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Modeling the evolution of weighted networks

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We present a general model for the growth of weighted networks in which the structural growth is coupled with the edges' weight dynamical evolution. The model is based on a simple weight-driven dynamics and a weights' reinforcement mechanism coupled to the local network growth. That coupling can be generalized in order to include the effect of additional randomness and nonlinearities which can be present in real-world networks. The model generates weighted graphs exhibiting the statistical properties observed in several real-world systems. In particular, the model yields a nontrivial time evolution of vertices' properties and scale-free behavior with exponents depending on the microscopic parameters characterizing the coupling rules. Very interestingly, the generated graphs spontaneously achieve a complex hierarchical architecture characterized by clustering and connectivity correlations varying as a function of the vertices' degree.

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I. INTRODUCTION

Networked structures appear in a wide array of systems belonging to domains as diverse as biology, ecology, social sciences, or large information infrastructures such as the Internet and the World Wide Web [1–4]. In recent years, many empirical findings have uncovered the general occurrence of a complex topological organization underlying many of these networks, triggering the interest of the research community. In particular, small-world properties [5] and large fluctuations in the connectivity pattern identifying the class of scale-free networks [1] have been repeatedly observed in real-world networks. These findings triggered a wealth of theoretical and empirical studies devoted to the characterization and modeling of these features. These studies have pointed out the importance of the evolution and growth of networks [3,4,6] and led to the formulation of a long list of models aimed at studying the architecture of complex networks and the dynamical processes which are taking place on their structure [7–10].

So far the research activity on networks has been mainly focused on graphs in which links are represented as binary states, i.e., either present or absent. More recently, however, the gathering of more complete data has allowed one to take into account the variation of the strength of the connections between nodes (i.e., the *weights* of the links), providing a more complete representation of some networked structures in terms of weighted graphs. Indeed, this diversity in the interaction intensity is of crucial interest in real networks. Studies of congestion phenomena on the Internet implies the knowledge and the characterization of its traffic [4] and the number of passengers in the airline networks is obviously basic information to assess the importance of an airline connection [2,11,12]. In the case of ecological networks [13], recent studies (see [14] and references cited therein) highlighted the importance of the strength of the predator-prey interaction in ecosystem stability. Metabolic reactions also carry fluxes that are essential to the understanding of metabolic networks, as shown in [15]. Finally, sociologists [16] showed already some time ago the importance of weak links in social networks. Very interestingly, the analysis of some paradigmatic weighted networks have revealed that in addition to a complex topological structure, real networks display a large heterogeneity in the capacity and intensity of the connections. In particular, broad distributions and nontrivial correlations between weights and topology were observed in different networks [11,12,17].

From the previous discussion, it appears clearly that there is a need for a modeling approach to complex networks that goes beyond the purely topological point. In this paper, we analyze in detail a general model for the evolution of weighted networks that couples the topology and weights dynamical evolution. Vertices entering the system draw new edges with an attachment dynamics driven by the weight properties of existing edges and vertices. In addition, in contrast with previous models [18,19] for which weights are statically assigned, we allow for the dynamical evolution of weights during the growth of the system. This dynamics is inspired by the evolution and reinforcements of interactions in natural and infrastructure networks. We provide a detailed analytical and numerical inspection of the model, considering different specific mechanisms—homogeneous, heterogeneous, nonlinear—for the evolution of weights. (A short report of the simplest linear and homogeneous case appeared in Ref. [20].) The obtained networks display heavy-tailed distributions of weight, degree, and strength. We determine analytically the exponents of the corresponding power laws showing that they depend on the unique parameter defining the model's dynamics. Interestingly, the model generates graphs that spontaneously develop a structural organization in which vertices with different degrees exhibit different levels of local clustering and correlations. These correlations can be shown to emerge as a direct consequence of the coupling between topology and dynamics. While the model we introduce here is possibly the simplest one in the class of weight-driven models, it generates a very rich phenomenology that captures many of the complex features emerging in the analysis of real networks. In this perspective it can be considered as a general starting point for more realistic models aimed at the representation of specific networks.

The paper is structured as follows. In Sec. II we review the necessary definitions and tools for the characterization of complex weighted networks. The general formulation of the model is reported in Sec. III. Section IV discusses the homogeneous reinforcement rules and reports the corresponding analytical and numerical analysis. In Sec. V, the dynamics is generalized in order to include the effect of local randomness. A further generalization to more complicated nonlinear reinforcement mechanisms for the weights' evolution is discussed and analyzed in Sec. VI.

II. WEIGHTED NETWORKS

The topological properties of a graph are fully encoded in its adjacency matrix a_{ij}, whose elements are 1 if a link connects node i to node j, and 0 otherwise. The indices i,j run from 1 to N where N is the size of the network and we use the convention $a_{ii}=0$. Similarly, a weighted network is entirely described by a matrix W whose entry w_{ij} gives the weight on the edge connecting the vertices i and j (and $w_{ij}=0$ if the nodes i and j are not connected). In the following we will consider only the case of symmetric weights $w_{ij}=w_{ji}$ while the directed case is considered in [21].

Important examples of weighted networks have been recently characterized. The first example is the worldwide airport network (WAN) [11,12,22] where the weight w_{ij} is the number of available seats on direct flight connections between the airports i and j. A second important case study is the scientific collaboration network (SCN) [24,25] where the nodes are identified with authors and the weight depends on the number of co-authored papers [12,24]. These two cases, which are paradigms of, respectively, large infrastructure and social networks, display complex features characterized by heavy tailed distributions for topological and weighted quantities. Another very important example of a weighted network is the biochemical network of metabolic reactions. For this network, the nodes are biochemical elements (enzymes, etc.) and a link between two nodes denotes the existence of an individual chemical reaction between them. The weight of a link can be characterized by the flux of this chemical reaction. A very recent study [15] has provided the first analysis of the weighted graph of a metabolic network, bringing further evidence for the heterogeneous complex aspect of weighted networks.

In the following we introduce a set of general quantities whose statistical analysis allows the mathematical characterization of the complex and heterogeneous nature of a weighted graph.

A. Weights and strength

The most commonly used topological information about vertices is their degree and is defined as the number k_i of neighbors

$$k_i = \sum_j a_{ij}. \tag{1}$$

A natural generalization in the case of weighted networks is the *strength* s_i defined as [12,18]

$$s_i = \sum_j w_{ij}. \tag{2}$$

Indeed, the strength of a node combines the information about its connectivity and the intensity of the weights of its links. In the case of the worldwide airport network, the strength s_i corresponds to the total traffic going through a vertex i and is therefore an indication of the importance of the airport i. In the case of the scientific collaboration network, the strength gives the number of papers authored by a given scientist (excluding single-author publications, see e.g., [12]).

