Mie scattering by a uniaxial anisotropic sphere

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(Received 24 April 2004; published 17 November 2004)

The field solution to the electromagnetic scattering of a plane wave by a uniaxial anisotropic sphere is obtained in terms of a spherical vector wave function expansion form. Using the source-free Maxwell's equations for uniaxial anisotropic media and making the Fourier transform of the field quantities, the electromagnetic fields in the spectral domain in uniaxial anisotropic media are assumed to have a form similar to the plane wave expanded also in terms of the spherical vector wave functions. Applying the continuous boundary conditions of electromagnetic fields on the surface between the air region and uniaxial anisotropic sphere, the coefficients of transmitted fields and the scattered fields in uniaxial anisotropic media can be obtained analytically in the expansion form of vector wave eigenfunctions. Numerical results for some special cases are obtained and compared with those of the classical Lorenz-Mie theory and the method of moments accelerated with the conjugate-gradient fast-Fourier-transform approach. We also present some new numerical results for the more general uniaxial dielectric material media.

DOI: 10.1103/PhysRevE.70.056609 PACS number(s): 03.50.De, 41.20.Jb, 42.25.Bs, 52.40.Db

I. INTRODUCTION

Recently, there has been a growing interest in the interaction between electromagnetic waves and anisotropic media. This is because there are many natural and artificial anisotropic materials and they have a variety of applications in optical signal processing, constructing signal processing, optical frequency elements and devices, enhancement and reduction of radar cross sections of various scatterers, characterization of antenna radomes, optimum design of optical fibers, synthesis of special types of radar absorbers, and fabrication of specific substrates for microwave devices (filters, dividers, and amplifiers) and microstrip antennas.

One of the basic problems to investigate waves in anisotropic media is to characterize the scattering properties of anisotropic objects. A rigorous solution of scattered fields can be obtained using the Lorenz-Mie theory of electromagnetic fields scattered by a homogeneous isotropic dielectric sphere [1,2], originated by Lorenz in 1890 and Mie in 1908. The Lorenz-Mie solution can be easily extended to treat radially inhomogeneous isotropic spheres [3–5]. Scattering by homogeneous anisotropic objects has attracted a great deal of interest in recent years—for instance, [6,7]. Numerical methods have been employed to analyze this problem based on integral equations [8,9], differential equations [10], and the analytical method using vector wave eigenfunctions expansions [11]. Although the efforts were primarily spent in the past on two-dimensional (2D) geometries, some progresses have been made in the analysis of three-dimensional (3D) anisotropic scatterers using the method of moments (MOM) [12,13], combined field (surface) integral equation (CFIE) formulation [14], the integral equation [15], the coupled dipole approximation method [16], the expansion of scalar fields [17], and the spectrum-domain Fourier-transform approach [18–20]. In contradistinction to these numerical methods and analytical solutions, we concentrate in this paper on the analytical solution to the three-dimensional scattering of a plane wave by uniaxial anisotropic sphere.

To obtain a solution of vector wave functions in uniaxial anisotropic media, we start from the electric field vector wave equation. Taking the Fourier transform of the electric field and substituting it into the vector wave equation, we obtain the characteristic equation. Solving this equation, we obtain the eigenvalues and corresponding vector wave eigenfunctions. Then, we obtain representative electromagnetic fields inside and outside the uniaxial anisotropic sphere similarly in terms of their respective spherical vector wave eigenfunctions with their scattering coefficients as unknowns [1–5] with the expansion of the plane wave factors in terms of spherical vector wave functions in an isotropic medium [21]. Application of the continuous boundary conditions of the tangential electric and magnetic field components on the uniaxial anisotropic spherical surface leads to the scattering unknown coefficients determined analytically where orthogonality relations of the Legendre polynomials are employed. Numerical results are obtained to gain more physical insight into this problem. After the results were validated by comparison with the existing data, some new results are computed and discussed.

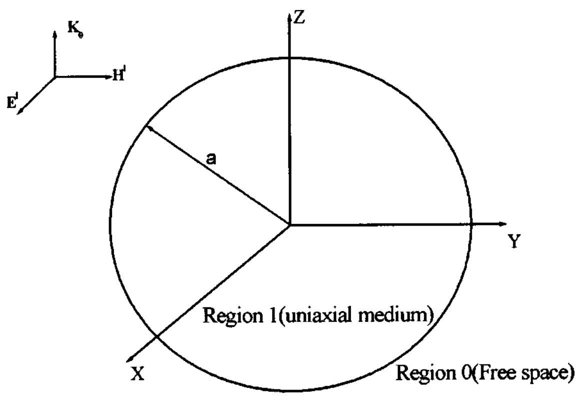

FIG. 1. Geometry for electromagnetic scattering of a plane wave by an uniaxial anisotropic sphere.

In the subsequent analysis, a time dependence of the form $\exp(-i\omega t)$ is assumed for the electromagnetic field quantities, but is suppressed throughout the treatment.

