PHYSICAL REVIEW E **70**, 066201 (2004)

Amplitude death in oscillators coupled by a one-way ring time-delay connection

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(Received 26 July 2004; published 1 December 2004)

The coupling induced stabilization of a steady state, which is called amplitude death, cannot be observed in identical nonlinear oscillators coupled by diffusive connections. However, it has been analytically confirmed that amplitude death can be induced by using a diffusive time-delay coupling. In this paper, a one-way ring time-delay coupled system consisting of N identical oscillators is proposed. This system is equivalent to a delayed-feedback control system when $N=1$, and to a time-delay coupled oscillator system when $N=2$. In the proposed system, amplitude death never occurs at a steady state when the Jacobi matrix evaluated at a fixed point has an odd number of real positive eigenvalues. Furthermore, a simple systematic graphical procedure to test the stability of the system is presented. This procedure is illustrated in two numerical examples: coupled Rössler oscillators and coupled Lorenz oscillators.

DOI: 10.1103/PhysRevE.70.066201 PACS number(s): 05.45.Xt, 05.45.Ra

I. INTRODUCTION

Diffusive coupled nonlinear oscillators exhibit several attractive phenomena [1]. Among them, coupling induced stabilization of an unstable fixed point, which is called amplitude death or oscillation death, has been actively investigated [1–7]. Previous studies have revealed that at least one of the following conditions is required to produce the death phenomenon in coupled nonlinear oscillators: (a) the oscillators have different parameter values [1]; (b) the coupling unit has its own dynamics [6]; or (c) the signal propagation in the coupling unit includes a time delay [7].

The death phenomenon under condition (c), discovered by Reddy *et al.* [7,8], has gained popularity [9]. Since the discovery of this phenomenon, it has been thoroughly investigated using experimental and theoretical approaches. It was been observed in real physical systems such as electronic circuits [10], living oscillators [11], and thermo-optical oscillators [12]. Atay enlarged the death region in the parameter space by introducing a distributed delayed connection [13] and derived the condition for death stability in networks of time-delay coupled two-dimensional oscillators [14]. Dodla *et al.* investigated the stability of phase-locked patterns and death in two-dimensional limit cycle oscillators coupled by a both-way ring time-delay connection [15]. In addition, a simple sufficient condition for avoiding time-delay induced death has also been derived [5,16].

Delayed-feedback control (DFC) is well known as a practical method for stabilizing unstable periodic orbits and fixed points embedded within a chaotic attractor [17]. In particular, stability analysis and controller design problem of the DFC method have generated considerable interest [18]. However, the DFC method has a crucial defect: a class of unstable periodic orbits and fixed points that satisfies the *odd number property* cannot be stabilized by the delayed feedback controller [19–22]. In order to overcome this defect, several modifications have been proposed [18].

It has been discovered that both time-delay induced amplitude death in coupled oscillators [5,16] and stabilization with the DFC method [19–22] never occur when the Jacobi matrix at a fixed point in oscillators satisfies the odd number property. In fact, there has been no theoretical framework for explaining this common feature. This paper proposes the m-dimensional nonlinear oscillators coupled by a one-way ring time-delay connection illustrated in Fig. 1. It is assumed that the individual oscillators are identical and the coupling scalar signal propagates in one direction. The proposed system with $N=2$ is equivalent to the time-delayed coupled oscillators proposed by Reddy *et al.* [7,16]. Moreover, the DFC system is identical to the proposed system with $N=1$ [21,22]. Therefore, the proposed system can be considered as an extension of both the time-delayed coupled oscillators and DFC systems. This paper also presents a framework for explaining the common feature. Two main results are presented. First, the odd number property is valid for any number of oscillators. Then, a simple systematic and graphical

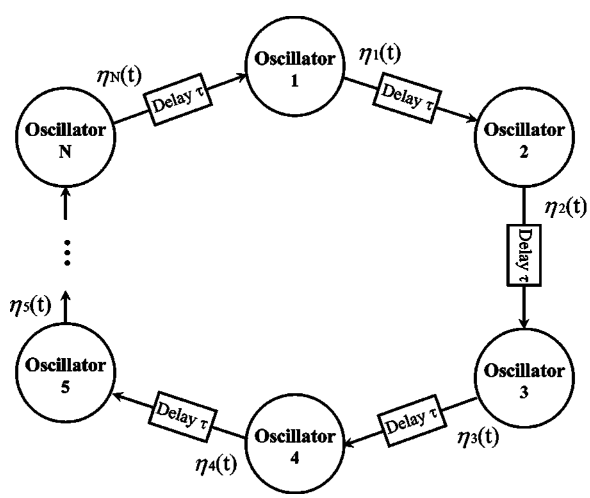

FIG. 1. Nonlinear oscillators coupled by one-way time-delay connection.