A natural characterization of the statistical properties of networks is provided by the probability $P(k)$ that any given vertex has a degree k. Many studies have revealed that networks display a heavy tailed probability distribution $P(k)$ that in many cases is well approximated by a power-law behavior $P(k)\sim k^{-\gamma}$ with $2\leqslant\gamma\leqslant 3$. This has led to the introduction of the class of scale-free networks [6], as opposed to the regular graphs with Poissonian degree distribution. Similar information on the statistical properties of weighted networks can be gathered at first instance by the analysis of the strength and weight distributions $P(s)$ and $P(w)$ which denote the probability of a vertex to have the strength s and of a link to have the weight w, respectively. Also for these distributions, recent measurements on weighted networks have uncovered the presence of heavy tails and power-law behaviors [11,12,15,17,22,23]. The heavy-tailed behavior of these distributions is an extremely relevant characteristic of complex networks indicating the presence of statistical fluctuations diverging with the graph size. This implies that the average values $\langle k\rangle$, $\langle w\rangle$, and $\langle s\rangle$ are not typical in the network and there is an appreciable probability of finding vertices with very high degree and strength. In other words, we are generally facing networks which are very heterogeneous. It is worth stressing that the correlations between the weight and topological properties are encoded in the statistical relations among these quantities. Indeed, s_i, which is a sum over all neighbors of i, is correlated with its degree k_i. In the simplest case of random, uncorrelated weights w_{ij} with average $\langle w\rangle$, the strength is $s\sim\langle w\rangle k$. In the presence of correlations between weights and topology, we may observe a more complicated behavior with $s\sim Ak^{\beta}$ with $\beta\neq 1$ or with $\beta=1$ and $A\neq\langle w\rangle$.

B. Clustering and correlations

Complex networks display an architecture imposed by the structural and administrative organization of these systems that is not fully characterized by the distributions $P(k)$ and $P(s)$. Indeed, the structural organization of complex networks is mathematically encoded in the various correlations existing among the properties of different vertices. For this reason, a set of topological and weighted quantities are customarily studied in order to uncover the network architecture. A first and widely used quantity is given by the *clustering* of vertices. The clustering of a vertex i is defined as

$$c_i = \frac{1}{k_i(k_i-1)}\sum_{j,h} a_{ij}a_{ih}a_{jh} \tag{3}$$

and measures the local cohesiveness of the network in the neighborhood of the vertex. Indeed, it yields the fraction of interconnected neighbors of a given vertex. The average over all vertices gives the network *clustering coefficient* C which describes the statistics of the density of connected triples. Further information can be gathered by inspecting the average clustering coefficient $C(k)$ restricted to classes of vertices with degree k:

$$C(k) = \frac{1}{NP(k)}\sum_{i/k_i=k} c_i. \tag{4}$$

In many networks, the degree-dependent clustering coefficient $C(k)$ is a decreasing function of k which shows that low-degree nodes generically belong to well interconnected communities while high-degree sites are linked to many nodes that may belong to different groups which are not directly connected [26,27]. This is generally the signature of a nontrivial architecture in which hubs, high degree vertices, play a distinct role in the network.

Another important source of information lies in the correlations of the degree of neighboring vertices [28,29]. Since the whole conditional distribution $P(k'|k)$ that a given site with degree k is connected to another site of degree k' is often difficult to interpret, the *average nearest-neighbor degree* has been proposed to measure these correlations [28]

$$k_{nn,i} = \frac{1}{k_i}\sum_{j=1}^{N} a_{ij}k_j. \tag{5}$$

Once averaged over classes of vertices with connectivity k, the average nearest-neighbor degree can be expressed as

$$k_{nn}(k) = \sum_{k'} k' P(k'|k), \tag{6}$$

providing a probe on the degree correlation function. If degrees of neighboring vertices are uncorrelated, $P(k'|k)$ is only a function of k' and thus $k_{nn}(k)$ is a constant. When correlations are present, two main classes of possible correlations have been identified: *assortative* behavior if $k_{nn}(k)$ increases with k, which indicates that large degree vertices are preferentially connected with other large degree vertices, and *disassortative* if $k_{nn}(k)$ decreases with k [30].

While the above quantities provide clear signatures of the structural organization, they are defined solely on topological grounds and the inclusion of weights and their correlations might be extremely important for a full understanding of the networks' architecture. For instance, Fig. 1 clearly shows that very different situations in terms of weights can have the same topological clustering: if the existing triples are formed by links with small weights, the (geometrical) clustering coefficient will overestimate their relevance in the network's organization. For this reason, generalizations of clustering

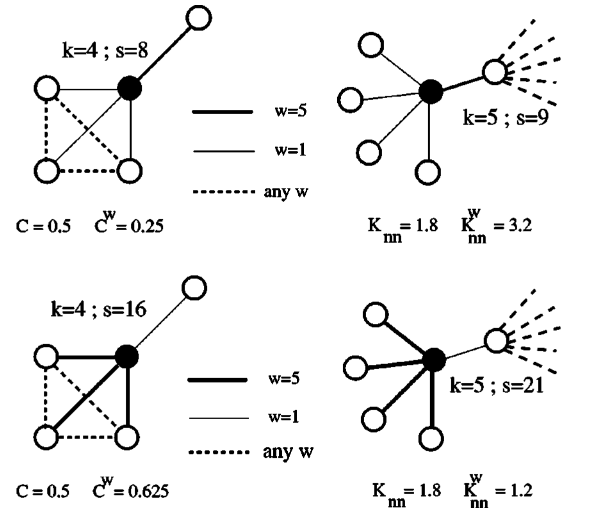

FIG. 1. Examples of local configurations whose topological and weighted quantities are different. Top: In both cases the central vertex (filled) has a very strong link with only one of its neighbors. Bottom: opposite situation. The weighted clustering and the weighted average nearest-neighbors degree (values in the figure correspond to the central vertex) capture more precisely than their topological counterparts the effective level of cohesiveness and affinity due to the actual interaction intensity as measured by the weights.

and correlations measurements to weighted networks have been put forward in [12].

The *weighted clustering coefficient* of a vertex is defined as [12]

$$c_i^w = \frac{1}{s_i(k_i-1)}\sum_{j,h}\frac{(w_{ij}+w_{ih})}{2}a_{ij}a_{ih}a_{jh}. \tag{7}$$

This quantity combines the measure of the existence of triples around vertex i with the intensity of the links emanating from i and participating to these triples. As we show in Fig. 1, c_i^w describes more accurately than c_i the relevance of these triples. The normalization $s_i(k_i-1)$ corresponds to the maximum possible value of the numerator and thus ensures that $c_i^w \in [0,1]$. The average over all sites, or over sites of a given degree k, define, respectively, the global weighted clustering coefficients C^w and $C^w(k)$. For random or uniform weights, these averages coincide with their geometrical counterparts. On the other hand, the comparison between C and C^w [and also between $C(k)$ and $C^w(k)$] conveys information on the repartition of weights. A larger weighted clustering $C^w > C$ signals that links with large weights have a tendency to form triples while the opposite case $C^w < C$ signals a lower relevance of the triangles.

Analogously, high degree vertices could be connected mainly to small degree vertices with a small intensity of connections and to a few large degree vertices with large weights: a topological disassortative character is therefore emerging while in terms of interactions one would conclude to an assortative behavior. In the same spirit as for the clustering, one can therefore define the *weighted average nearest-neighbor degree* as [12]

$$k^w_{nn,i} = \frac{1}{s_i}\sum_{j=1}^{N} w_{ij}k_j. \quad (8)$$

This quantity is the natural generalization of the usual assortativity $k_{nn,i}$ and balances the nearest-neighbor degree with the normalized weight of the connecting edge w_{ij}/s_i. It reduces to $k_{nn,i}$ for uniform or random weights. Comparing $k^w_{nn,i}$ with $k_{nn,i}$ informs us if the larger weights point to the neighbors with larger degree (if $k^w_{nn,i} > k_{nn,i}$) or on the contrary to the ones with smaller degree (see Fig. 1). The behavior of $k^w_{nn}(k)$ thus measures the effective *affinity* to connect with high or low degree neighbors according to the magnitude of the actual interactions.

