(\omega)\{|I_1(\omega)|^2 + |I_2(\omega)|^2\}}{\cosh^2(\pi\eta/2)}d\omega, \tag{110}$$

where

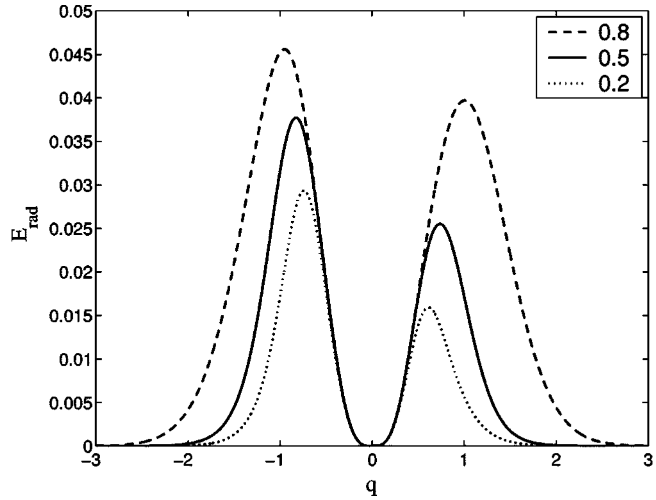

FIG. 2. The frequency distribution of radiation for different values of the parameter ω_c.

$$I_1(\omega) = 2 - 2c_1(1 - i\eta) + \frac{c_1^2}{4}(3 - \eta^2 - 4i\eta), \tag{111}$$

$$I_2(\omega) = -\frac{a^2}{\zeta^2}\left(2 - 2c_2(1 - i\eta) + \frac{c_2^2}{4}(3 - \eta^2 - 4i\eta)\right), \tag{112}$$

with

$$\eta = \frac{2[\omega + k(\zeta)]}{\nu}, \quad c_1 = \frac{\nu(1 + \zeta_1/\zeta)}{\nu - ik(\zeta)}, \quad c_2 = c_1\frac{\zeta}{\zeta_1}. \tag{113}$$

In accordance with the property (28), the function $\langle|S_{21}(\zeta)|^2\rangle$ has singularities at $\zeta = \pm a$. However, as one can see from Eqs. (106) and (107), the spectral densities of the momentum and the energy are finite, and, moreover, equal zero at $\zeta = \pm a$, that is, at the frequency $q = 0$. The frequency distribution of the radiative energy, when the noise correlator has the Gaussian form

$$C(\omega) = (D_0/\omega_c)\exp(-\omega^2/\omega_c^2), \tag{114}$$

where $\omega_c = 1/\tau_c$, is shown in Fig. 2 for different values of the parameter ω_c with $D_0 = 0.1$, $\nu = 1$. The distribution has two asymmetrical peaks and exponentially decaying tails.

B. Linear gain and two-photon absorption

In this subsection we consider spatial dark solitons. As an example of the external perturbation we take the simultaneous action of two-photon absorption and gain. This is the usual situation in the problem of the propagation of spatial solitons [5,7]. The corresponding equation has the form

$$i\partial_t u + \partial_x^2 u - 2|u|^2 u = i\alpha u - i\beta|u|^2 u, \tag{115}$$

where on the right-hand side the first term represents the constant gain contribution and the second one accounts for the intensity-dependent saturation of the gain (e.g., due to the absorption). In the absence of solitons, the background may be stabilized by the simultaneous action of gain and absorption [14,23]. One can see that Eq. (115) has a stationary solution in the form of a stable continuous-wave background $u(t)=\rho_0\exp(-2i\rho_0^2 t)$ with the amplitude

$$\rho_0 = \sqrt{\alpha/\beta}. \tag{116}$$

After the substitution $u=\psi\exp(-2i\rho_0^2 t)$, where ρ_0 is defined by Eq. (116), we get Eq. (5) with the perturbation term

$$p = i\beta(|\psi|^2 - \rho_0^2)\psi. \tag{117}$$

Substituting Eq. (117) into Eq. (75) yields the adiabatic equation for the slowly varying soliton phase θ,

$$\frac{\partial \theta}{\partial t} = -\frac{\alpha}{3}\sin\theta. \tag{118}$$

Equation (118) has the solution

$$\theta(t) = 2\arctan[e^{-\alpha t/3}\tan(\theta_0/2)], \tag{119}$$

where $\theta_0=\theta(0)$ is the initial phase of the soliton. Equation (118) was first obtained in Ref. [14] with the aid of the renormalized integrals of motion.

