

Quasiperiodic localized oscillating solutions in the discrete nonlinear Schrödinger equation with alternating on-site potential

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We present an example of an exact quasiperiodic localized stable solution with spatially symmetric large-amplitude oscillations in a nonintegrable Hamiltonian lattice model. The model is a one-dimensional discrete nonlinear Schrödinger equation with alternating on-site energies, modeling, e.g., an array of optical waveguides with alternating widths. The solution bifurcates from a stationary discrete gap soliton, and in a regime of large oscillations its intensity oscillates periodically between having one peak at the central site and two symmetric peaks at the neighboring sites with a dip in the middle. Such solutions, termed “pulsons,” are found to exist in continuous families ranging arbitrarily close to both the anticontinuous and continuous limits. Furthermore, it is shown that they may be linearly stable also in a regime of large oscillations.

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The discrete nonlinear Schrödinger (DNLS) equation is one of the most studied examples of a nonintegrable Hamiltonian lattice model. It is of great interest as much from a general nonlinear dynamics point of view, where it provides a particularly simple system to analyze fundamental phenomena arising from competition of nonlinearity and discreteness such as energy localization, wave instabilities, etc., as from a more applied viewpoint describing, e.g., arrays of nonlinear optical waveguides [1] or Bose-Einstein condensates in external periodic potentials [2]. For recent reviews of the history, properties, and applications of DNLS-like models, see Refs. [3,4].

Recent attention has been given to DNLS-like models having, in addition to the fundamental periodicity given by the lattice constant, also a superlattice modulation creating a second period. In particular, the DNLS equation with an additional term corresponding to alternating on-site energies has been proposed to model an optical waveguide array where the individual waveguides have alternating widths [5]. Creating a two-branch linear dispersion relation, the superperiodicity thus provides the possibility for existence of a type of nonlinear localized modes called *discrete gap solitons* (or *discrete gap breathers*), with frequencies in the gap between the upper and lower branches. These modes appear as *stationary* solutions to the DNLS equation (i.e., with constant intensity and a purely harmonic time dependence which can be removed by transforming into a rotating frame), and their properties were recently analyzed in detail in Ref. [6], to which we also refer for further references on the topic. Very recently, they have also been experimentally observed [7].

However, it is known that as the DNLS equation in one aspect is nongeneric among nonlinear Hamiltonian lattice models, with a second conserved quantity being the total norm of the excitation, there exist also localized *quasiperiodic* solutions which may have two (generally) incommen-

surate frequencies. Here the first frequency corresponds to harmonic oscillations at constant intensity as for stationary solutions, while the second frequency corresponds to time-periodic *oscillations of the intensity* in the frame rotating with the first frequency. The existence of such solutions was proposed in Ref. [8], and later explicit examples were constructed and analyzed by continuation of multisite breathers from the “*anticontinuous*” limit of zero intersite coupling [9] (note that similar ideas were used already in Ref. [10] for finite systems). A rigorous approach to the connection between existence of quasiperiodic solutions and additional conserved quantities was given in Ref. [11]. A slightly different approach was taken in Ref. [12] (see also [3]), where the existence of exact quasiperiodic solutions bifurcating from localized eigenmodes of the *linearized* equations of motion around some particular stationary solutions was shown. In all cases, to guarantee localization it is necessary that a *nonresonance* condition is satisfied, so that no higher harmonics enter the continuous linear spectrum.

However, all known examples of stable localized exact quasiperiodic solutions to the ordinary DNLS equation could be considered as rather special, since (i) they only exist in bounded parameter intervals at weak intersite coupling, and (ii) the intensity oscillations are typically quite small compared to their average values. Furthermore, to the best of our knowledge no explicit example of a stable quasiperiodic breather with *spatial symmetry* has been given (although one of the modes presented in Ref. [12] possibly could yield such a solution). On the other hand, it was shown already in Ref. [13] that for the two-dimensional DNLS model a state with large-amplitude symmetric intensity oscillations, with the intensity maximum periodically oscillating between the central site and four surrounding sites, was created in an intermediate stage in the process of “quasicollapse” of a broad excitation to a highly localized breather. This entity, termed a *pulson*, typically disappeared after 3–4 oscillations, transforming into an on-site localized stationary breather with slowly decaying small-amplitude internal mode oscillations (the decay of which can be calculated similarly as for the one-dimensional case in Ref. [14]).