In the following we will make use of all these quantities in order to provide a thorough characterization of the weighted graphs generated by our model and to assess the relevance of the weights in their structural organization.

III. THE MODEL

Previous models of weighted growing networks [18,19] were considering the growth as driven by the topological features only, with weights statically assigned to the links; i.e., w_{ij} is chosen at the creation of the link *i*-*j* and does not evolve afterwards. This mechanism leads to topological heterogeneities, however, it lacks any dynamical feature of the network's weights. Indeed, it is rather intuitive to consider that the addition of new vertices and links will perturb, at least locally, the existing weights. This phenomenon can be easily understood by considering the example of the airline network: a new airline connection arriving at airport A will generally modify (increase) the traffic activity between airport A and its neighboring airports. Passengers brought by the new connection will eventually get on connection flights, increasing the passenger flow on the other routes. In the Internet as well, it is easy to realize that the introduction of a new connection to a router corresponds to an increase in the traffic handled on the other router's links. Indeed in many technological, large infrastructure and social networks we are generally led to think about a reinforcement of the weights due to the network's growth. In this spirit we consider here a model for a growing weighted network that takes into account the coupled evolution in time of topology and weights [20] and leaves room for accommodating different mechanisms for the reinforcement of interactions.

The definition of the model is based on two coupled mechanisms: the topological growth and the weights' dynamics.

(i) *Growth*. Starting from an initial seed of N_0 vertices connected by links with assigned weight w_0, a new vertex n is added at each time step. This new site is connected to m previously existing vertices (i.e., each new vertex will have initially exactly m edges, all with equal weight w_0), choosing preferentially sites with large strength; i.e., a node i is chosen according to the probability

$$\Pi_{n\to i} = \frac{s_i}{\sum_j s_j}. \quad (9)$$

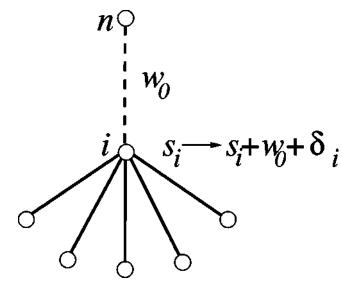

FIG. 2. Illustration of the construction rule. A new node n connects to a node i with probability proportional to $s_i/\Sigma_j s_j$. The weight of the new edge is w_0 and the total weight on the existing edges connected to i is modified by an amount equal to δ_i.

This rule, of *strength driven attachment*, generalizes the usual preferential attachment mechanism driven by the topology, to weighted networks. Here, new vertices connect more likely to vertices which are more central in terms of the strength of interactions.

(ii) *Weights' dynamics*. The weight of each new edge (n,i) is initially set to a given value w_0. The creation of this edge will introduce variations of the traffic across the network. For the sake of simplicity we limit ourselves to the case where the introduction of a new edge on node i will trigger only local rearrangements of weights on the existing neighbors $j \in \mathcal{V}(i)$, according to the rule

$$w_{ij} \to w_{ij} + \Delta w_{ij}, \quad (10)$$

where in general Δw_{ij} depends on the local dynamics and can be a function of different parameters such as the weight w_{ij}, the connectivity or the strength of i, etc. In the following we focus on the case where the addition of a new edge with weight w_0 induces a total increase δ_i of the total outgoing traffic and where this perturbation is proportionally distributed among the edges according to their weights [see Fig. 2]

$$\Delta w_{ij} = \delta_i \frac{w_{ij}}{s_i}. \quad (11)$$

This rule yields a total strength increase for node i of $\delta_i + w_0$, implying that $s_i \to s_i + \delta_i + w_0$. After the weights have been updated, the growth process is iterated by introducing a new vertex, i.e., going back to step (i) until the desired size of the network is reached.

The mechanisms (i) and (ii) have simple physical and realistic interpretations. Equation (9) corresponds to the fact that new sites try to connect to existing vertices with the largest strength. This is a plausible mechanism in many real-world networks. For instance, in the Internet new routers connect to routers that have larger bandwidth and traffic handling capabilities. In the case of the airport's networks, new connections are generally established to airports with a large passenger traffic. In contrast to the connectivity preferential attachment of the "rich get richer" type, the mechanism here relies on the importance of the traffic and could be more adequately described as "busy get busier." At the same time, the weights' dynamics Eqs. (10) and (11) couples the addition of new edges and vertices with the evolution of weight and strength and corresponds to different scenarios according to the value of δ_i.

(1) For $\delta_i < w_0$, the new link does not have a large influence. This may be the case for scientific collaborations where the birth of a new collaboration (co-authorship) is very likely not going to strengthen the activity on previous collaborations.

(2) $\delta_i \approx w_0$ corresponds to situations for which the new created traffic (on the new link *n*-*i*) is transferred onto the already existing connections in a "conservative" way.

(3) $\delta_i > w_0$ is an extreme case in which a new edge generates a sort of multiplicative effect that is bursting the weight or traffic on neighbors.

We have considered only the case of a reinforcement mechanism, i.e., $\delta_i > 0$. In certain particular cases, like, e.g., a network of power distribution, the addition of a new link could in fact *decrease* the traffic of other links. This would correspond to possibly negative δ_i and would imply the definition of more complicated rules to ensure positivity of traffic. We will not consider these cases in this paper, leaving them for future investigations.

The quantity w_0 sets the scale of the weights and we can therefore use the rescaled quantities w_{ij}/w_0, s_i/w_0, and δ_i/w_0, or equivalently set $w_0=1$. The model then depends only on the dimensionless parameter δ_i. The generalization to arbitrary w_0 is simply obtained by replacing δ_i, w_{ij}, and s_i, respectively, by δ_i/w_0, w_{ij}/w_0, and s_i/w_0 in all results.

The model is very general and the properties obtained for the generated networks will strongly depend on the kind of coupling between topology and weights as specified by the parameter δ_i and its variations depending on the vertex' properties. In the following we will provide analytical and numerical inspections of three prototypical situations that can be used as starting points for further generalizations.

IV. HOMOGENEOUS COUPLING

In this section, we will focus on the simplest form of coupling with $\delta_i = \delta = \text{const}$. This case amounts to a very homogeneous system in which all the vertices have an identical coupling between the addition of new edges and the corresponding weights' increase. Such a growing model can be analytically studied through the time evolution of the *average* value of $s_i(t)$ and $k_i(t)$ of the ith vertex at time t, neglecting fluctuations and thus working at a "mean-field" level. In addition, we use numerical simulations in order to provide a direct statistical analysis of the generated graph and substantiate the analytical findings.