Let us consider radiative effects, which are described by the off-diagonal term S_{21} (or, equivalently, S_{12}) of the monodromy matrix **S**. These effects include, in particular, emission of radiation by the soliton and distortion of the soliton shape. The emission intensity is characterized by its power, i.e., the energy (or the momentum) emission rate. As follows from Eq. (106), the momentum emission power spectral density $W_P(\zeta)\equiv dP_{rad}/dt$ is

$$W_P(\zeta) = \frac{\zeta}{4\pi}\left(1-\frac{a^2}{\zeta^2}\right)^2\frac{1}{1+|S_{21}(\zeta)|^2}\mathrm{Re}\left\{S_{21}^*\frac{dS_{21}}{dt}\right\}. \tag{120}$$

The energy emission power spectral density $W_E(\zeta)\equiv dE_{rad}/dt$ is

$$W_E(\zeta) = \frac{\zeta}{2}\left(1+\frac{a^2}{\zeta^2}\right)W_P(\zeta). \tag{121}$$

Inserting the perturbation (117) into the general perturbation-induced evolution equation (78) for the coefficient $S_{21}(\zeta,t)$ and calculating the integrals, one can obtain for $s(\zeta,t)=S_{21}(\zeta,t)\exp[i\Omega(\zeta)t]$

$$\frac{ds(\zeta)}{dt} = \frac{e^{-i\theta/2}(\zeta-\zeta_1^*)}{\Delta(\zeta)(\zeta-\zeta_1)}A(\zeta)e^{i(\Omega-kv)t-ikx_0}, \tag{122}$$

where $A(\zeta)$ is some function that can be written in an explicit form. For example, for the soliton with $\theta\sim\pi$, that is, the one which is close to the motionless (or absolutely dark) soliton

$$\psi_s = -\rho_0\tanh(\rho_0 x), \tag{123}$$

the function $A(\zeta)$ takes the form

$$A(\zeta) = \frac{2\pi\alpha k(\zeta)[ik(\zeta)+\nu]a(\zeta^2+a^2)}{3\nu^3\zeta^2\sinh[\pi k(\zeta)/\nu]}. \tag{124}$$

Let us integrate Eq. (122), the right-hand side of which should be multiplied by $\exp(\epsilon t)$ with an infinitely small $\epsilon > 0$. As usual, this implies adiabatically turning on a perturbation that was absent at $t=-\infty$. Thus, we get

$$s^* = \frac{ie^{i\theta/2}(\zeta-\zeta_1)A^*}{\Delta(\Omega-kv+i\epsilon)(\zeta-\zeta_1^*)}e^{-i(\Omega-kv)t+ikx_0}. \tag{125}$$

Then, making use of the relation $\lim_{\epsilon\to 0}(y-i\epsilon)^{-1}=\mathrm{P}(1/y)+i\pi\delta(y)$, where P is the symbol of the principal value, one can find

$$|S_{21}(\zeta)|^2 = \frac{|A(\zeta)|^2}{(\Omega-kv)\Delta^2(\zeta)} \tag{126}$$

and

$$\mathrm{Re}\left\{S_{21}^*\frac{dS_{21}}{dt}\right\} = \frac{\pi|A(\zeta)|^2}{\Delta^2(\zeta)}\delta(\Omega-kv). \tag{127}$$

Equations (120) and (121) together with Eqs. (126) and (127) give the spectral distribution of the emitted momentum and energy rates in terms of the spectral parameter ζ. The wave number q of the emitted waves is connected with ζ by the relation (108). Since $v<a$, one can see that the emission is concentrated at one point of the spectrum $q=0$.