II. FORMULATION OF THE SCATTERING PROBLEM

Assume that a homogeneous, uniaxial anisotropic sphere of radius a is center located in the free space and is shown in Fig. 1 in the spherical coordinates. The permittivity and permeability tensors are characterized by the two matrices

$$\overline{\boldsymbol{\epsilon}} = \epsilon_t(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) + \epsilon_z\hat{\mathbf{z}}\hat{\mathbf{z}} = \begin{bmatrix} \epsilon_t & 0 & 0 \\ 0 & \epsilon_t & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}, \tag{1a}$$

$$\overline{\boldsymbol{\mu}} = \mu_t(\hat{\mathbf{x}}\hat{\mathbf{x}} + \hat{\mathbf{y}}\hat{\mathbf{y}}) + \mu_z\hat{\mathbf{z}}\hat{\mathbf{z}} = \begin{bmatrix} \mu_t & 0 & 0 \\ 0 & \mu_t & 0 \\ 0 & 0 & \mu_z \end{bmatrix}. \tag{1b}$$

The **E**-field vector wave equation can be obtained by substituting the above constitutive relations into the source-free Maxwell's equations [11,18–20]—that is,

$$\nabla \times [\overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r})] - \omega^2\overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) = 0. \tag{2}$$

The solution to Eq. (2) can be obtained by the Fourier transform

$$\mathbf{E}(\mathbf{r}) = \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} dk_z, \tag{3}$$

where the wave number is denoted by $\mathbf{k}=k_x\hat{\mathbf{x}}+k_y\hat{\mathbf{y}}+k_z\hat{\mathbf{z}}$, while the space vector is identified as $\mathbf{r}=x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}}$, with $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$ being the unit vectors of a Cartesian coordinate system. By substituting Eq. (3) into Eq. (2), the wave equation is transformed into

$$\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \int_{-\infty}^{\infty} \overline{\mathbf{K}}(\mathbf{k}) \cdot \mathbf{E}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}} dk_z = 0, \tag{4}$$

where

$$\overline{\mathbf{K}}(\mathbf{k}) = \frac{1}{\mu_t}\begin{bmatrix} k_z^2 + \mu k_y^2 - a_1 & -\mu k_x k_y & -k_x k_z \\ -\mu k_x k_y & k_z^2 + \mu k_x^2 - a_1 & -k_y k_z \\ -k_x k_z & -k_y k_z & k_y^2 + k_x^2 - a_2 \end{bmatrix}, \tag{5}$$

with

$$a_1 = \omega^2\epsilon_t\mu_t, \tag{6a}$$

$$a_2 = \omega^2\epsilon_z\mu_t, \tag{6b}$$

$$\mu = \mu_t/\mu_z. \tag{6c}$$

Letting the $\mathbf{E}(\mathbf{k})$ have nontrivial solutions, we find that the following characteristic equation is satisfied:

$$\mathrm{Det}[\overline{\mathbf{K}}(\mathbf{k})] = 0. \tag{7}$$

In explicit form, the characteristic equation is rewritten as

$$A(\theta_k,\phi_k)k^4 - B(\theta_k,\phi_k)k^2 + C = 0, \tag{8}$$

where

$$A(\theta_k,\phi_k) = a_2\cos^4\theta_k + \mu a_1\sin^4\theta_k + (a_1 + \mu a_2)\sin^2\theta_k\cos^2\theta_k, \tag{9a}$$

$$B(\theta_k,\phi_k) = (a_1^2 + \mu a_1 a_2)\sin^2\theta_k + 2a_1 a_2\cos^2\theta_k, \tag{9b}$$

$$C = a_1^2 a_2, \tag{9c}$$

with

$$k^2 = k_x^2 + k_y^2 + k_z^2, \tag{10a}$$

$$\theta_k = \tan^{-1}(\sqrt{k_x^2 + k_y^2}/k_z), \tag{10b}$$

$$\phi_k = \tan^{-1}(k_y/k_x). \tag{10c}$$

Equation (8) is a biquadratic algebraic equation and has the following four roots of k_ℓ (where $\ell=1, 2, 3$, or 4) for the radial wave vectors:

$$k_{1,3}^2 = \frac{a_1}{\cos^2\theta_k + \mu\sin^2\theta_k}, \tag{11a}$$

$$k_{2,4}^2 = \frac{a_1 a_2}{a_1\sin^2\theta_k + a_2\cos^2\theta_k}. \tag{11b}$$

So the corresponding **E**-field eigenvectors can be easily obtained [18,20,22] from Eq. (5) and are given, for $q=1, 2, 3$, and 4, as follows:

$$\mathbf{E}_q = \mathbf{F}_q^e f_q(\theta_k,\phi_k) = [F_{qx}^e(\theta_k,\phi_k)\hat{\mathbf{x}} + F_{qy}^e(\theta_k,\phi_k)\hat{\mathbf{y}} + F_{qz}^e(\theta_k,\phi_k)\hat{\mathbf{z}}]f_q(\theta_k,\phi_k), \tag{12}$$

where

$$F_{qx}^e = \begin{cases} -\sin\phi, & q = 1,3, \\ W_q^{(e)}(\theta_k)\cos\phi, & q = 2,4, \end{cases} \tag{13a}$$

$$F_{qy}^e = \begin{cases} \cos\phi, & q=1,3, \\ W_q^{(e)}(\theta_k)\sin\phi, & q=2,4, \end{cases} \tag{13b}$$

$$F_{qz}^e = \begin{cases} 0, & q=1,3, \\ 1, & q=2,4, \end{cases} \tag{13c}$$

with

$$W_q^{(e)}(\theta_k) = \frac{k_q^2 \sin\theta_k \cos\theta_k}{k_q^2 \cos^2\theta_k - a_1}, \quad q=2,4. \tag{14}$$

With those obtained eigenvalues and their associated formulas, the $\mathbf{E}$ field in Eq. (3) is then given as follows:

$$\mathbf{E}(\mathbf{r}) = \sum_{q=1}^{2} \int_0^{\pi} \int_0^{2\pi} \mathbf{F}_q^e(\theta_k,\phi_k) f_q(\theta_k,\phi_k) e^{i\mathbf{k}_q\cdot\mathbf{r}} k_q^2 \sin\theta_k d\theta_k d\phi_k, \tag{15}$$

where

$$\mathbf{k}_q = \hat{\mathbf{x}} k_q \sin\theta_k \cos\phi_k + \hat{\mathbf{y}} k_q \sin\theta_k \sin\phi_k + \hat{\mathbf{z}} k_q \cos\theta_k$$

and $f_q(\theta_k,\phi_k)$ denotes the unknown angular spectrum amplitude. Equation (15) is also known as the eigen-plane-wave spectrum representation of the electric field in a homogeneous uniaxial anisotropic medium. From Eq. (3), it is also evident that the integration over the radial wave vector component is reduced to a summation of four terms corresponding to the roots of Eq. (8), which are the only permissible solutions. The symmetric roots—i.e., $k=-k_q$ of $k=k_q$ ($q=1,2$)—are taken into account automatically as θ spans from 0 to π while ϕ spans from 0 to 2π. Physically, we need to sum up for only two of the four components—namely, k_1 and k_2.