A. Evolution of strength and weights

The network growth starts from an initial seed of N_0 nodes and continues with the addition of one node per unit time, until a size N is reached. When a new vertex n is added to the network, an already present vertex i can be affected in two ways: (i) It is chosen with probability (9) to be connected to n; then its connectivity increases by 1, and its strength by $1+\delta$. (ii) One of its neighbors $j \in \mathcal{V}(i)$ is chosen to be connected to n. Then the connectivity of i is not modified but w_{ij} is increased according to the rule Eq. (10), and thus s_i is increased by $\delta w_{ij}/s_j$. This dynamical process modulated by the respective occurrence probabilities $s_i(t)/\Sigma_l s_l(t)$ and $s_j(t)/\Sigma_l s_l(t)$ is thus described by the following evolution equations for s_i and k_i:

$$\frac{ds_i}{dt} = m\frac{s_i(t)}{\sum_l s_l(t)}(1+\delta) + \sum_{j\in\mathcal{V}(i)} m\frac{s_j(t)}{\sum_l s_l(t)}\delta\frac{w_{ij}(t)}{s_j(t)},$$

$$\frac{dk_i}{dt} = m\frac{s_i(t)}{\sum_l s_l(t)}, \tag{12}$$

where we have considered the continuous approximation that treats k, s, and the time t, as continuous variables [1,3]. These equations may be written in a more compact form by noticing that the addition of a node results in the addition of m links obtaining that the total degree at time t is given by $\Sigma_{i=1}^t k_i(t) \approx 2mt$. Similarly, each added link increases the total strength by an amount equal to $2+2\delta$, so that $\Sigma_{i=1}^t s_i(t) \approx 2m(1+\delta)t$. By plugging this result into the equations (12), and integrating the resulting equations with initial conditions $k_i(t=i)=s_i(t=i)=m$, we obtain

$$s_i(t) = m\left(\frac{t}{i}\right)^{(2\delta+1)/(2\delta+2)}, \quad k_i(t) = \frac{s_i(t)+2m\delta}{2\delta+1}. \tag{13}$$

The strength and degree of vertices are thus related by the following expression

$$s_i = (2\delta+1)k_i - 2m\delta \tag{14}$$

that implies for large degree a proportionality between strength and degree. It is worth noticing, however, that this relation indicates the existence of correlations that are not present in the case of randomly assigned weights. Indeed, at each new link created the sum of weights is incremented by $1+\delta$ and therefore $\langle w\rangle = 1+\delta$. As previously mentioned, a random assignment of weights would then lead to $s_i = k_i(1+\delta)$. Equation (14) instead reveals a different proportionality factor signaling correlations between the two quantities. The proportionality relation $s \sim k$ also indicates that the weight-driven dynamics generates in Eq. (9) an effective degree preferential attachment. This model thus displays a microscopic mechanism accounting for the presence of the preferential attachment dynamics in growing networks.

In order to check the analytical predictions we performed numerical simulations of networks generated by using the present model with different values of δ, minimum degree m and varying network size N. The analytically predicted behavior is recovered (see [20]).

The time evolution of the weights w_{ij} can also be computed analytically along the lines used for the study of $s_i(t)$ and $k_i(t)$. Indeed, w_{ij} evolves each time a new node connects to either i or j and the corresponding evolution equation can be written as

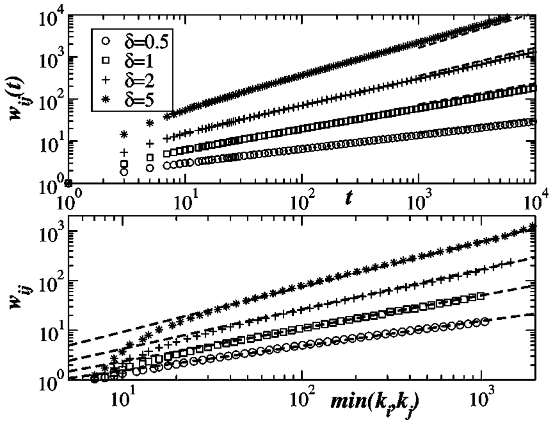

FIG. 3. Top: Time evolution of w_{ij} during the growth of the network, for different values of δ. The functional behavior is consistent with the predicted power law t^b, $b=\delta/(1+\delta)$, shown as dashed lines. Data are averaged over 200 networks with $m=2$ and $N=10^4$. Bottom: w_{ij} vs $\min(k_i,k_j)$; the dashed lines are the theoretical predictions ($m=8$, $N=10^4$, data averaged over 1000 samples).

$$\frac{dw_{ij}}{dt}=m\frac{s_i}{\sum_l s_l}\delta\frac{w_{ij}}{s_i}+m\frac{s_j}{\sum_l s_l}\delta\frac{w_{ij}}{s_j}=\frac{\delta}{(1+\delta)}\frac{w_{ij}}{t}. \quad (15)$$

The link (i,j) is created at $t_{ij}=\max(i,j)$ with initial condition $w_{ij}(t_{ij})=1$, so that

$$w_{ij}(t)=\left(\frac{t}{t_{ij}}\right)^{\delta/(\delta+1)}. \quad (16)$$

At fixed time t, this result implies that $w_{ij}(t)\sim\min(i,j)^{\delta/(\delta+1)}$, and since $k_i(t)\sim i^{-(2\delta+1)/(2\delta+2)}$, we obtain

$$w_{ij}\sim\min(k_i,k_j)^{2\delta/(2\delta+1)}. \quad (17)$$

In this case also, the numerical simulations of the model reproduce the behaviors predicted by the analytical calculations. The time evolution of some randomly chosen weights is displayed in Fig. 3 and compared with the prediction of Eq. (16). Figure 3 also show the validity of Eq. (17) for the correlation between w_{ij} and the connectivities of i and j.

B. Probability distributions

The knowledge of the time evolution of the various quantities allows us to compute their statistical properties. Indeed, the time $t_i=i$ at which the node i enters the network is uniformly distributed in $[0,t]$ and the degree probability distribution can be written as

$$P(k,t)=\frac{1}{t+N_0}\int_0^t\delta(k-k_i(t))dt_i, \quad (18)$$

where $\delta(x)$ is the Dirac delta function. Using equation $k_i(t)\sim(t/i)^a$ obtained from Eq. (13) one obtains in the infinite size limit $t\to\infty$ the distribution $P(k)\sim k^{-\gamma}$ with $\gamma=1+1/a$:

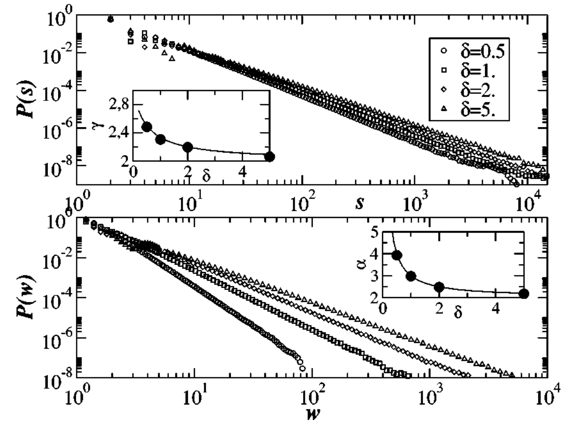

FIG. 4. Top: Probability distribution $P(s)$. Data are consistent with a power-law behavior $s^{-\gamma}$. Bottom: Probability distribution of the weights $P(w)\sim w^{-\alpha}$. In the insets we report the value of α obtained by data fitting (filled circles) and the analytic expressions $\gamma=(3+4\delta)/(1+2\delta)$ and $\alpha=2+1/\delta$ (solid lines). The data are averaged over 200 networks of size $N=10^5$.

$$\gamma=\frac{4\delta+3}{2\delta+1}. \quad (19)$$

Since s and k are proportional, the same behavior $P(s)\sim s^{-\gamma}$ is obtained for the strength distribution. $P(s)$ is displayed in Fig. 4, showing that the obtained graph is a scale-free network both for topology and strength and described by an exponent $\gamma\in[2,3]$ that depends on the value of the parameter δ. As expected, if the addition of a new edge does not affect the existing weights ($\delta=0$), we recover the Barabási-Albert (BA) model [6] with the value $\gamma=3$. As δ increases, the distributions get broader with $\gamma\to 2$ when $\delta\to\infty$, i.e., in a range of values usually observed in the empirical analysis of networked structures [1,3,4]. This result could be an explanation of the lack in real-world networks of any universality of the degree distribution exponent. Our model indeed predicts that all the exponents will be nonuniversal and depend on the local processes which take place at nodes receiving new links.