The radiative part of the field in physical space can be determined from Eqs. (84) and (85). It follows from Eqs. (77) and (78) that in the first order the reflection coefficient $\tilde{r}=-S_{12}/S_{11}$ is

$$\tilde{r}(\zeta) = -\frac{iA^*(\zeta)e^{ik(\zeta)x_0}}{\Delta(\zeta)[\Omega(\zeta)-k(\zeta)v]}(e^{ik(\zeta)vt}-e^{i\Omega(\zeta)t}). \tag{128}$$

Expression (128) has singularities at $\zeta=\pm a$. The detailed structure and evolution of the radiative tail are described by Eqs. (84) and (85) with the use of the more correct form of $\tilde{r}(\zeta)$, when the singularities are absent. However, an approximate asymptotic estimate at times $\sim O(\varepsilon)$, where the small parameter ε characterizes the perturbation, can be obtained in a way similar to the one suggested in Refs. [24,25], where the well-known problem of a perturbation-induced shelf generation by a soliton of the Korteweg–de Vries (KdV) equation was studied. In that problem the corresponding reflection coefficient also has a singularity in the first approximation of perturbation theory. We simply substitute Eq. (128) and the corresponding one-soliton Jost functions into Eqs. (84) and (85). The main contribution in the appearing integrals arises from the vicinities of the points $\zeta=\pm a$. The function $A(\zeta)$ has no singularities, and $A(a)=A(-a)$. In particular, for $A(\zeta)$ defined by Eq. (33), we get $A(a)=2\alpha a/3\nu$. For example, at $\zeta\to a$ we can write

$$\tilde{r}(\zeta) \sim \frac{iaA(a)[e^{iv(\zeta-a)t}-e^{ia(\zeta-a)t}]}{2(\zeta-a)^2(v-a)}, \tag{129}$$

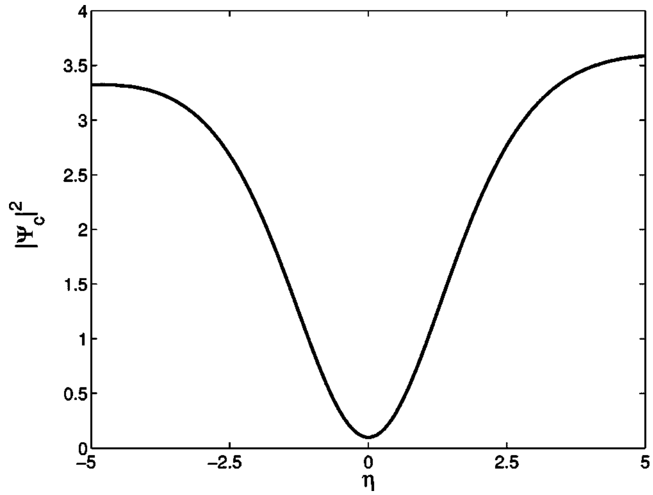

FIG. 3. The normalized intensity of the radiation field vs $\eta=x-x_0-vt$ for $a=10$, $\theta=2.5$, $t=1.0$.

$$(M_{21}^-)^2(\zeta,x,t) \sim -e^{-i(\zeta-a)x}\left\{1-\frac{(1+a/\zeta_1)}{\nu}w(x,t)\right\}^2, \tag{130}$$

and similar expressions for $\zeta\to -a$. Here, the function $w(x,t)$ is defined by Eq. (A8). The slowly varying parts are evaluated at $\zeta=\pm a$ and taken outside the integrals in Eq. (84). The rapidly varying parts are integrated. Under this, we make use of the formula

$$\int_{-\infty}^{\infty}\frac{\cos p\xi-\cos q\xi}{\xi^2}d\xi=\pi(|q|-|p|). \tag{131}$$

As a result, for the radiative field one can obtain

$$\psi_c=-\frac{iaA(a)}{8}\left\{\left[1+\frac{\exp(i\theta/2)}{1+\exp(-\eta)}\right]^2\frac{p_1}{a-v}+\left[1+\frac{\exp(-i\theta/2)}{1+\exp(-\eta)}\right]^2\frac{p_2}{a+v}\right\}, \tag{132}$$

where $\eta=\nu(x-x_0-vt)$, and $p_1=|\eta+(v-a)t|-|\eta|$, $p_2=|\eta+(v+a)t|-|\eta|$. The structure of the radiation with corresponding normalization [by the factor before $\{\cdots\}$ in Eq. (132)] is presented in Fig. 3 for $a=10$, $\theta=2.5$, $t=1.0$. It is necessary to stress that this picture has only qualitative character. As one can see, the radiation has asymmetrical tails. Unlike the KdV soliton, there is no shelf behind the soliton in our case.