It is noted that the unknown angular spectrum amplitude $f_q(\theta_k,\phi_k)$ is periodic with θ_k and ϕ_k, respectively. So we can use surface harmonics of the first kind of expansion for the $f_q(\theta_k,\phi_k)$ as

$$f_q(\theta_k,\phi_k) = \sum_{m',n'} G_{m'n'q} P_{n'}^{m'}(\cos\theta_k) e^{im'\phi_k}, \tag{16}$$

where $P_n^m(x)$ denotes the associated Legendre function of indices n and m, and n' is summed from 0 to $+\infty$, while m' is summed from $-n'$ to n', and $\mathbf{k}$ is pointing in the (θ_k,ϕ_k) direction in the spherical coordinates. Substituting Eq. (16) into Eq. (15), we obtain

$$\mathbf{E}(\mathbf{r}) = \sum_{q=1}^{2} \sum_{m',n'} G_{m'n'q} \int_0^{\pi} \int_0^{2\pi} \mathbf{F}_q^e(\theta_k,\phi_k) \times P_{n'}^{m'}(\cos\theta_k) e^{im'\phi} e^{i\mathbf{k}_q\cdot\mathbf{r}} k_q^2 \sin\theta_k d\theta_k d\phi_k. \tag{17}$$

This specific form of Eq. (17) suggests the use of the well-known identity [1,2,21]

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(kr) \left[\sum_{m=0}^{n} \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta_k) \times P_n^m(\cos\theta) e^{im(\phi-\phi_k)} \sum_{m=1}^{n} \frac{(n-m)!}{(n+m)!} \times P_n^m(\cos\theta_k) P_n^m(\cos\theta) e^{-im(\phi-\phi_k)} \right]. \tag{18}$$

After substituting Eq. (18) into Eq. (17), we obtain the solution of $\mathbf{E}(\mathbf{r})$ for homogeneous uniaxial anisotropic media and express it in terms of the scalar spherical wave functions. In order to have a compact and explicit solution to the boundary value problem involving the spherical structures of anisotropic materials, however, it is necessary to introduce the spherical vector wave functions as follows [1–5,18–21]:

$$\mathbf{M}_{mn}^{(l)} = z_n^{(l)}(kr) \left[im \frac{P_n^m(\cos\theta)}{\sin\theta} \hat{\boldsymbol{\theta}} - \frac{dP_n^m(\cos\theta)}{d\theta} \hat{\boldsymbol{\phi}} \right] e^{im\phi}, \tag{19a}$$

$$\mathbf{N}_{mn}^{(l)} = n(n+1) \frac{z_n^{(l)}(kr)}{kr} P_n^m(\cos\theta) e^{im\phi} \hat{\mathbf{r}} + \frac{1}{kr} \times \frac{d(r z_n^{(l)}(kr))}{dr} \left[\frac{dP_n^m(\cos\theta)}{d\theta} \hat{\boldsymbol{\theta}} + im \frac{P_n^m(\cos\theta)}{\sin\theta} \hat{\boldsymbol{\phi}} \right] e^{im\phi}, \tag{19b}$$

$$\mathbf{L}_{mn}^{(l)} = k \left\{ \frac{dz_n^{(l)}(kr)}{d(kr)} P_n^m(\cos\theta) e^{im\phi} \hat{\mathbf{r}} + \frac{z_n^{(l)}(kr)}{kr} \times \left[\frac{dP_n^m(\cos\theta)}{d\theta} \hat{\boldsymbol{\theta}} + im \frac{P_n^m(\cos\theta)}{\sin\theta} \hat{\boldsymbol{\phi}} \right] e^{im\phi} \right\}, \tag{19c}$$

where $z_n^{(l)}(x)$ (where $l=1$, 2, 3, or 4) denotes an appropriate kind of spherical Bessel functions—that is, j_n, y_n, $h_n^{(1)}$, or $h_n^{(2)}$, respectively. Because of the complete property of the vector wave functions given in Eqs. (19a)–(19c), we obtain the expressions

$$\mathbf{F}_q^e(\theta,\phi) e^{i\mathbf{k}_q\cdot\mathbf{r}} = \sum_{mn} [A_{mnq}^e(\theta_k) \mathbf{M}_{mn}^{(1)}(\mathbf{r},k_q) + B_{mnq}^e(\theta_k) \mathbf{N}_{mn}^{(1)}(\mathbf{r},k_q) + C_{mnq}^e(\theta_k) \mathbf{L}_{mn}^{(1)}(\mathbf{r},k_q)] e^{-im\phi_k}, \tag{20}$$

where n is summed from 0 to $+\infty$, while m is summed from $-n$ to n, and $\mathbf{k}$ is pointing in the (θ_k,ϕ_k) direction, while $\mathbf{r}$ is pointing in the (θ,ϕ) direction in the spherical coordinates. The other interparameters, $A_{mnq}^e(\theta_k)$, $B_{mnq}^e(\theta_k)$, and $C_{mnq}^e(\theta_k)$ are provided in the Appendix.