It is the purpose of the present Brief Report to provide an

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example of an *exact* stable solution with such pulson properties, which we have found to exist in the one-dimensional binary modulated DNLS equation. In fact, we remarked already in Ref. [6] that similar arguments as in Ref. [12] prove the existence of exact quasiperiodic solutions bifurcating from certain internal modes of the stationary gap breathers; here the crucial property (which is not satisfied, e.g., for single-site breathers in the nonmodulated DNLS model [14]) is that these internal modes have frequencies *above* the continuous spectrum, and consequently also all higher harmonics will lie outside the continuous spectrum. Referring to the notation in Fig. 3 of Ref. [6], we here concentrate on the spatially symmetric mode denoted as P1, which may exist arbitrarily close to the anticontinuous as well as the continuous limit, and show that the families of quasiperiodic solutions bifurcating from this mode indeed exhibit pulson properties as the oscillation amplitude increases.

Using for convenience a slightly different notation than in Ref. [6], we consider the DNLS equation in the form

$$i\dot{\psi}_n - V_0(-1)^n\psi_n + C(\psi_{n+1} + \psi_{n-1}) + |\psi_n|^2\psi_n = 0, \quad (1)$$

with the two conserved quantities Hamiltonian

$$H = \sum_n \left[V_0(-1)^n |\psi_n|^2 - C(\psi_n\psi_{n+1}^* + \psi_n^*\psi_{n+1}) - \frac{1}{2}|\psi_n|^4 \right]$$

and norm $\mathcal{N} = \sum_n |\psi_n|^2$. Writing a stationary solution as $\psi_n(t) = \phi_n^{(\Lambda)} e^{i\Lambda t}$ with time-independent $\phi_n^{(\Lambda)}$, the linear dispersion relation becomes $\Lambda = \pm \sqrt{V_0^2 + 4C^2 \cos^2 q}$ with gap $\Lambda \in [-|V_0|, |V_0|]$. Assuming $V_0 > 0$, a discrete gap soliton then bifurcates from the *lower* gap edge $\Lambda = -V_0$, and corresponds in the limit $C \rightarrow 0$ to a single excited *even* site with intensity $|\psi_n|^2 = \Lambda + V_0$.

Exact quasiperiodic solutions with two independent frequencies ω_0 and ω_b can then be found numerically to computer precision by using essentially the same method as in Ref. [9]. First, we transform into a frame rotating with the frequency ω_0 , $\psi_n(t) = \phi_n(t) e^{i\omega_0 t}$, yielding

$$i\dot{\phi}_n - [\omega_0 + V_0(-1)^n]\phi_n + C(\phi_{n+1} + \phi_{n-1}) + |\phi_n|^2\phi_n = 0, \quad (2)$$

whereupon standard Newton-type schemes are used to find time-periodic solutions to Eq. (2) satisfying $\phi_n(t + 2\pi/\omega_b) = \phi_n(t)$, where ω_b is the second frequency. In addition, linear stability is simply investigated numerically by standard Floquet analysis as described in Ref. [9].

When a stationary gap breather of frequency Λ has an internal linear eigenmode of frequency ω_l (in the frame rotating with frequency Λ), the family of quasiperiodic solutions bifurcating from it at this point has $\omega_b = \omega_l$ and $\omega_0 = \Lambda - \omega_l$. Thus, by using as initial trial solution the stationary gap breather perturbed with the relevant linear eigenmode, the two-parameter family of solutions (for given V_0 and C) can then be followed through parameter space by standard continuation techniques. (Note that, in contrast to Ref. [9], once the stationary gap breather is known we here obtain directly quasiperiodic solutions at finite C , without invoking the anticontinuous limit $C \rightarrow 0$.) For a small enough continuation step ($\leq 10^{-3}$), we typically with standard double-precision FORTRAN routines for numerical integration and matrix inver-