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procedure is proposed to test for death stability. Numerical examples using the Rössler and Lorenz oscillators are presented in order to verify the theoretical results.

II. COUPLED OSCILLATORS

This section describes the m-dimensional oscillators and the coupling structure. Consider the individual oscillators

$$\begin{aligned}\dot{\boldsymbol{\xi}}_i &= \mathbf{F}(\boldsymbol{\xi}_i) + \mathbf{b}u_i \\ \eta_i &= \mathbf{c}\boldsymbol{\xi}_i\end{aligned} \quad (i = 1,2,\ldots,N). \tag{1}$$

The ith oscillator has the system variable $\boldsymbol{\xi}_i = [\xi_i^{(1)}\xi_i^{(2)}\cdots\xi_i^{(m)}]^T \in \mathbb{R}^m$, the coupling input signal $u_i \in \mathbb{R}$, and the coupling output signal $\eta_i \in \mathbb{R}$. The input and output vectors are denoted by $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^{1\times m}$, respectively. $N>0$ is the total number of oscillators. Suppose that the nonlinear function $\mathbf{F}:\mathbb{R}^m \to \mathbb{R}^m$ has the fixed point $\boldsymbol{\xi}_f = \{\boldsymbol{\xi} \in \mathbb{R}^m : \mathbf{F}(\boldsymbol{\xi}) = \mathbf{0}\}$. The individual oscillators are connected by the coupling signals

$$u_1 = k[\eta_N(t-\tau) - \eta_1(t)],$$

$$u_i = k[\eta_{i-1}(t-\tau) - \eta_i(t)] \quad (i = 2,3,\ldots,N), \tag{2}$$

where $k \in \mathbb{R}$ represents the coupling strength. The one-way ring coupled oscillators described by Eqs. (1) and (2) are illustrated in Fig. 1. The ith oscillator is influenced by only the delayed signal $\eta_{i-1}(t-\tau)$ of the $(i-1)$th oscillator. Due to diffusive coupling, this coupled system has the steady state

$$[\boldsymbol{\xi}_1^T\ \boldsymbol{\xi}_2^T\ \cdots\ \boldsymbol{\xi}_N^T]^T = [\boldsymbol{\xi}_f^T\ \boldsymbol{\xi}_f^T\ \cdots\ \boldsymbol{\xi}_f^T]^T \tag{3}$$

which is identical to that of the individual oscillators without coupling.

When $N=2$, the one-way time-delay coupled system is equivalent to the delay coupled oscillators proposed by Reddy *et al.* [6,7]. When $N=1$, it is equivalent to the delayed-feedback control system [21,22]. Thus, amplitude death for $N=1$ corresponds to the stabilization of a fixed point in a nonlinear oscillator controlled by delayed feedback. Accordingly, the one-way time-delay coupled system can be considered as an extension of both the delay coupled oscillators and DFC systems.

III. INSTABILITY CONDITION

This section presents an instability condition under which death never occurs via linear stability analysis. Let $\mathbf{x}_i := \boldsymbol{\xi}_i - \boldsymbol{\xi}_f$ and $y_i := \mathbf{c}(\boldsymbol{\xi}_i - \boldsymbol{\xi}_f)$ so that the origin $\mathbf{x}_i = \mathbf{0}$ becomes a fixed point. It is obvious that the behavior of $\boldsymbol{\xi}_i$ $(i=1,2,\ldots,N)$ near fixed point (3) is equivalent to that of $\mathbf{x}_i$ $(i=1,2,\ldots,N)$ governed by

$$\dot{\mathbf{x}}_i = \mathbf{A}\mathbf{x}_i + \mathbf{b}u_i$$

$$y_i = \mathbf{c}\mathbf{x}_i \tag{4}$$

with the connection

$$u_1 = k[y_N(t-\tau) - y_1(t)],$$

$$u_i = k[y_{i-1}(t-\tau) - y_i(t)] \quad (i = 2,3,\ldots,N). \tag{5}$$

The Jacobian matrix

$$\mathbf{A} := \left.\frac{\partial \mathbf{F}(\boldsymbol{\xi})}{\partial \boldsymbol{\xi}}\right|_{\boldsymbol{\xi}=\boldsymbol{\xi}_f} \tag{6}$$

is evaluated at the fixed point $\boldsymbol{\xi}_f$. As a result, we say that the stability analysis of $\boldsymbol{\xi}_i$ can be reduced to the stability of a linear system consisting of (4) and (5). It is assumed throughout this paper that $\mathbf{A}$ does not have an eigenvalue on the origin.