Substituting Eq. (20) into Eq. (17), integrating with respect to ϕ_k, and after some straightforward algebraic manipulations, we end up with

$$\mathbf{E}(\mathbf{r}) = \sum_{q=1}^{2}\sum_{mn}\sum_{n'} 2\pi G_{mn'q}\int_0^{\pi} [A_{mnq}^{e}(\theta_k)\mathbf{M}_{mn}^{(1)}(\mathbf{r},k_q) + B_{mnq}^{e}(\theta_k)\mathbf{N}_{mn}^{(1)}(\mathbf{r},k_q) + C_{mnq}^{e}(\theta_k)\mathbf{L}_{mn}^{(1)}(\mathbf{r},k_q)] \times P_{n'}^{m}(\cos\theta_k)k_q^2 \sin\theta_k d\theta_k. \tag{21}$$

Equation (21) is the eigenfunction representation of electric field in uniaxial anisotropic media. The representation of magnetic field in uniaxial anisotropic media is very similar to that of electric field in Eq. (21) and can be easily obtained by changing **E**-field eigenvectors for **H**-field eigenvectors following Eqs. (15)–(20). The **H**-field eigenvectors can be derived from the **E**-field eigenvectors shown in Eqs. (8)–(11) by using the source-free Maxwell's equations in the spectral domain. Because the equations of the **H** field are very similar to those of the **E** field, it is only given relative to the **H**-field eigenvectors (i.e., $\mathbf{F}_q^h$) and **E**-field eigenvectors (i.e., $\mathbf{F}_q^e$) in the Cartesian coordinate system

$$\mathbf{F}_q^h = \begin{bmatrix} 0 & -\cos\theta_k & \sin\theta_k\sin\phi_k \\ \cos\theta_k & 0 & -\sin\theta_k\cos\phi_k \\ -\mu\sin\theta_k\sin\phi_k & \mu\sin\theta_k\cos\phi_k & 0 \end{bmatrix} \times \frac{k_q}{\omega\mu_t}\mathbf{F}_q^e, \tag{22}$$

where $q=1,2$.

From the resulting equation (21), it is shown that the solutions to the source-free Maxwell's equations for the uniaxial anisotropic medium are expanded in terms of the first kind of spherical vector functions. Because spherical Bessel functions of different kinds satisfy the same differential equation and the same recursive relations, the first kind of vector wave functions in Eq. (21) can be changed easily to the other three ones. So we can use the field expressions given in Eq. (21) to characterize the scattering and radiation involving the layered structures of the uniaxial anisotropic media.

Assume that the electric field of an incident plane wave is given by $\mathbf{E}=\hat{\mathbf{x}}E_0e^{ik_0z}$. The incident wave fields (designated by the superscript *inc*) may be expanded into an infinite series of spherical vector wave functions for an isotropic medium as follows [1–5,18,19,21]:

$$\mathbf{E}^{inc} = E_0\sum_{m,n}[\delta_{m,1}+\delta_{m,-1}][a_{mn}^{x}\mathbf{M}_{mn}^{(1)}(\mathbf{r},k_0) + b_{mn}^{x}\mathbf{N}_{mn}^{(1)}(\mathbf{r},k_0)], \tag{23a}$$

$$\mathbf{H}^{\mathrm{inc}} = \frac{k_0}{i\omega\mu_0}E_0\sum_{mn}[\delta_{m,1}+\delta_{m,-1}] \times[a_{mn}^{x}\mathbf{N}_{mn}^{(1)}(\mathbf{r},k_0) + b_{mn}^{x}\mathbf{M}_{mn}^{(1)}(\mathbf{r},k_0)], \tag{23b}$$

where

$$a_{mn}^{x} = \begin{cases} i^{n+1}\dfrac{2n+1}{2n(n+1)}, & m=1, \\ i^{n+1}\dfrac{2n+1}{2}, & m=-1, \end{cases}$$

$$b_{mn}^{x} = \begin{cases} i^{n+1}\dfrac{2n+1}{2n(n+1)}, & m=1, \\ -i^{n+1}\dfrac{2n+1}{2}, & m=-1, \end{cases} \tag{24a}$$

$$\delta_{s,l} = \begin{cases} 1, & s=l, \\ 0, & s\neq l. \end{cases} \tag{24b}$$

According to the radiation condition of an outgoing wave and asymptotic behavior of spherical Bessel functions, only $h_n^{(1)}$ should be retained in the radial function; therefore, the scattering fields (designated by the superscript s) are expanded as

$$\mathbf{E}^{s} = \sum_{mn}[A_{mn}^{s}\mathbf{M}_{mn}^{(3)}(\mathbf{r},k_0) + B_{mn}^{s}\mathbf{N}_{mn}^{(3)}(\mathbf{r},k_0)], \tag{25a}$$

$$\mathbf{H}^{s} = \frac{k_0}{i\omega\mu_0}\sum_{mn}[A_{mn}^{s}\mathbf{N}_{mn}^{(3)}(\mathbf{r},k_0) + B_{mn}^{s}\mathbf{M}_{mn}^{(3)}(\mathbf{r},k_0)], \tag{25b}$$

where A_{mn}^{s} and B_{mn}^{s} (with n being from 0 to $+\infty$ and m being from $-n$ to n) are unknown coefficients, $\mathbf{M}_{mn}^{(l)}(\mathbf{r},k_0)$ and $\mathbf{N}_{mn}^{(l)}(\mathbf{r},k_0)$ are solenoidal spherical vector wave functions given in Eqs. (19a) and (19b), respectively, and $k_0=\omega(\epsilon_0\mu_0)^{1/2}$, ϵ_0, and μ_0 denote the wave number, permittivity, and permeability in free space, respectively.