The weight distribution $P(w)$ can be analogously calculated yielding the power-law behavior

$$P(w)\sim w^{-\alpha}, \quad \alpha=2+\frac{1}{\delta}. \quad (20)$$

The distribution $P(w)$ therefore is even more sensitive to the parameter δ and evolves from a delta function for $\delta=0$ (no evolution of the weights) to a very broad power law as $\delta\to\infty$. The value $\delta=1$ corresponds to $\alpha=3$, i.e., to the boundary between finite and unbounded fluctuations of the weights. In Fig. 4, the weight distributions obtained from numerical simulations at different values of the parameter δ are reported along with the comparison between the values of the measured exponent α and the analytical prediction. The precise microscopic dynamics ruling the network's growth and the rearrangement of weights is therefore very relevant to the final distribution of weights, even if it affects only in a much milder way degree and strength.

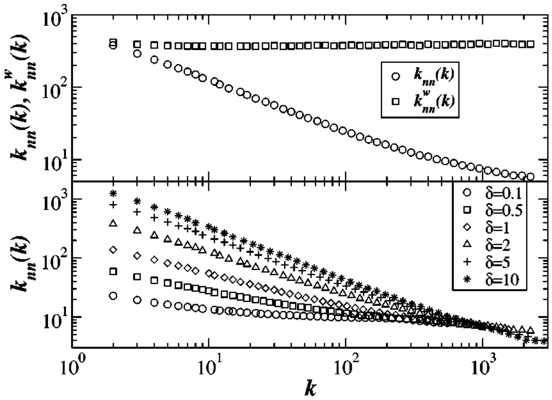

FIG. 5. $m=2$; top: $k_{nn}(k)$ and $k_{nn}^w(k)$ for $\delta=2$; bottom: $m=2$; evolution of $k_{nn}(k)$ for increasing δ.

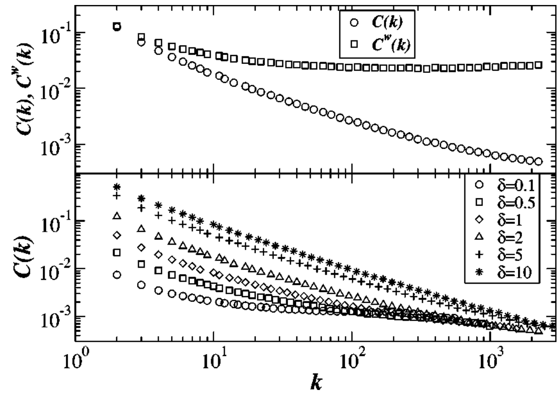

FIG. 6. $m=2$; top: $C(k)$, $C^w(k)$ for $\delta=2$; bottom: $C(k)$ for various δ.

C. Correlations and clustering

The previous analytical study of the model does not provide information on the correlations generated by the growing process. In order to have a direct inspection of these properties we therefore consider the graphs generated by the model for different values of δ, m and N and measure the quantities defined in Sec. II that characterize the clustering and correlation properties.

The model exhibits also in this case properties which are depending on the basic parameter δ. More precisely, for small δ, the average nearest-neighbor degree $k_{nn}(k)$ is quite flat as in the BA model. The disassortative character emerges as δ increases and gives rise to a power-law behavior of $k_{nn}(k)$ as shown in Fig. 5 [31]. Remarkably, the weighted average nearest-neighbor degree displays for any δ a flat behavior. We note that, in some recently studied weighted real-world networks [12], topological and weighted k_{nn} show a clear assortative behavior, opposite to what we find in the present model. In other cases such as the Internet and the World Wide Web, a disassortative behavior is instead found, consistently with our results. Assortative behavior may be originated in the community structure and geographical aspects intrinsic to the SCN and WAN, which are not considered in our model (inclusion of geography, i.e., space, in the model is considered in [32]). Remarkably, however, our model reproduces the important feature that the weighted assortativity $k_{nn}^w(k)$ is significantly larger than $k_{nn}(k)$, as in real networks (see Fig. 5): the larger weights contribute to the links towards vertices with larger connectivities.

Analogous properties are obtained for the clustering spectrum. At small δ, the clustering coefficient of the network is small and $C(k)$ is flat. As δ increases, however, the global clustering coefficient increases and $C(k)$ becomes a decreasing power law similar to real networks data [33]. Figure 6 clearly shows that the increase in clustering is determined by small k vertices. The weighted clustering C^w also increases and is larger than the topological C, with an essentially flat $C^w(k)$. Especially at large k, it is clear that the usual clustering coefficient underestimates the importance of triples in the network since, for the hubs, the edges with the highest weights belong in great part to the interconnected triples. In other words, interconnected vertices are joined by edges that have weights larger than the average value found in the network.

Interestingly, correlations and clustering spectrum can be qualitatively understood by considering the dynamical process leading to the formation of the network. Indeed, vertices with large connectivities and strengths are the ones that entered the system at the early times as shown by Eq. (13). Newly arriving vertices attach to preexisting vertices with large strength which on their turn are reinforced by the rearrangement of the weights. This process naturally builds up a hierarchy among the nodes: "old" vertices have neighbors that are more likely to be "young," i.e., with small connectivity. On the contrary, newly arriving vertices have neighbors with high degree and strength, generally leading to a disassortative behavior. This effect gets stronger as δ increases and $P(k)$ broadens leading to disassortative properties. As well, edges among "old" vertices are the ones that gets more reinforced by the weights dynamics indicating that the edges between older nodes, with large connectivities, will be typically stronger than the average. This means that the weighted assortativity will be larger than the topological assortativity, leading to $k_{nn}^w(k) > k_{nn}(k)$, especially for large k.

Similarly, the increase of C with δ is also directly related to the mechanism which rearranges the weights after the addition of a new edge. Since vertices with large strength and degree are generally connected among them, a new vertex has more probability to attach to the extremities of a given edge. Triangles will typically be made of two "old" nodes and a "young" one. Therefore $C(k)$ increases faster for "younger" (low degree) nodes when δ increases generating the observed spectrum. On the other hand, the edges between "old" and "young" vertices are the most recent ones and do not have large weights. This feature implies that for low degree vertices c_i and c_i^w are rather close. In contrast, high degree vertices are connected to each other by edges with large weights, leading to a weighted clustering coefficient larger than the topological one.