Substituting the coupling signals (5) into the linearized system (4) and defining $\overline{\mathbf{A}} := \mathbf{A} - \overline{\mathbf{b}}$, $\overline{\mathbf{b}} := \mathbf{b}k\mathbf{c}$, the linear system can be written as

$$\begin{bmatrix}\dot{\mathbf{x}}_1\\ \dot{\mathbf{x}}_2\\ \vdots\\ \dot{\mathbf{x}}_{N-1}\\ \dot{\mathbf{x}}_N\end{bmatrix} = \begin{bmatrix}\overline{\mathbf{A}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \overline{\mathbf{A}} & \ddots & \mathbf{0} & \mathbf{0}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \ddots & \overline{\mathbf{A}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \overline{\mathbf{A}}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \vdots\\ \mathbf{x}_{N-1}\\ \mathbf{x}_N\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \overline{\mathbf{b}}\\ \overline{\mathbf{b}} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \ddots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & \overline{\mathbf{b}} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(t-\tau)\\ \mathbf{x}_2(t-\tau)\\ \vdots\\ \mathbf{x}_{N-1}(t-\tau)\\ \mathbf{x}_N(t-\tau)\end{bmatrix}. \tag{7}$$

The characteristic function of system (7) is

$$g(\lambda,\tau)=\det\begin{bmatrix}\lambda\mathbf{I}_m-\overline{\mathbf{A}} & \mathbf{0} & \cdots & \mathbf{0} & -e^{-\lambda\tau}\overline{\mathbf{b}}\\ -e^{-\lambda\tau}\overline{\mathbf{b}} & \lambda\mathbf{I}_m-\overline{\mathbf{A}} & \ddots & \mathbf{0} & \mathbf{0}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \ddots & \lambda\mathbf{I}_m-\overline{\mathbf{A}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & -e^{-\lambda\tau}\overline{\mathbf{b}} & \lambda\mathbf{I}_m-\overline{\mathbf{A}}\end{bmatrix}$$

$$=\det\begin{bmatrix}\lambda\mathbf{I}_m-\overline{\mathbf{A}}-e^{-\lambda\tau}\overline{\mathbf{b}} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0}\\ -e^{-\lambda\tau}\overline{\mathbf{b}} & \lambda\mathbf{I}_m-\overline{\mathbf{A}}+e^{-\lambda\tau}\overline{\mathbf{b}} & \ddots & e^{-\lambda\tau}\overline{\mathbf{b}} & e^{-\lambda\tau}\overline{\mathbf{b}}\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \mathbf{0} & \mathbf{0} & \ddots & \lambda\mathbf{I}_m-\overline{\mathbf{A}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & -e^{-\lambda\tau}\overline{\mathbf{b}} & \lambda\mathbf{I}_m-\overline{\mathbf{A}}\end{bmatrix}$$

$$=g_1(\lambda,\tau)g_2(\lambda,\tau),$$

where

$$g_1(\lambda,\tau)=\det[\lambda\mathbf{I}_m-\overline{\mathbf{A}}-e^{-\lambda\tau}\overline{\mathbf{b}}]=\det[\lambda\mathbf{I}_m-\mathbf{A}+\mathbf{b}(1-e^{-\lambda\tau})],$$

$$g_2(\lambda,\tau):=\det\begin{bmatrix}\lambda\mathbf{I}_m-\overline{\mathbf{A}}+e^{-\lambda\tau}\overline{\mathbf{b}} & e^{-\lambda\tau}\overline{\mathbf{b}} & \cdots & e^{-\lambda\tau}\overline{\mathbf{b}} & e^{-\lambda\tau}\overline{\mathbf{b}}\\ -e^{-\lambda\tau}\overline{\mathbf{b}} & \lambda\mathbf{I}_m-\overline{\mathbf{A}} & \cdots & \mathbf{0} & \mathbf{0}\\ \vdots & \ddots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \ddots & \lambda\mathbf{I}_m-\overline{\mathbf{A}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \cdots & -e^{-\lambda\tau}\overline{\mathbf{b}} & \lambda\mathbf{I}_m-\overline{\mathbf{A}}\end{bmatrix}.$$

The following result is obtained from the above characteristic function.