The expressions of the electromagnetic fields inside the uniaxial anisotropic sphere are given in Eq. (21), and the continuity of the tangential electric and magnetic field components at $r=a$ yields

$$\sum_{q=1}^{2}\sum_{n'=0}^{\infty} 2\pi G_{mn'q}\int_0^{\pi} Q_{mnq}P_{n'}^{m}(\cos\theta_k)k_q^2\sin\theta_k d\theta_k = [\delta_{m,1}+\delta_{m,-1}]E_0 a_{mn}^{x}\frac{i}{(k_0a)^2}, \tag{26a}$$

$$\sum_{q=1}^{2}\sum_{n'=0}^{\infty} 2\pi G_{mn'q}\int_0^{\pi} R_{mnq}P_{n'}^{m}(\cos\theta_k)k_q^2\sin\theta_k d\theta_k = [\delta_{m,1}+\delta_{m,-1}]E_0 b_{mn}^{x}\frac{i}{(k_0a)^2}, \tag{26b}$$

where

$$Q_{mnq} = \left\{A_{mnq}^{e}\frac{1}{k_0r}\frac{d}{dr}[rh_n^{(1)}(k_0r)]j_n(k_qr) - \frac{i\omega\mu_0}{k_0} \times\left[B_{mnq}^{h}\frac{1}{k_qr}\frac{d}{dr}[rj_n(k_qr)] + C_{mnq}^{h}\frac{j_n(k_qr)}{r}\right]h_n^{(1)}(k_0r)\right\}_{r=a}, \tag{27a}$$

$$R_{mnq} = \left\{ \frac{i\omega\mu_0}{k_0} A^h_{mnq} \frac{1}{k_0 r} \frac{d}{dr}[r h^{(1)}_n(k_0 r)] j_n(k_q r) - \left[B^e_{mnq} \frac{1}{k_q r} \frac{d}{dr}[r j_n(k_q r)] + C^e_{mnq} \frac{j_n(k_q r)}{r} \right] h^{(1)}_n(k_0 r) \right\}_{r=a}. \tag{27b}$$

The scattering coefficients—i.e., A^s_{mn} and B^s_{mn}—are thus expressed as

$$A^s_{mn} = \frac{1}{h^{(1)}_n(k_0 a)} \left[\sum_{n'=0}^{\infty} \sum_{q=1}^{2} 2\pi G_{mnq} \times \int_0^{\pi} A^e_{mnq} j_n(k_q a) P^m_{n'} k_q^2 \sin\theta_k d\theta_k - [\delta_{m,1} + \delta_{m,1}] E_0 a^x_{mn} j_n(k_0 a) \right], \tag{28a}$$

$$B^s_{mn} = \frac{1}{h^{(1)}_n(k_0 a)} \left[\frac{i\omega\mu_0}{k_0} \sum_{n'=0}^{\infty} \sum_{q=1}^{2} 2\pi G_{mnq} \times \int_0^{\pi} A^h_{mnq} j_n(k_q a) P^m_{n'} k_q^2 \sin\theta_k d\theta_k - [\delta_{m,1} + \delta_{m,1}] b^x_{mn} j_n(k_0 a) \right]. \tag{28b}$$

From the coefficients of scattered fields by the uniaxial anisotropic sphere, the radar cross sections (RCS's) can be calculated—i.e.,

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \frac{|E^s|^2}{|E^i|^2} = \frac{4\pi}{k_0^2} \left[\left| \sum_{n=1}^{\infty} (-i)^n \times \left\{ \frac{P^1_n}{\sin\theta} \left(A^s_{1n} e^{i\phi} + \frac{A^s_{-1n}}{n(n+1)} e^{-i\phi} \right) + \frac{dP^1_n}{d\theta} \left(B^s_{1n} e^{i\phi} - \frac{B^s_{-1n}}{n(n+1)} e^{-i\phi} \right) \right\} \right|^2 + \left| \sum_{n=1}^{\infty} (-i)^{n+1} \left\{ \frac{dP^1_n}{d\theta} \left(A^s_{1n} e^{i\phi} - \frac{A^s_{-1n}}{n(n+1)} e^{-i\phi} \right) + \frac{P^1_n}{\sin\theta} \left(B^s_{1n} e^{i\phi} + \frac{B^s_{-1n}}{n(n+1)} e^{-i\phi} \right) \right\} \right|^2 \right]. \tag{29}$$

III. NUMERICAL RESULTS AND DISCUSSION

All results presented in this section are for nonmagnetic (i.e., $\mu_t = \mu_z = \mu_0$) spherical scatterers of radius a and with permittivity tensor $\bar{\bar{\boldsymbol{\epsilon}}}$. The incident field is a plane wave with electric field amplitude equal to unity, polarized parallel to the $\hat{\mathbf{x}}$ direction, and that propagates in the positive $\hat{\mathbf{z}}$ direction.

To demonstrate the accuracy of the solutions achievable by using the present method, we compare bistatic RCS's in E plane (xoz plane as shown in Fig. 1) and H plane (yoz plane as shown in Fig. 1) with the Lorenz-Mie theory [1,2,18,19] and the MOM conjugate-gradient fast-Fourier-transform (CG-FFT) method [13], as shown in Figs. 2 and 3. The series in Eqs. (26a) and (26b) converges rapidly, and it is sufficient to take $N=4$ as the upper limit of the summation indices n and n'. Certainly, it should be pointed out that the convergence rate or the upper limit of the summation depends on the electrical dimension of the sphere (with respect to the wavelength). In Fig. 2, the RCS's in both E and H planes using the formulations in this paper are compared with those of Lorenz-Mie theory. An excellent agreement of the RCS results is achieved between those two methods, where the permittivity tensor elements are characterized by $\epsilon_t = \epsilon_z = 2\epsilon_0$ and the electric size of the uniaxial anisotropic sphere is chosen as $k_0 a = 0.5\pi$ or $a = \lambda/4$. It is shown that the obtained solution is stable even for almost isotropic scatterers, since