These qualitative arguments confirm the importance of considering weighted correlations since topological correlations do not fully reveal the intrinsic coupling between topology and weights that may lead to very different behavior of the correlation and clustering spectrum.

V. HETEROGENEOUS COUPLING

In the model described in the previous section as well as in most models of growing networks, connectivities and strengths of different vertices grow with the same exponent [see Eq. (13)]. Therefore vertices entering the system at the early times have always the largest connectivities and strengths. One can, however, imagine that a newly arriving vertex has intrinsic properties which make it more attractive than older ones so that its connectivity and strength could grow faster than its predecessors. This feature is certainly very important in many real systems where individuals are not identical and has been put forward in the so-called fitness model [34] (see also [35] for a static definition of a fitness model). In a very similar spirit, we introduce here a node-dependent δ_i implying that the perturbation of weights created by any new edge attached to the node i will depend on the very local properties of i. This amounts to introducing a general heterogeneity in the dynamical properties of the elements of the system.

In this case, each vertex entering the system is tagged with its own δ_i that we will assume are independent random variables taken from a given distribution $\rho(\delta)$ characterizing the system's heterogeneity. The preferential attachment [Eq. (9)] is not modified and the redistribution of weights now reads

$$\Delta w_{ij} = \delta_i \frac{w_{ij}}{s_i}. \tag{21}$$

A large value of δ_i does not favor immediately the attractiveness of the vertex but the addition of a new link to i modifies its total strength by a large amount,

$$s_i \to s_i + w_0 + \delta_i. \tag{22}$$

In the long run, larger δ_i yield therefore larger increases Δs_i when i is chosen for the addition of a new edge. Since the model's dynamics is driven by a strength driven attachment the vertices with larger δ_i will be progressively favored in the establishment of new connections, therefore achieving a faster degree and strength growth as time goes by.

Similarly to the homogeneous model of the previous section, the evolution equations of s_i and k_i can be written as

$$\frac{ds_i}{dt} = m\frac{s_i(t)}{\sum_l s_l(t)}(1+\delta_i) + \sum_{j\in\mathcal{V}(i)} m\frac{s_j(t)}{\sum_l s_l(t)}\delta_j\frac{w_{ij}(t)}{s_j(t)},$$

$$\frac{dk_i}{dt} = m\frac{s_i(t)}{\sum_l s_l(t)}, \tag{23}$$

where now the explicit dependence from the specific δ_i of each vertex is properly considered. For each newly added edge from the vertex n to the existing vertex i, $\Sigma_j s_j$ is increased by $2+2\delta_i$. On the average it is therefore natural to consider that the total strength is increasing linearly with time as

$$\sum_{i=1}^{t} s_i(t) \simeq 2m(1+\delta')t. \tag{24}$$

We assume that δ' is a well-defined constant, which is certainly the case if $\rho(\delta)$ is bounded. It is worth remarking that $\delta' \neq \langle\delta\rangle$ since during the growth process, vertices with larger δ_i will be preferentially chosen. The quantity δ' will be determined self-consistently by the general solution.

We use Eq. (24) in the evolution Eqs. (23) and since $\Sigma_{j\in\mathcal{V}(i)} w_{ij}\delta_j$ is a sum over the neighbors of *i that are chosen by the strength driven dynamics*, we assume that $\delta_j \simeq \delta'$ and therefore $\Sigma_{j\in\mathcal{V}(i)} w_{ij}\delta_j \simeq \delta' s_i$. We then obtain

$$\frac{ds_i}{dt} = \frac{\delta_i+\delta'+1}{2\delta'+2}\frac{s_i(t)}{t},$$

$$\frac{dk_i}{dt} = \frac{s_i(t)}{2(1+\delta')t}. \tag{25}$$

The integration of these equations with the initial conditions $k_i(t=i)=s_i(t=i)=m$ yields

$$s_i(t) = m\left(\frac{t}{i}\right)^{a_i}, \tag{26}$$

$$k_i(t) = \frac{s_i(t)+m(\delta_i+\delta')}{\delta_i+\delta'+1}. \tag{27}$$

The strength and degree of the vertices therefore grow as power laws, with an exponent that depends on their ability to redistribute weights:

$$a_i = \frac{1+\delta_i+\delta'}{2(1+\delta')}. \tag{28}$$

Nodes arriving later but having large δ_i will thus grow faster and overcome older nodes with smaller δ values. Equation (27) also shows that also in the heterogeneous model the strength and connectivity are proportional for large degree, with a coefficient depending on δ_i,

$$s_i = k_i(1+\delta_i+\delta') - m(\delta_i+\delta'). \tag{29}$$

The knowledge of the time behavior of $s_i(t)$ makes it possible to obtain the probability distribution of connectivities and strengths. For example, the distribution of strengths can be written as

$$P_s(s,t) = \int d\delta\rho(\delta)\frac{1}{t+N_0}\int_0^t \delta(s-s_i(t))dt_i, \tag{30}$$

where $\delta(s-s_i(t))$ is the Dirac delta function (not to be confused with the heterogeneity parameter). Since $s_i(t) \propto (t/t_i)^{a(\delta)}$ we obtain

$$P(s) \propto \int d\delta\frac{\rho(\delta)}{a(\delta)}\left(\frac{1}{s}\right)^{1/a(\delta)+1}, \tag{31}$$

which shows that the precise form of $P(s)$ depends on $\rho(\delta)$. Finally, the proportionality of k_i and s_i ensures that their distributions have the same form.

Along the same lines we can obtain the dynamical evolution of the weights. Indeed, w_{ij} grows each time a new node connects to either i or j, its evolution equation being

$$\frac{dw_{ij}}{dt} = m\frac{s_i}{\sum_l s_l}\delta_i\frac{w_{ij}}{s_i} + m\frac{s_j}{\sum_l s_l}\delta_j\frac{w_{ij}}{s_j} = \frac{\delta_i + \delta_j}{2(1+\delta')}\frac{w_{ij}}{t}. \tag{32}$$

This readily implies that $w_{ij}(t) \propto t^{b_{ij}}$ with

$$b_{ij} = \frac{\delta_i + \delta_j}{2(1+\delta')}. \tag{33}$$

The behavior of the model depends explicitly on the value δ' that has to be self-consistently determined. The consistency of the solution is obtained using Eqs. (26)–(28) which give

$$\sum_{i=1}^{t} s_i(t) \approx \int d\delta\rho(\delta)\int_1^t dt_0 m\left(\frac{t}{t_0}\right)^{a(\delta)} = m\int d\delta\rho(\delta)\frac{t - t^{a(\delta)}}{1 - a(\delta)}, \tag{34}$$

with $a(\delta)=(1+\delta+\delta')/[2(1+\delta')]$. Since s_i cannot grow faster than t, $a(\delta)$ has to be less than 1 and using Eq. (24), we obtain from Eq. (34) in the large time limit

$$2m(1+\delta') = m\int \frac{d\delta\rho(\delta)}{1 - \dfrac{1+\delta+\delta'}{2(1+\delta')}} \tag{35}$$

or

$$\int \frac{d\delta\rho(\delta)}{1+\delta'-\delta} = 1, \tag{36}$$

which determines the value of δ'. Finally, we note that these results are valid only if the quantity δ_i is bounded: if it is not the case, the basic assumption that $\Sigma_i s_i$ grows linearly is no longer true [34].