Lemma 1. Suppose that the fixed point $\boldsymbol{\xi}_f$ of individual oscillators without coupling is unstable; that is, the Jacobian matrix $\mathbf{A}$ is an unstable matrix. Steady state (3) is unstable (i.e., amplitude death never occurs) for any k, $\mathbf{b}$, $\mathbf{c}$, and N, if the coupling signals u_i include no time delay ($\tau=0$).

Proof. For no time delay ($\tau=0$), the instability condition for steady state (3) is that at least one root λ of the characteristic equation $g(\lambda,0)=0$ is in the open right half of the complex plane. It is obvious that the roots of $g(\lambda,0)=0$ include those of $g_1(\lambda,0)=0$ and $g_2(\lambda,0)=0$. Since the equation $g_1(\lambda,0)=0$ can be rewritten as

$$g_1(\lambda,0)=\det[\lambda\mathbf{I}_m-\mathbf{A}]=0,$$

its roots are equivalent to the eigenvalues of the unstable matrix $\mathbf{A}$. Thus, for $\tau=0$, there exists at least one root of $g(\lambda,0)=0$ in the open right half of the complex plane for any k, $\mathbf{b}$, $\mathbf{c}$, and N. ■

Lemma 1 agrees with the death stability analysis [7]. When $N=2$, Lemma 1 is derived for both discrete-time and continuous-time dynamics [5,16]. The following theorem summarizes this paper's main result.

Theorem 1. Steady state (3) is unstable (i.e., amplitude death never occurs) for any k, $\mathbf{b}$, $\mathbf{c}$, τ, and N, if the Jacobian matrix $\mathbf{A}$ has an odd number of real positive eigenvalues (odd-number property).

Proof. The roots of $g_1(\lambda,\tau)=0$ are the primary focus. The function $g_1(\lambda,\tau)$ is continuous in λ; hence $\lim_{\lambda\to\infty}g_1(\lambda,\tau)=\infty$. Substituting $\lambda=0$ into $g_1(\lambda,\tau)$,

$$g_1(0,\tau)=\det[-\mathbf{A}]=\prod_{q=1}^{m}(-\sigma_q),$$

where σ_q are the eigenvalues of $\mathbf{A}$. If $\mathbf{A}$ has an odd number of real positive eigenvalues, then $g_1(0,\tau)<0\ \forall\ \tau>0$. Accordingly, the equation $g_1(\lambda,\tau)=0$ has at least one positive root. This means that steady state (3) is unstable. ■

When $N=1$, this theorem reduces to the odd-number property of a DFC system derived in [21,22]. Furthermore, when $N=2$, this theorem becomes a limitation to amplitude death in time-delay coupled oscillators [6,16]. This theorem guarantees that death never occurs for any coupling strength, input and output vectors, time delay, and number of oscillators. Therefore, if it is desirable to avoid amplitude death, the oscillators should be designed so that the odd-number property is always maintained.

IV. STABILITY CRITERION

This section proposes a simple systematic and graphical procedure that guarantees the stability of steady state (3).

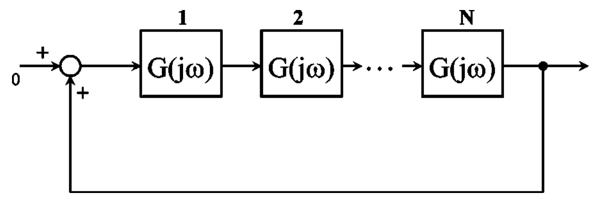

FIG. 2. Coupled linearized systems in frequency domain description.

This procedure is based on the frequency domain description. In individual oscillators, the transfer function from $y_{i-1}(t-\tau)$ to $y_i(t)$ is

$$H(j\omega) \coloneqq \mathbf{c}(j\omega\mathbf{I}_m - \mathbf{A} + \mathbf{b}k\mathbf{c})^{-1}\mathbf{b}k,$$

where $j \coloneqq \sqrt{-1}$. Therefore, the transfer function from $y_{i-1}(t-\tau)$ to $y_i(t-\tau)$ is

$$G(j\omega) \coloneqq e^{-j\omega\tau}H(j\omega),$$

where $(\mathbf{A}-\mathbf{b}k\mathbf{c},\mathbf{b}k,\mathbf{c})$ is assumed to be controllable and observable [23]. Figure 2 illustrates the coupled linearized system consisting of Eqs. (4) and (5) in the frequency domain description. According to the famous Nyquist criterion [23], the following lemma holds.