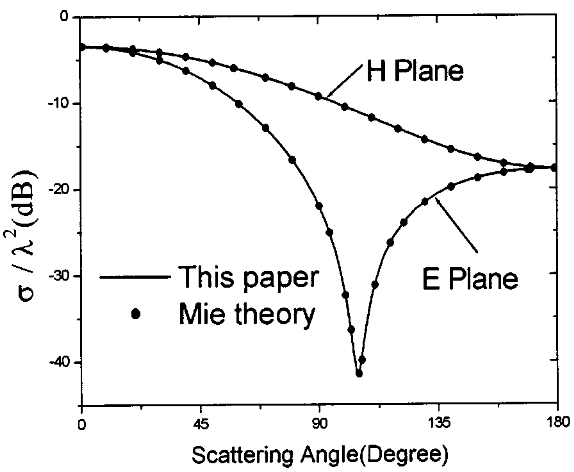

FIG. 2. Radar cross sections (RCS's) versus scattering angle θ (in degrees): results of this paper (solid curve) and of the Lorenz-Mie theory. The electric dimension is chosen as $k_0 a = 0.5\pi$ while the permittivity tensor elements are assumed to be $\epsilon_t = \epsilon_2 = 2\epsilon_0$.

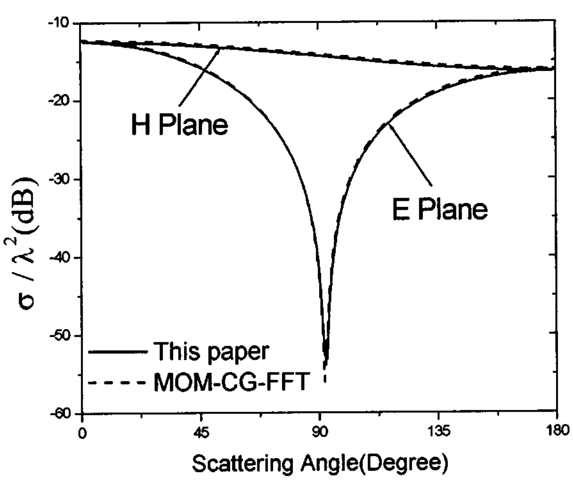

FIG. 3. Radar cross sections (RCS's) versus scattering angle θ (in degrees): results of this paper (solid curve) and of the MOM with CG-FFT fast algorithm (dashed curve). The electric dimension is chosen as $k_0 a = 0.3\pi$ while the permittivity tensor elements are assumed to be $\epsilon_t = 3\epsilon_0$ and $\epsilon_z = 2\epsilon_0$.

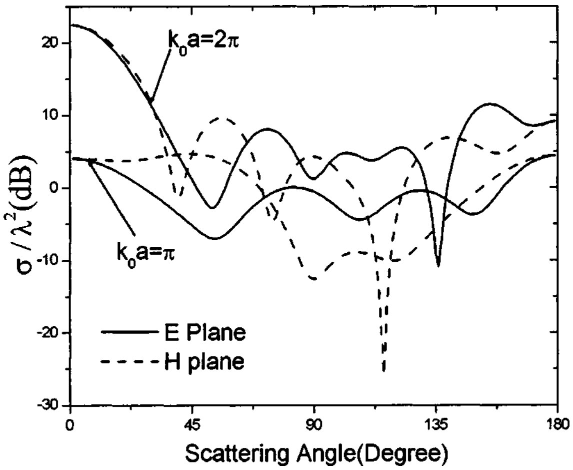

FIG. 4. Radar cross sections (RCS's) versus scattering angle θ (in degrees) in the E plane (solid curve) and in the H plane (dashed curve). The electric dimension is chosen as $k_0a=\pi$ and $k_0a=2\pi$, respectively, while the permittivity tensor elements are assumed to be $\epsilon_t=5.3495\epsilon_0$ and $\epsilon_z=4.9284\epsilon_0$.

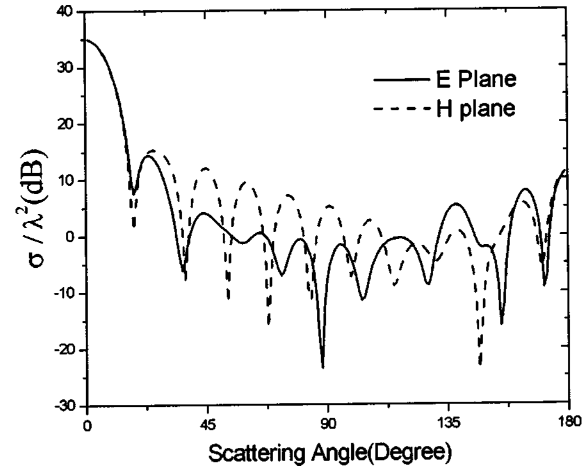

FIG. 5. Radar cross sections (RCS's) versus scattering angle θ (in degrees) in the E plane (solid curve) and in the H plane (dashed curve). The electric dimension is chosen as $k_0a=4\pi$ while the permittivity tensor elements are assumed to be $\epsilon_t=(2+0.1i)\epsilon_0$ and $\epsilon_3=(4+0.2i)\epsilon_0$.

the proposed solution is an analytical one of the uniaxial anisotropic media, and the result of the Lorenz-Mie theory is a special case of the present method. In Fig. 3, the RCS of the uniaxial anisotropic sphere is computed using both our formulation and the MOM-CG-FFT technique. Since the MOM-CG-FFT approach is very efficient for the electrically small size objects, the electric size of the uniaxial is chosen as the $k_0a=0.3\pi$, the permittivity tensor elements are characterized by $\epsilon_t=3\epsilon_0$ and $\epsilon_z=2\epsilon_0$, and the uniaxial anisotropic sphere is lossless. From Fig. 3, it is depicted that the RCS results obtained using the two methods are in good agreement; thus, it partially verifies that the proposed method and the FORTRAN code developed in this paper are correct.