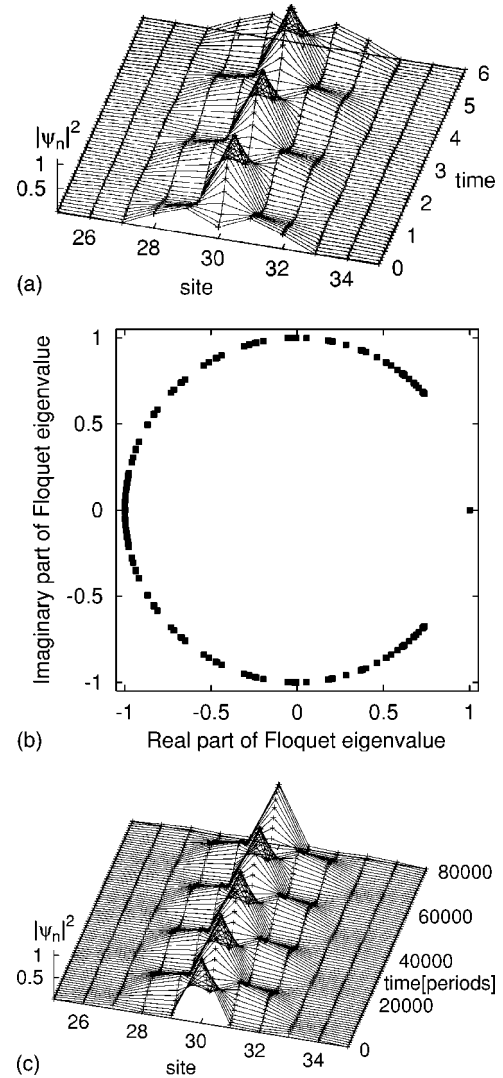


FIG. 1. (a) The first three oscillations of an exact localized pulson solution to Eq. (1) with $V_0=1$, $C=1$. The dynamical parameters are $\omega_b=3.18$, $\omega_0=-3.722$, $\mathcal{N} \approx 1.90867$. Only the central part is shown; the solution decays exponentially outside this region. (b) Linear stability eigenvalues for the solution in (a). (c) Snapshots, at integer multiples of the original period $2\pi/\omega_b$, from long-time integration using the solution in (a), with a random perturbation $\sim 10^{-3}$, as initial condition.

sion obtain convergence to solutions with $\|\phi_n(t + 2\pi/\omega_b) - \phi_n(t)\| < 10^{-12}$ in five or fewer Newton iterations. In fact, since norm is a conserved quantity, it is generally more instructive to use \mathcal{N} and ω_b as independent parameters, and practically continuations versus ω_b at constant norm may be performed by varying ω_0 to keep \mathcal{N} constant. It is also convenient to work with rescaled quantities amounting to setting $C=1$ in Eqs. (1) and (2); consequently the three relevant independent parameters for the two-frequency solutions can be chosen as V_0/C , \mathcal{N}/C (or, alternatively, ω_0/C), and ω_b/C .

Figure 1 illustrates the dynamics of a typical exact exponentially localized pulson solution, obtained by continuation (versus ω_b/C at constant ω_0/C and V_0/C) of the linear P1 mode of a stationary gap breather with $\Lambda = -0.5V_0$ (i.e., with stationary frequency in the middle of the lower half of the

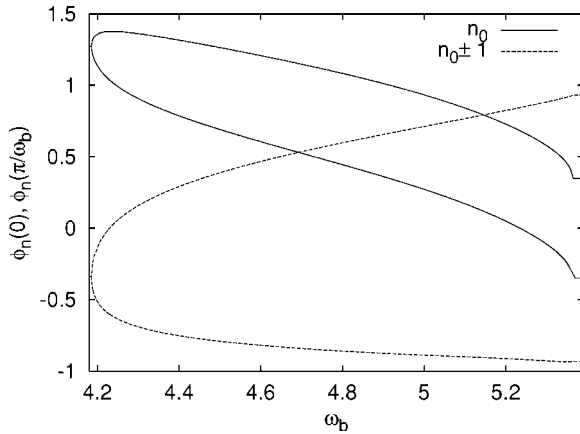


FIG. 2. Maximum and minimum amplitudes for the central (even) site n_0 and its neighboring (odd) sites for continuation at constant norm $\mathcal{N}=2$ of the family of two-frequency solutions bifurcating from a stationary gap breather with $\Lambda \approx -0.905$ at $\omega_b \approx 4.186$ (leftmost part of the figure). Here $V_0=2$ and $C=1$. The solution is unstable for $\omega_b \geq 4.47$.

gap). The intensity oscillates periodically, from having a minimum at the central (even) site and two symmetric maxima at the neighboring (odd) sites at $t=0$, to having one single maximum at the central site at $t=\pi/\omega_b$. The solution is linearly stable, as is seen from the numerically calculated Floquet eigenvalues [Fig. 1(b)], which are all on the unit circle. The main effect of a small but non-negligible perturbation is illustrated by Fig. 1(c): the solution almost perfectly retains its pulson character over very large time scales but with a slight shift in frequency (a consequence of the four-fold degenerate Floquet eigenvalue at $+1$ corresponding to drifts of the two arbitrary phases [9]), making visible the pulson oscillations also in the stroboscopic plot.