Lemma 2. Suppose that $\mathbf{A}-\mathbf{b}k\mathbf{c}$ is a stable matrix. Steady state (3) is stable if the vector locus of $G(j\omega)^N$ ($\omega \in [0,\infty)$) does not intersect with the half line

$$L_0 = \{1 + \mu : \mu \in \mathbb{R} \geqslant 0\}$$

on the complex plane.

Lemma 2 requires the calculation and plotting of the vector locus $G(j\omega)^N$. Therefore, if N is large, the calculation is not easy. Furthermore, if N is changed, the vector locus must be recalculated and replotted. Instead, the following theorem provides a simple systematic procedure that guarantees stability by using the vector locus $G(j\omega)$.

Theorem 2. Suppose that $\mathbf{A}-\mathbf{b}k\mathbf{c}$ is a stable matrix. Steady state (3) is stable, if the vector locus of $G(j\omega)$ ($\omega \in [0,\infty)$) does not intersect with any of N half lines

$$L(l,N) = \{\mu e^{j2(l-1)\pi/N} : \mu \in \mathbb{R} \geqslant 1\} \ (l = 1,2,\ldots,N)$$

on the complex plane.

Proof. It is obvious that the following two statements are equivalent: the vector locus $G(j\omega)^N$ intersects the half line L_0 of Lemma 2 and the vector locus $G(j\omega)$ intersects with at least one of N half lines $L(l,N)$ ($l=1,2,\ldots,N$). Hence, if the vector locus $G(j\omega)$ does not intersect with any of N half lines $L(l,N)$, then steady state (3) is stable. ■

Figure 3 illustrates the half lines $L(l,N)$ for $N=1, 2, \ldots, 5$. Lemma 2 is based on the sufficient condition for the stability of coupled linearized systems. Therefore, there is a possibility that a system would be stable even if Theorem 2 is not satisfied.

V. NUMERICAL EXAMPLES

The previously described procedure is illustrated using two numerical examples.

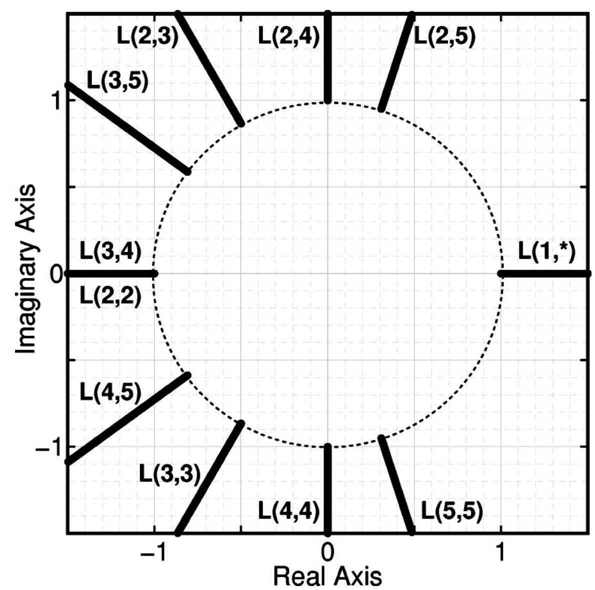

FIG. 3. Half lines $L(l,N)$ on the complex plane. $L(1,*)$ denotes $L(1,n)$ for $n=1, 2, \ldots, N$.