It is clear from the above solution that the graph's properties will depend upon the particular form of the coupling distribution $\rho(\delta)$. In order to compare with numerical simulations of the model we analyze the specific case of a uniform distribution $\rho(\delta)$ in the interval between δ_{min} and δ_{max}. The equation for δ' can be explicitly solved, obtaining

$$\delta' = \frac{(\delta_{\max}-1)\exp(\delta_{\max}-\delta_{\min}) + 1 - \delta_{\min}}{\exp(\delta_{\max}-\delta_{\min}) - 1}. \tag{37}$$

We are therefore in the position to provide an explicit value of the exponent $a(\delta)$ for the evolution of s and k during the growth of the network. Similarly, the strength probability distribution can be written as

$$P(s) \propto \int_{a_{\min}}^{a_{\max}} \frac{da}{a}\left(\frac{m}{s}\right)^{1+1/a}, \tag{38}$$

whose behavior at large s is

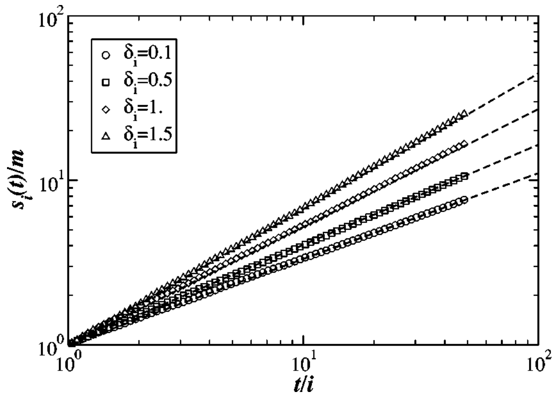

FIG. 7. Model with random δ_i: evolution of s_i/m for a given node i with various values of its redistribution parameter δ_i. $m=2$, $N=10^4$, average over 1000 networks with the same realization of $\delta_j \in [0;2]$ $(j \neq i)$. The dashed lines correspond to the predicted power laws $(t/i)^{a(\delta_i)}$.

$$P(s) \propto \frac{1}{s^{1+1/a_{\max}}\log(s)}, \tag{39}$$

where $a_{max}=(1+\delta_{max}+\delta')/[2(1+\delta')]$ is the largest possible value of the exponent $a(\delta)$.

A test of the analytical results is obtained by the direct inspection of networks obtained by numerical simulations of the model in the case of heterogenous coupling. In particular we consider networks generated by uniform distribution of the coupling constants in specific intervals as reported in the figure captions. The striking agreement of the analytical predictions [Eqs. (26), (28), and (37)] with the numerical results is shown in Figs. 7 and 8. In particular, various curves cross in Fig. 8 since nodes arriving later may have a larger ability to redistribute weights and thus overcome older nodes in the long run. It is worth noticing the remarkable agreement shown in these figures, obtained without any free parameter. The statistical distributions of the quantities of interest are also in good agreement with the

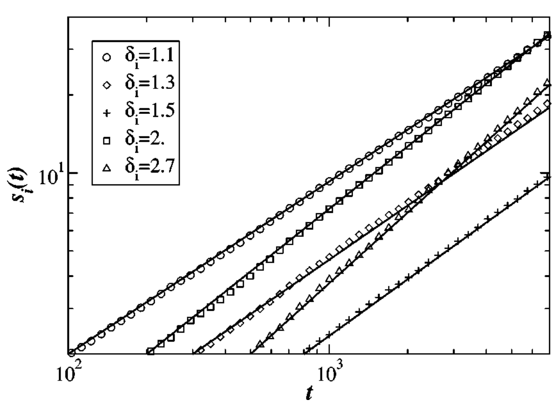

FIG. 8. Model with random δ_i: evolution of s_i for various nodes $(i=100,200,300,500,800)$. $m=2$, $N=10^4$. Symbols: results of numerical simulations, average over 1000 networks with the same realization of $\delta_j \in [1;3]$. Nodes arriving later can overcome older nodes. Lines are the prediction (26) with the exponent given by Eqs. (28) and (37).

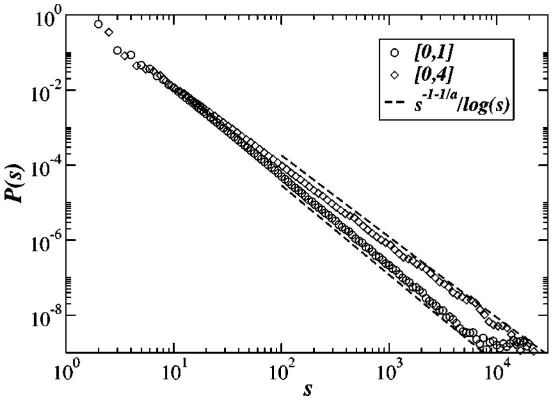

FIG. 9. Model with random δ_i: distribution of strengths $P(s)$ for uniform distributions $\rho(\delta)$, $\delta_j \in [0;1]$ and $\delta_j \in [0;4]$. Dashed lines are the theoretical predictions (39) $s^{-1-1/a_{\max}}/\log(s)$. $N=10^5$, $m=2$, average over 1000 realizations.

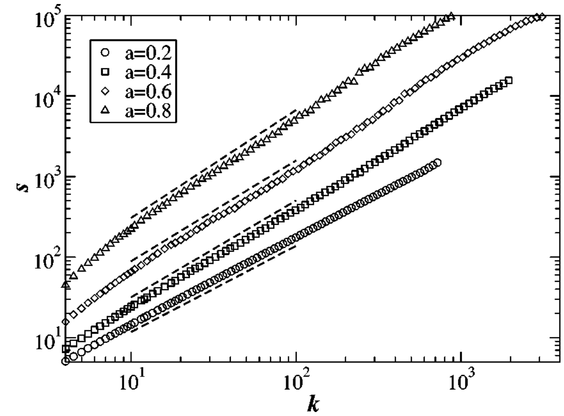

FIG. 11. $N=5000$; $s_0=10^4$. s vs k for $a=0.2, 0.4, 0.6, 0.8$. Dashed lines have slope 1.07, 1.21, 1.25, and 1.34 (from bottom to top). For values of s larger than s_0 there is a crossover towards $\beta=1$.

analytical prediction as shown in Fig. 9 for the strength distribution. Finally, as shown in Fig. 10, the correlation and clustering properties are nontrivial as well. As in the case of the homogeneous coupling the average nearest-neighbors degree and the clustering coefficient have a clear structure with a hierarchical ordering of high and low degree vertices (we only display k_{nn} and k_{nn}^w in Fig. 10 since the clustering properties are very similar to the ones displayed in Fig. 6). Also in this situation the inclusion of weights in the characterization of correlations provide additional information on the structure of the network. While more cumbersome because of the heterogeneous nature of the coupling, a general understanding of the observed properties can be obtained along the reasoning reported for the homogeneous coupling model; in all cases, the dynamical growth process itself is at the origin of the complex architecture and structure of the generated networks.