The results of more systematic investigations of the properties of such families of solutions are summarized in Figs. 2–4. For convenience, we now choose a “moderate” constant value of the norm, $\mathcal{N}/C=2$, and discuss the behavior for various values of V_0/C . In particular, this value is large enough for the stationary gap breather frequency Λ to be sufficiently far away from the lower gap boundary so as to always have a localized P1 mode, but small enough for Λ to be in the lower half of the gap, thus avoiding additional complications such as oscillatory instabilities and nonmonotonic continuation which may appear for stationary breathers in the upper gap half [6].

Figure 2, for $V_0/C=2$, shows the typical behavior for larger values of V_0/C . The continuation of the two-frequency solution is monotonic, and ends in a bifurcation with another (unstable) stationary solution with frequency $\Lambda=2\omega_b+\omega_0$. With the notation of Ref. [6], this solution is denoted as $\{0\uparrow(00)\}_S^O$, meaning that, in the anticontinuous limit $C\rightarrow 0$, it corresponds to symmetric in-phase oscillations only for the two (odd) sites $n_{0\pm 1}$, with frequency *above* the upper linear band. The pulson character of the two-frequency solution appears when the absolute value of the minimum (negative) value of $\phi_{n_{0\pm 1}}$ (lower part of dashed curve in Fig. 2) exceeds the minimum value of ϕ_{n_0} (lower part of solid curve), which for the case in Fig. 2 happens for $\omega_b \geq 4.43$. However, a

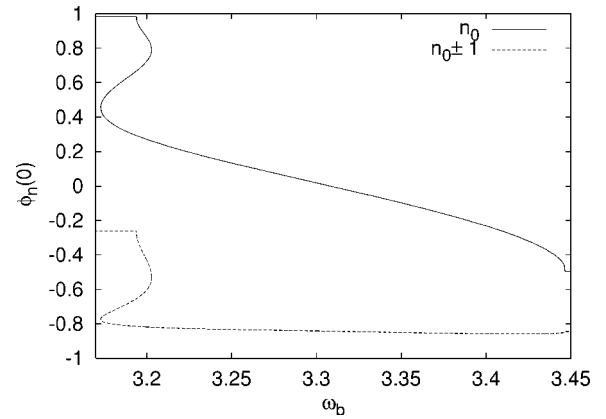


FIG. 3. Same as Fig. 2 but for $V_0=1$ and with only minimum amplitudes. The two-frequency solution family bifurcates from a stationary gap breather with $\Lambda \approx -0.558$ at $\omega_b \approx 3.194$, and is unstable for $\omega_b \geq 3.21$ on the lower branch.

stable pulson solution appears in this case only in a small frequency interval, since for $\omega_b \geq 4.47$ the solution is unstable through a symmetry-breaking instability (with real Floquet eigenvalue). For larger values of V_0/C , when the

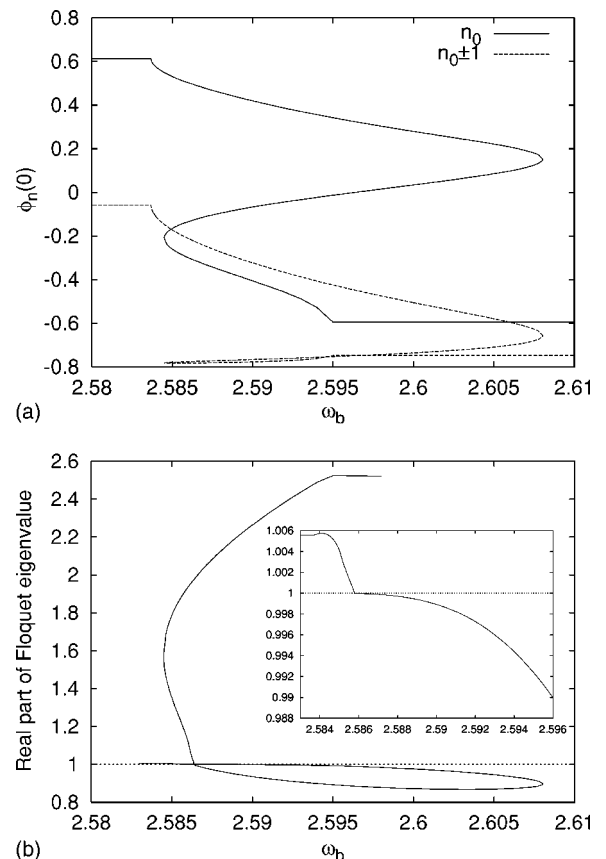


FIG. 4. (a) Same as Fig. 3 but for $V_0=0.5$. The two-frequency solution family bifurcates from the stationary gap breather with $\Lambda \approx -0.3147$ at $\omega_b \approx 2.5837$, and (b) becomes unstable for $\omega_b \approx 2.5862$ on the middle branch. Magnification in inset of (b) illustrates how the weakly unstable stationary gap breather gets stabilized by the intensity oscillations. In (b), only the largest real part of the eigenvalue pair corresponding to the symmetry-breaking localized mode is shown.