A. Rössler oscillators

For the first example, the Rössler equation

$$\mathbf{F}(\boldsymbol{\xi}_i) = \begin{bmatrix} -\xi_i^{(2)} - \xi_i^{(3)} \\ \xi_i^{(1)} + p_1\xi_i^{(2)} \\ p_2 + \xi_i^{(3)}(\xi_i^{(1)} - p_3) \end{bmatrix}$$

is used for the individual oscillators, where $p_{1,2,3}$ are the system parameters. One of the two fixed points is

$$\boldsymbol{\xi}_f = [\xi_f^{(1)}\ \xi_f^{(2)}\ \xi_f^{(3)}]^T = \left[\frac{1}{2}\gamma \quad -\frac{1}{2p_1}\gamma \quad \frac{1}{2p_1}\gamma\right]^T,$$

$$\gamma = p_3 - \sqrt{p_3^2 - 4p_1p_2}.$$

The purpose of this example is to know whether amplitude death occurs. The input and output vectors are $\mathbf{b}=[0\ 1\ 0]^T$ and $\mathbf{c}=[0\ 1\ 0]$, and the Jacobi matrix evaluated at the fixed point $\boldsymbol{\xi}_f$ is given by

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & p_1 & 0 \\ \xi_f^{(3)} & 0 & \xi_f^{(1)} - p_3 \end{bmatrix}.$$

Assume that the system parameters are set to $p_1=p_2=0.2$, $p_3=5.7$. For the no-coupling case ($k=0$), the individual oscillators exhibit the famous Rössler attractor. Since the eigenvalues of $\mathbf{A}$ are $\sigma_1=-5.687$ and $\sigma_{2,3}=0.097\pm j0.995$, $\mathbf{A}$ is an unstable matrix and does not satisfy the odd-number property presented in Theorem 1. Therefore, Lemma 1 guarantees that death never occurs at $\boldsymbol{\xi}_f$ for any k, $\mathbf{b}$, $\mathbf{c}$, and N when the coupling does not include a time delay. Theorem 1 does not guarantee that steady state (3) is stabilized by the time-delay coupling. For $k=1.5$, the matrix $\mathbf{A}-\mathbf{b}k\mathbf{c}$ is stable (eigenvalues -5.687, $-0.653\pm j0.763$) and $(\mathbf{A}-\mathbf{b}k\mathbf{c},\mathbf{b}k,\mathbf{c})$ is controllable and observerable. Therefore,

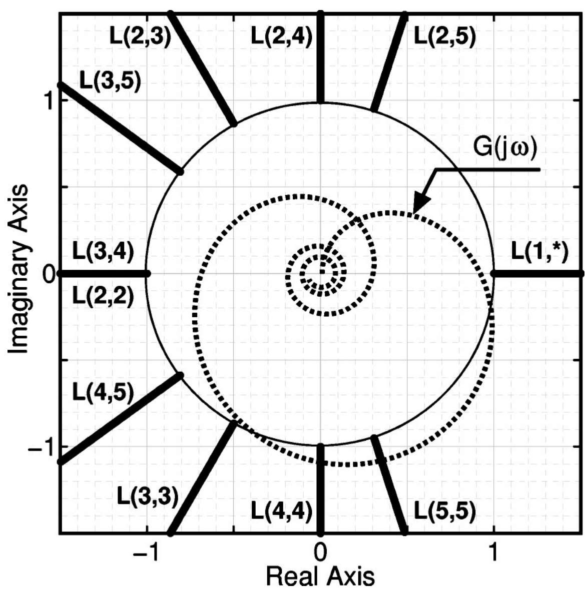

FIG. 4. Vector locus (dashed line) of $G(j\omega)$ for the coupled Rössler oscillators ($k=1.5, \tau=1.0$).

Theorem 2 can be employed. The vector locus of $G(j\omega)$ with $\tau=1.0$ is plotted in Fig. 4. Because it does not intersect any of the half lines $L(1,*)$, $L(2,2)$, $L(2,3)$, and $L(3,3)$, Theorem 2 guarantees that steady state (3) is stable for $N=1$, 2, and 3. However, because the vector locus intersects the half lines $L(4,4)$ and $L(5,5)$, the stability of (3) cannot be guaranteed for $N=4$ and 5.

Figure 5 shows the temporal behavior of the coupled Rössler oscillators ($N=1$, 2, …, 5). The individual oscillators without coupling behave independently until $t=50$. At $t=50$ one-way time-delay coupling is achieved. Figure 5(a) reveals that the oscillator variables $\xi_i^{(2)}$ for $N=1$, 2, and 3 converge to the fixed point $\boldsymbol{\xi}_f$ after coupling: That is, amplitude death occurs. This result agrees with the stability analysis shown in Fig. 4. On the other hand, the variables $\xi_i^{(2)}$ for $N=4$ and 5 do not converge to a fixed point. This fact does not contradict the stability analysis.