Figure 4 presents radar cross sections for a more general lossless uniaxial anisotropic medium. It is assumed that the permittivity tensor elements are $\epsilon_t=5.3495$ and $\epsilon_z=4.9284$. The electrical dimension of the uniaxial sphere is chosen as $k_0a=\pi$ and $k_0a=2\pi$, respectively, where the convergence number is $N=6$ for the electrical dimension of $k_0a=\pi$ and it becomes $N=10$ for the electrical dimension of $k_0a=2\pi$. From Fig. 4, it is shown that the larger the electrical size, the sharper the bistatic RCS becomes.

To illustrate applicability of this analytical solution to the uniaxial anisotropic sphere of the electrically large size (for example, in the resonance region), the RCS's of a relatively large uniaxial anisotropic sphere with $k_0a=4\pi$ under a plane-wave incidence are given in Fig. 5. The permittivity tensor elements are chosen as $\epsilon_t=2+0.1i$ and $\epsilon_z=4+0.2i$. It is seen that the RCS's vary with scattering angle. When the dimensions are increased, the convergence number $N=20$ is also increased.

IV. CONCLUSIONS

In this paper, an analytical solution to source-free Maxwell's equations in uniaxial anisotropic media has been obtained in terms of the spherical vector wave functions for uniaxial anisotropic media. The method is developed based on the expansion of a plane-wave factor of the field and the Fourier transform where the unknown angular spectrum amplitude is determined. The three-dimensional electromagnetic scattering of a plane wave by an uniaxial anisotropic sphere has been theoretically formulated, physically characterized, and numerically discussed. Numerical results for some special cases are also obtained and compared with those of the Lorenz-Mie theory and the method of moments accelerated with the conjugate-gradient fast-Fourier-transform approach, and a very good agreement is achieved. We also present numerical results in the resonance region for the lossy uniaxial media.

ACKNOWLEDGMENTS

The authors are greatly indebted to Professor Chang-Yin Jiang at the China Research Institute of Radiowave Propagation (CRIRP), Xinxiang, China, for his helpful discussions. This work is partially supported by Grant No. 60071025 of National Natural Science Foundation of China (NSFC) and Grant No. 20030625 of Zhejiang Province Education Bureau.

APPENDIX: SCATTERING COEFFICIENTS OF EIGENEXPANSIONS IN Eqs. (20), (27a), and (27b)

We let

$$\mathbf{F}_q^e(\theta,\phi)e^{i\mathbf{k}_q\cdot\mathbf{r}} = \sum_{mn}[A_{mnq}^e(\theta_k)\mathbf{M}_{mn}^{(1)}(\mathbf{r},k_q) + B_{mnq}^e(\theta_k)\mathbf{N}_{mn}^{(1)}(\mathbf{r},k_q) + C_{mnq}^e(\theta_k)\mathbf{L}_{mn}^{(1)}(\mathbf{r},k_q)]e^{-im\phi_k}. \tag{A1}$$

Because the spherical vector wave functions $\mathbf{M}_{mn}$, $\mathbf{N}_{mn}$, and $\mathbf{L}_{mn}$ are orthogonal and self-contained, the plane-wave factor can be expanded as follows:

$$\hat{\mathbf{x}}e^{i\mathbf{k}_q\cdot\mathbf{r}} = \sum_{mn}[a_{mn}^{x}\mathbf{M}_{mn}^{(1)}(\mathbf{r},k_q) + b_{mn}^{x}\mathbf{N}_{mn}^{(1)}(\mathbf{r},k_q) + C_{mn}^{x}\mathbf{L}_{mn}^{(1)}(\mathbf{r},k_q)], \tag{A2a}$$

$$\hat{\mathbf{y}}e^{i\mathbf{k}_q\cdot\mathbf{r}} = \sum_{mn}[a_{mn}^{y}\mathbf{M}_{mn}^{(1)}(\mathbf{r},k_q) + b_{mn}^{y}\mathbf{N}_{mn}^{(1)}(\mathbf{r},k_q) + C_{mn}^{y}\mathbf{L}_{mn}^{(1)}(\mathbf{r},k_q)], \tag{A2b}$$

$$\hat{\mathbf{z}}e^{i\mathbf{k}_q\cdot\mathbf{r}} = \sum_{mn}[a_{mn}^{z}\mathbf{M}_{mn}^{(1)}(\mathbf{r},k_q) + b_{mn}^{z}\mathbf{N}_{mn}^{(1)}(\mathbf{r},k_q) + C_{mn}^{z}\mathbf{L}_{mn}^{(1)}(\mathbf{r},k_q)]. \tag{A2c}$$

The coefficients of the plane-wave factors a_{mn}^p, b_{mn}^p, and c_{mn}^p (where $p=x,y,z$) are the functions of θ_k and ϕ_k. Their detailed reduction and formulation have been derived in Ref. [21], and we will provide here only the coefficients of A_{mnq}^p, B_{mnq}^p, and C_{mnq}^p (where $p=e,h$, and $q=1,2$) in Eqs. (20), (27a), and (27b). The coefficients are the following: when $q=1$ and $m\geqslant 0$,

$$A_{mnq}^{e} = i^n\frac{2n+1}{2n(n+1)}\frac{(n-m)!}{(n+m)!}[(n+m)(n-m+1)P_n^{m-1}(\cos\theta_k) - P_n^{m+1}(\cos\theta_k)], \tag{A3a}$$