VI. NONLINEAR COUPLING

In the previous sections we considered the coupling term δ_i as independent of the topological and weight properties of the vertex i. We can, however, think of different situations in which the perturbation depends on the centrality of the node as measured by its strength or degree. In the airline network, for example, this might mimic the fact that the airport is larger and the increase of traffic is larger with a much larger response to the creation of a new connection compared to a smaller airport. It is thus natural to investigate the consequence of nonlinear coupling forms.

The simplest nonlinear coupling consists in considering that δ_i is proportional to s_i. In order to avoid unrealistic divergences a cutoff s_0 is, however, needed to bound the coupling, leading to the reinforcement rule

$$\Delta w_{ij} = \delta \frac{w_{ij}}{s_i}\left[s_0 \tanh\left(\frac{s_i}{s_0}\right)\right]^a. \tag{40}$$

This relation simply expresses that the larger the traffic on the vertex i, the larger will be the traffic attracted to it during the weights' dynamics. The total change in s_i when a link is added is now $\delta[s_0 \tanh(s_i/s_0)]^a$. For strength smaller than the

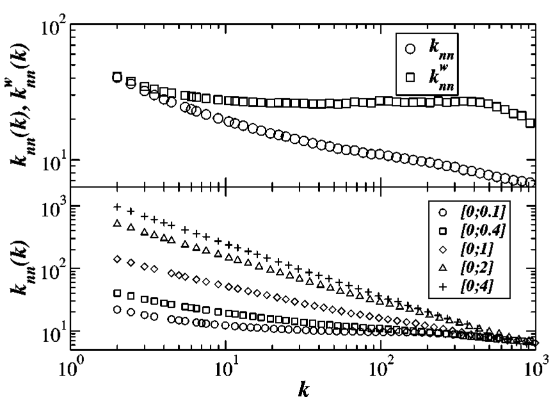

FIG. 10. $m=2$; top: $k_{nn}(k)$ and $k_{nn}^w(k)$ for a uniform distribution $\rho(\delta)$ with $\delta_j \in [0;0.4]$; bottom: $k_{nn}(k)$ for uniform distributions $\rho(\delta)$ in various intervals $[\delta_{\min}, \delta_{\max}]$.

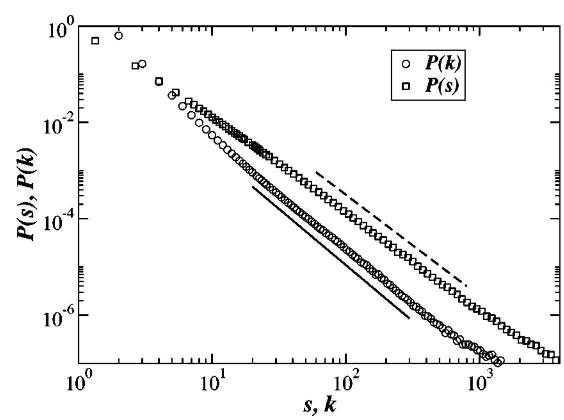

FIG. 12. $N=5000$; $s_0=10^4$. $P(s)$ and $P(k)$ for $a=0.6$, $\delta=0.1$. The data for $P(k)$ have been shifted vertically for clarity. Continuous and dashed lines correspond to the power laws $k^{-2.33}$ and $s^{-2.1}$, respectively.

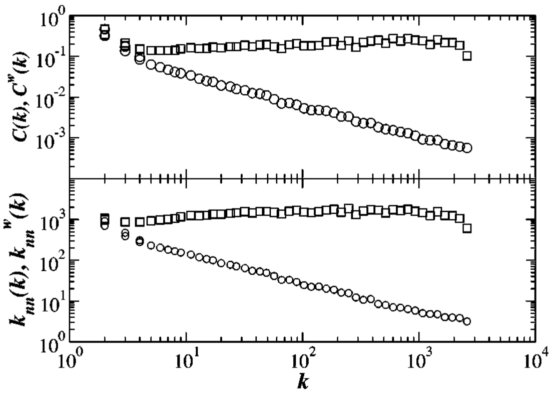

FIG. 13. $N=5000$; $s_0=10^4$. Correlations and clustering for $a=0.6$, $\delta=0.1$. Unweighted quantities [$k_{nn}(k)$ and $C(k)$] are shown by circles, weighted quantities [$k_{nn}^w(k)$ and $C^w(k)$] by squares.

cutoff s_0, this change grows as s_i^a and saturates to δs_0^a for very large strengths.

The nonlinear mechanism makes the analytical solution very difficult to find and we rely on a numerical study of the model to inspect its topological and weight properties. We find that $\Sigma s_i(t)$ now grows faster than t, seemingly like $\exp(t^b)$. Analogously, we observe that vertices' degree and the strength grow faster than simple power laws. Very interestingly, we observe that the strength grows as a power law with the connectivity, $s \sim k^\beta$ with $\beta > 1$, as shown from the numerical simulations reported in Fig. 11. The exponent β increases with a but it is independent from δ. This result raises the possibility that some real-world networks, in which a value $\beta > 1$ is observed, are governed by local nonlinear reinforcement processes. This could be the case for the airport network where $\beta \simeq 1.5$ is observed [12].

An interesting consequence is then observed for the degree and strength probability distribution. While both distributions still behave as power laws, $P(k) \sim k^{-\gamma_k}$ and $P(s) \sim s^{-\gamma_s}$, as shown in Fig. 12, they exhibit different exponents γ_s and $\gamma_k > \gamma_s$ in contrast with all the situations considered previously where we found $\gamma_s = \gamma_k$. This is obviously linked to the fact that $\beta \neq 1$ and it is not difficult to show that

$$\gamma_s = \frac{\gamma_k}{\beta} + \frac{\beta - 1}{\beta}. \tag{41}$$

The weight distribution $P(w)$ is also power-law distributed and in addition, all distributions get broader as either a or δ are increased. Finally, we note that the correlations and clustering properties exhibit also in this case a nontrivial spectrum as a function of the degree k, very similar to the case of linear coupling, as shown in Fig. 13, signaling the presence of a hierarchical architecture also in the presence of a nonlinear coupling.

It is clear that the results obtained for nonlinear coupling mechanisms are depending upon the detailed form of the coupling and further studies are needed in order to understand the variations of time behavior and distribution exponents as a function of the various parameters defining the reinforcement dynamics. Obviously, a detailed study of all nonlinear coupling mechanisms is impossible and each modeling effort must be driven by specific insights on the dynamics of the real systems under examination.

VII. CONCLUSIONS

In this paper we have presented a general model of growing networks that considers the effect of the coupling between topology and weights' dynamics. We investigated in details several coupling mechanisms including the effect of randomness and nonlinearity in the redistribution process.

The model produces graphs which display nontrivial complex and scale-free behavior that depend on the detailed coupling form. In particular, different quantities such as strength, degree, and weights are distributed according to power laws with exponents which are not universal and depend on the specific parameters that control the local microscopic weights' dynamics. This result hints to a simple explanation of the lack of any universality observed in real-world networks. In addition, the dynamics generates spontaneously a nontrivial architecture in which nodes with different degrees are arranged in a hierarchical way as indicated by the clustering and correlation properties measured in the obtained networks.

While many other parameters and dynamical features may be entering the dynamics of real-world networks, we believe that the present model might provide a general starting point for the realistic modeling of several systems where the interplay of topology and traffic is a key point in the determination of the global network's properties.

ACKNOWLEDGMENTS

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