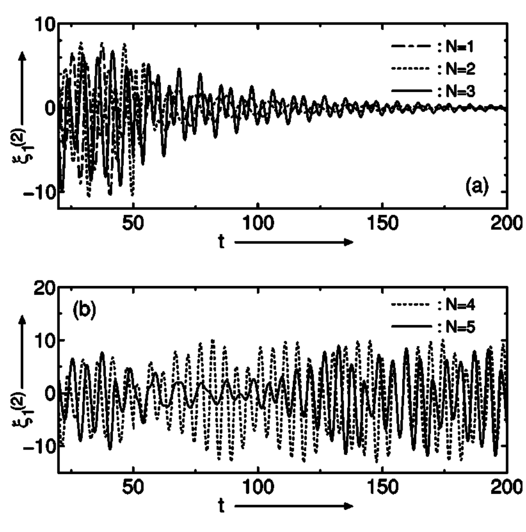

FIG. 5. Temporal behavior of the coupled Rössler oscillators ($k=1.5, \tau=1.0$). (a) $N=1$, 2, and 3, (b) $N=4$ and 5.

B. Lorenz oscillators

The Lorenz equation

$$\mathbf{F}(\boldsymbol{\xi}_i) = \begin{bmatrix} p_1(\xi_i^{(2)} - \xi_i^{(1)}) \\ -\xi_i^{(1)}\xi_i^{(3)} + p_2\xi_i^{(1)} - \xi_i^{(2)} \\ \xi_i^{(1)}\xi_i^{(2)} - p_3\xi_i^{(3)} \end{bmatrix}$$

has three system parameters denoted by $p_{1,2,3}$. For $p_2 > 1$, there are three fixed points:

$$\boldsymbol{\xi}_{f0}:[0\ 0\ 0]^T,$$

$$\boldsymbol{\xi}_{f+}:[\sqrt{p_3(p_2-1)}\ \sqrt{p_3(p_2-1)}\ p_2-1]^T,$$

$$\boldsymbol{\xi}_{f-}:[-\sqrt{p_3(p_2-1)}\ -\sqrt{p_3(p_2-1)}\ p_2-1]^T.$$

The input and output vectors are $\mathbf{b}=[0\ 1\ 0]^T$ and $\mathbf{c}=[0\ 1\ 0]$, and the Jacobian matrix

$$\mathbf{A} = \begin{bmatrix} -p_1 & p_1 & 0 \\ p_2-\xi_f^{(3)} & -1 & -\xi_f^{(1)} \\ \xi_f^{(2)} & \xi_f^{(1)} & -p_3 \end{bmatrix}$$

is evaluated at the fixed points. The system parameters are set to $p_1=10.0$, $p_2=28.0$, and $p_3=8.0/3.0$. Without coupling, the individual oscillators exhibit the famous Lorenz attractor.

For the fixed point $\boldsymbol{\xi}_{f0}$, the eigenvalues of $\mathbf{A}$ are estimated to be $\sigma_1=11.828$, $\sigma_2=-2.667$, and $\sigma_3=-22.828$; therefore, $\mathbf{A}$ is an unstable matrix that satisfies the odd-number property in Theorem 1. According to Lemma 1 and Theorem 1, both systems with no time-delay coupling and time-delay coupling never induce amplitude death at the steady state

$$[\boldsymbol{\xi}_1^T\ \boldsymbol{\xi}_2^T\ \cdots\ \boldsymbol{\xi}_N^T]^T = [\boldsymbol{\xi}_{f0}^T\ \boldsymbol{\xi}_{f0}^T\ \cdots\ \boldsymbol{\xi}_{f0}^T]^T.$$

For the fixed points $\boldsymbol{\xi}_{f\pm}$, on the other hand, $\mathbf{A}$ has the eigenvalues $\sigma_1=-13.855$ and $\sigma_{2,3}=0.094\pm j10.195$, which do not satisfy the odd-number property. For $k=2.0$ and $\tau=0.5$, the matrix $(\mathbf{A}-\mathbf{b}k\mathbf{c})$ is stable and $(\mathbf{A}-\mathbf{b}k\mathbf{c},\mathbf{b}k,\mathbf{c})$ is controllable and observable. From Fig. 6, it can be seen that the vector locus of $G(j\omega)$ does not intersect the half lines $L(1,*)$, $L(2,2)$, $L(2,3)$, and $L(3,3)$. This fact guarantees that both steady states