solution becomes more discrete, the solution becomes unstable before it attains its pulson character, and thus in the strongly discrete case the stable two-frequency solutions only correspond to relatively small oscillations of the central-site intensity for the stationary gap breather.

For smaller values of V_0/C , the continuation versus ω_b of the two-frequency solutions at fixed norm $\mathcal{N}/C=2$ still starts and ends in the same stationary solutions as before, but becomes nonmonotonous as shown for $V_0/C=1$ in Fig. 3 [for clarity only minimum amplitudes $\phi_n(0)$ are shown]. The solution now attains its pulson character on the intermediate branch, where it is always stable, and becomes unstable only on the lower branch.

For even smaller values of V_0/C , the stationary gap breathers broaden and assume a more “continuouslike” shape, which in particular implies that the odd sites with largest $|\phi_n^{(A)}|$ will change from $n_0 \pm 1$ to $n_0 \pm 3$, and further to $n_0 \pm 5$, etc. [6]. For $V_0/C=0.5$ (Fig. 4), the largest “odd” $|\phi_n^{(A)}|$ for the stationary breather is at $n_0 \pm 5$; however already for rather small intensity oscillations [$\omega_b \gtrsim 2.5856$ on the upper branch of Fig. 4(a)] of the corresponding P1 family of two-frequency solutions, the maximum “odd” intensity is again at $n_0 \pm 1$. The solution now attains its pulson character already on the upper branch ($\omega_b \gtrsim 2.5926$), and the pulson remains stable until it reaches $\omega_b \approx 2.5862$ on the middle branch [Fig. 4(b)].

Another very interesting effect regarding stability is seen in Fig. 4(b). It is known [6] that as V_0/C decreases, there are certain intervals of “inversion of stability” for stationary gap breathers, where the symmetric stationary breather discussed here becomes unstable (through a translational “depinning” instability), and instead the otherwise unstable *antisymmetric* gap breather gains stability. $V_0/C=0.5$ belongs to such an interval for $\mathcal{N}/C=2$, and thus the stationary gap breather from which the two-frequency family in Fig. 4 bifurcates is itself unstable. However, as seen from the inset in Fig. 4(b), already at $\omega_b \approx 2.5856$ the unstable eigenvalue returns to the unit circle, and thus even rather small symmetric intensity

oscillations may *stabilize* the stationary solution with respect to its antisymmetric instability. We are not aware of any reported similar scenario.

To conclude, we have presented explicit examples of exact stable quasiperiodic pulson solutions with large-amplitude intensity oscillations, in the binary modulated DNLS equation describing, e.g., coupled waveguides of alternating widths. Thus, there should be good possibilities for experimental observation of such states in this context. Although we here focused on one particular family of symmetric solutions bifurcating from the symmetric stationary gap breather, the analysis illustrated by Fig. 3 in Ref. [6] suggests that solutions with similar properties may bifurcate from other, symmetric or antisymmetric, internal modes with frequencies above the continuous spectrum (P2, P3, P4, P5, etc.), existing for the symmetric as well as the antisymmetric gap breathers. It seems likely that such solutions also should exist for other types of multicomponent lattices with (at least) two conserved quantities; one interesting candidate being second-harmonic-generating lattices of the type considered, e.g., in Ref. [15], where also the fundamental discrete soliton was found to exhibit an internal mode with frequency above the linear spectrum. A challenge for future research is to obtain analytical expressions for the pulson solutions in the continuum limit. This is a nontrivial task, since although the relevant linear eigenmodes of the stationary gap breather apparently persist arbitrarily close to the continuum limit (cf. Fig. 3 of Ref. [6]), they cannot exist in the standard continuous twofield model for gap solitons (which is the same for DNLS as for diatomic Klein-Gordon lattices; see, e.g., Refs. [16,17]), since the continuous spectrum of such models extends to infinity leaving no room for localized modes above it. A more sophisticated continuous approximation would therefore appear necessary to this end.

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