VI. CONCLUSION

In conclusion, we have developed a perturbation theory based on the IST for perturbed dark NLSE solitons. This approach fully uses the natural separation of the discrete and continuous degrees of freedom of the unperturbed NLSE. N-soliton Jost solutions were calculated, and equations describing the dynamics of discrete (solitonic) and continuous (radiative) scattering data in the presence of perturbations were derived for the N-soliton case. Adiabatic equations for the soliton parameters and the perturbation-induced radiative field were obtained. The evolution equation for the dark NLSE soliton phase perfectly reduces to that obtained earlier in Ref. [14] with the aid of integrals of motion. We pointed out also that the threshold creation of new solitons with small amplitudes is possible under the action of a perturbation. As applications of the developed theory, we considered a temporal one-soliton pulse with random initial perturbation, and a spatial soliton with linear gain and two-photon absorption. The spectral distribution of the radiation was calculated in both cases.

The general approach presented in this paper may be useful for other physical systems described by the NLSE with nonvanishing boundary conditions and supporting propagation of dark solitons.

APPENDIX A

The scattering data corresponding to the one-soliton solution (44) are

$$S_{11}(\zeta)=e^{i\theta/2}\frac{\zeta-\zeta_1}{\zeta-\zeta_1^*},\quad \zeta_1\equiv v+i\nu=-ae^{-i\theta/2}, \tag{A1}$$

$$S_{12}(\zeta,t)=0\quad(\zeta\text{ is real}), \tag{A2}$$

$$\gamma_1(t)=\gamma_1(0)\exp(-2\rho_0^2 t\sin\theta)\quad(\gamma_1\text{ is real}). \tag{A3}$$

The one-soliton Jost solutions can be calculated from Eqs. (39)–(42). They are (t dependence is omitted)

$$M_{11}^-(x,\zeta)=e^{-ik(\zeta)x/2}\left\{1-\frac{(1+\zeta_1/\zeta)}{[\nu-ik(\zeta)]}w(x)\right\}, \tag{A4}$$

$$M_{21}^-(x,\zeta)=\frac{iae^{-ik(\zeta)x/2}}{\zeta}\left\{1-\frac{(1+\zeta/\zeta_1)}{[\nu-ik(\zeta)]}w(x)\right\}, \tag{A5}$$

$$\begin{pmatrix}M_{11}^+(x,\zeta)\\ M_{21}^+(x,\zeta)\end{pmatrix}=e^{i\theta/2}\frac{(\zeta-\zeta_1^*)}{(\zeta-\zeta_1)}\begin{pmatrix}M_{11}^-(x,\zeta)\\ M_{21}^-(x,\zeta)\end{pmatrix}, \tag{A6}$$

$$M_{22}^{\pm}=(M_{11}^{\pm})^*,\quad M_{12}^{\pm}=(M_{21}^{\pm})^*, \tag{A7}$$

where Im $\zeta=0$ and $k(\zeta)$ is the same as in Eq. (14). Here we have introduced the notation

$$w(x)=\frac{\nu}{1+e^{\nu(z-x)}}=\frac{\nu e^{\nu x}}{\gamma_1+e^{\nu x}}. \tag{A8}$$