$$[\boldsymbol{\xi}_1^T\ \boldsymbol{\xi}_2^T\ \cdots\ \boldsymbol{\xi}_N^T]^T = [\boldsymbol{\xi}_{f-}^T\ \boldsymbol{\xi}_{f-}^T\ \cdots\ \boldsymbol{\xi}_{f-}^T]^T, \tag{8}$$

$$[\boldsymbol{\xi}_1^T\ \boldsymbol{\xi}_2^T\ \cdots\ \boldsymbol{\xi}_N^T]^T = [\boldsymbol{\xi}_{f+}^T\ \boldsymbol{\xi}_{f+}^T\ \cdots\ \boldsymbol{\xi}_{f+}^T]^T \tag{9}$$

are stable for $N=1$, 2, and 3. However, since the vector locus intersects with the half lines $L(4,4)$ and $L(5,5)$, the stability of the steady states for $N=4$ and 5 cannot be guaranteed.

The system variables of the coupled Lorenz oscillators are shown in Fig. 7. The oscillators remain uncoupled until $t=50$. As shown in Fig. 7(a), steady states (3) for $N=1$, 2, and 3 are stable. The oscillators converge on (8) or (9), depending on the initial condition. In contrast, all of the oscillators do not converge to the steady states for $N=4$ and 5, as shown in Fig. 7(b). These facts also do not contradict the stability analysis in Fig. 6.

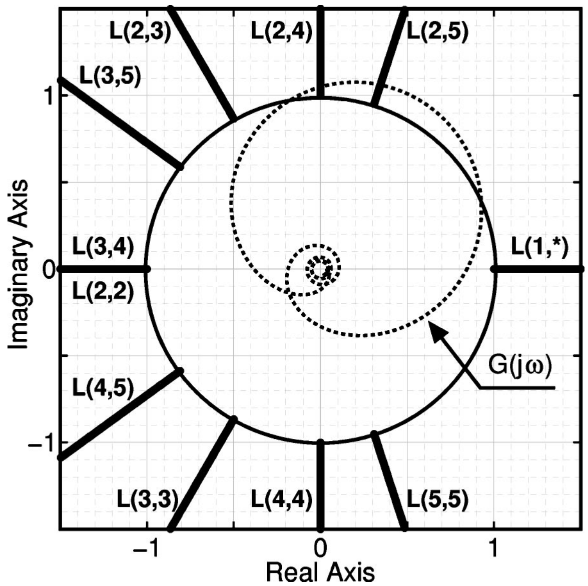

FIG. 6. Vector locus (dashed line) of $G(j\omega)$ for the coupled Lorenz oscillators $(k=2.0, \tau=0.5)$.

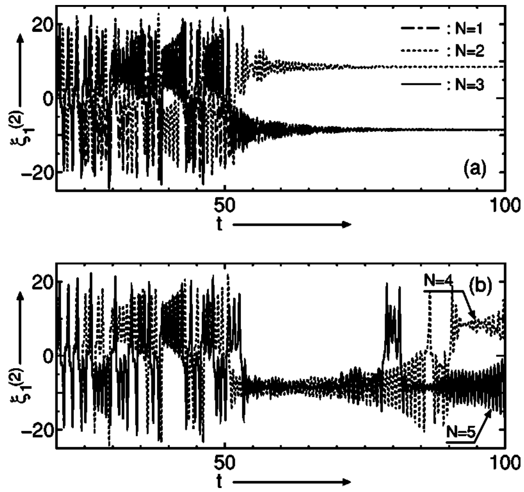

FIG. 7. Temporal behavior of the coupled Lorenz oscillators $(k=2.0, \tau=0.5)$. (a) $N=1$, 2, and 3, (b) $N=4$ and 5.

VI. CONCLUSIONS

This paper has proposed a system consisting of N oscillators coupled by a one-way ring time-delay connection, which is an extension of both the time-delay coupled oscillator and the DFC systems. It was verified that the odd-number property is a sufficient condition under which amplitude death never occurs in the proposed system. Furthermore, a systematic and graphical procedure to check the stability of the steady state on the basis of the Nyquist criterion has been proposed. Numerical simulations using the Rössler and Lorenz oscillators have been conducted to confirm the theoretical results.

ACKNOWLEDGMENTS

This research was supported by a Grant-in-Aid for Young Scientists (No. 15760326) from the Japanese Ministry of Education, Culture, Sports, Science, and Technology and by the Special Research Funds of Future University–Hakodate.

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