Since $k(\zeta_1)=i\nu$, we have also

$$M_{11}^-(x,\zeta_1)=\frac{e^{\nu x/2}}{1+e^{\nu(x-z)}}, \tag{A9}$$

$$M_{21}^-(x,\zeta_1)=\frac{ia}{\zeta_1}M_{11}^-(x,\zeta_1). \tag{A10}$$

In addition, it follows from Eq. (24) that

$$M_2^+(x,\zeta_1)=(i/\gamma_1)M_1^-(x,\zeta_1). \tag{A11}$$

APPENDIX B

The integral equation Eq. (82) for $\delta\Gamma_{21}$ and $\delta\Gamma_{22}$ can be written as

$$\delta\Gamma_{21}(x,y) + C_1(x)f_1(y) + C_2(x)f_2^*(y) = \Phi_{21}(x,y), \quad \text{(B1)}$$

$$\delta\Gamma_{22}(x,y) + C_1(x)f_2(y) + C_2(x)f_1(y) = \Phi_{22}(x,y), \quad \text{(B2)}$$

where

$$\Phi_{21}(x,y) = -\frac{1}{8\pi}\int_{-\infty}^{\infty}\tilde{r}*(\zeta)e^{iky/2}M_{22}^{-}(x,\zeta)d\zeta, \quad \text{(B3)}$$

$$\Phi_{22}(x,y) = -\frac{1}{8\pi}\int_{-\infty}^{\infty}\tilde{r}(\zeta)e^{-iky/2}M_{21}^{-}(x,\zeta)d\zeta, \quad \text{(B4)}$$

and we have introduced the notation

$$C_1(x) = \int_{-\infty}^{x}\delta\Gamma_{21}(x,y')e^{\nu y'/2}dy', \quad f_1(y) = \frac{\nu e^{\nu y/2}}{2\gamma_1},$$

$$C_2(x) = \int_{-\infty}^{x}\delta\Gamma_{22}(x,y')e^{\nu y'/2}dy', \quad f_2(y) = \frac{\nu\zeta_1 e^{\nu y/2}}{2i\gamma_1 a}.$$

The functions $M_{21}^{-}(x,\zeta)$ and $M_{22}^{-}(x,\zeta)$ are defined by Eqs. (A5) and (A7). Equations (B3) and (B4) follow from Eqs. (15), (16), and (83). Multiplying Eqs. (B1) and (B2) by $\exp(\nu y/2)$, and then integrating them over y from $-\infty$ to x, we get a linear algebraic system of equations for the unknown coefficients $C_1(x)$ and $C_2(x)$:

$$C_1(x)\left(1 + \frac{1}{2\gamma_1}e^{\nu x}\right) + C_2(x)\frac{i\zeta_1^*}{2\gamma_1 a}e^{\nu x} = \Phi_1(x),$$

$$C_2(x)\left(1 + \frac{1}{2\gamma_1}e^{\nu x}\right)C_1(x)\frac{i\zeta_1}{2\gamma_1 a}e^{\nu x} = \Phi_2(x), \quad \text{(B5)}$$

where

$$\Phi_1(x) = -\frac{e^{\nu x/2}}{4\pi}\int_{-\infty}^{\infty}\frac{\tilde{r}*(\zeta)e^{ik(\zeta)x/2}}{\nu + ik(\zeta)}M_{22}^{-}(x,\zeta)d\zeta, \quad \text{(B6)}$$

$$\Phi_2(x) = -\frac{e^{\nu x/2}}{4\pi}\int_{-\infty}^{\infty}\frac{\tilde{r}(\zeta)e^{-ik(\zeta)x/2}}{\nu - ik(\zeta)}M_{21}^{-}(x,\zeta)d\zeta. \quad \text{(B7)}$$

Having solved the system (B5) and using Eq. (B1), we have

$$\delta\Gamma_{21}(x,x) = \Phi_{21}(x,x) - \frac{\gamma_1[f_1(x)\Phi_1(x) + f_2^*(x)\Phi_2(x)]}{\gamma_1 + e^{\nu x}}. \quad \text{(B8)}$$

Then we insert the equations

$$\frac{e^{ikx/2}w}{\nu + ik} = \frac{e^{ikx/2} - M_{22}}{1 + \zeta_1^*/\zeta}, \quad \text{(B9)}$$

$$\frac{e^{-ikx/2}w}{\nu - ik} = \frac{e^{-ikx/2} + i\zeta M_{21}/a}{1 + \zeta/\zeta_1} \quad \text{(B10)}$$

into Eqs. (B6), (B7), and (B8). When making the change of variable $\zeta \to a^2/\zeta$ in the integrand containing a single power of M_{21}, and using the involution properties (21) and (27), some integrals are canceled, and after some manipulations we get Eq. (84).

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