$$B_{mnq}^{e} = i^n\frac{1}{2n(n+1)}\frac{(n-m)!}{(n+m)!}[(n+1)(n+m)(n+m-1)P_{n-1}^{m-1}(\cos\theta_k) + (n+1)P_{n-1}^{m+1}(\cos\theta_k) + n(n-m+2)\times(n-m+1)P_{n+1}^{m-1}(\cos\theta_k) + nP_{n+1}^{m+1}(\cos\theta_k)], \tag{A3b}$$

$$C_{mnq}^{e} = i^n\frac{1}{2k_g}\frac{(n-m)!}{(n+m)!}[(n+m)(n+m-1)P_{n-1}^{m-1}(\cos\theta_k) + P_{n-1}^{m+1}(\cos\theta_k) - (n-m+2)(n-m+1)P_{n+1}^{m-1}(\cos\theta_k) - P_{n+1}^{m+1}(\cos\theta_k)], \tag{A3c}$$

and $q=1$ and $m>0$,

$$A_{-mnq}^{e} = (-1)^m\frac{(n+m)!}{(n-m)!}A_{mnq}^{e}, \tag{A4a}$$

$$B_{-mnq}^{e} = (-1)^{m+1}\frac{(n+m)!}{(n-m)!}B_{mnq}^{e}, \tag{A4b}$$

$$C_{-mnq}^{e} = (-1)^{m+1}\frac{(n+m)!}{(n-m)!}C_{mnq}^{e}. \tag{A4c}$$

Similarly, when $q=2$ and $m>0$,

$$A_{mnq}^{e} = i^{n+1}\frac{2n+1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\left\{\frac{W_q^{(e)}(\theta_k)}{2}[(n+m)\times(n-m+1)P_n^{m-1}(\cos\theta_k) + P_n^{m+1}(\cos\theta_k)] - mP_n^m(\cos\theta_k)\right\} \tag{A5a}$$

$$B_{mnq}^{e} = i^{n+1}\frac{1}{n(n+1)}\frac{(n-m)!}{(n+m)!}\left\{\frac{W_q^{(e)}(\theta_k)}{2}[(n+1)(n+m)\times(n+m-1)P_{n-1}^{m-1}(\cos\theta_k) - (n+1)P_{n-1}^{m+1}(\cos\theta_k) + n(n-m+2)(n-m+1)P_{n+1}^{m-1}(\cos\theta_k) - nP_{n+1}^{m+1}(\cos\theta_k)] + [n(n-m+1)P_{n+1}^{m}(\cos\theta_k) - (n+1)\times(n+m)P_{n-1}^{m}(\cos\theta_k)]\right\}, \tag{A5b}$$

$$C_{mnq}^{e} = i^n\frac{1}{k_q}\frac{(n-m)!}{(n+m)!}\left\{\frac{W_q^{(e)}(\theta_k)}{2}[(n+m)(n+m-1)P_{n-1}^{m-1}(\cos\theta_k) - P_{n-1}^{m+1}(\cos\theta_k) - (n-m+2)(n-m+1)P_{n+1}^{m-1}(\cos\theta_k) + P_{n+1}^{m+1}(\cos\theta_k)] - (2n+1)\cos\theta_k P_n^m(\cos\theta_k)\right\}, \tag{A5c}$$

and $q=2$ and $m>0$,

$$A_{-mnq}^{e} = (-1)^{m+1}\frac{(n+m)!}{(n-m)!}A_{mnq}^{e}, \tag{A6a}$$

$$B_{-mnq}^{e} = (-1)^{m}\frac{(n+m)!}{(n-m)!}B_{mnq}^{e}, \tag{A6b}$$

$$C_{-mnq}^{e} = (-1)^{m}\frac{(n+m)!}{(n-m)!}C_{mnq}^{e}. \tag{A6c}$$

Similar to the above, the eigenvector expansion coefficients of the **H** field can be obtained.

[1] H. C. Van de Hulst, *Light Scattering by Small Particles* (Dover, New York, 1981).
[2] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941).
[3] A. L. Aden and M. Kerker, J. Appl. Phys. **22**, 1242 (1951).
[4] O. B. Toon and T. P. Ackerman, Appl. Opt. **20**, 3657 (1981).
[5] Z. S. Wu and Y. P. Wang, Radio Sci. **26**, 1393 (1991).
[6] R. J. Tarento, K.-H. Bennemann, P. Joyes, and J. Van de Walle, Phys. Rev. E **69**, 026606 (2004).
[7] You-Lin Geng, Xin-Bao Wu, and Le-Wei Li, Radio Sci. **38**, 1104/1–12 (2003).
[8] R. D. Graglia and P. L. E. Uslenghi, IEEE Trans. Antennas Propag. **AP-32**, 867 (1989).
[9] B. Beker and K. R. Umashankar, Electromagnetics **9**, 215 (1989).
[10] A. Taflove, Wave Motion **10**, 547 (1988).
[11] X. B. Wu and K. Yasumoto, J. Appl. Phys. **82**, 1996 (1997).
[12] R. D. Graglia, P. L. E. Uslenghi, and R. S. Zich, Proc. IEEE

77, 750 (1989).
[13] X. Q. Zhu, Y. L. Geng, and X. B. Wu, Chin. J. Radio Sci. **17**, 209 (2002) (in Chinese).
[14] B. Bekar, Ph.D. thesis, University of Illinois at Chicago, Chicago, IL, 1988.
[15] S. N. Papadakis, N. K. Uzunoglu, and C. N. Capsalis, J. Opt. Soc. Am. A **7**, 991 (1990).
[16] V. V. Varadan, A. Lakhtakia, and V. K. Varadran, IEEE Trans. Antennas Propag. **AP-37**, 800 (1989).
[17] K.-L. Wong and H.-T. Chen, IEE Proc., Part H: Microwaves, Antennas Propag. **139**, 314 (1992).
[18] W. C. Chow, *Waves and Fields in Inhomogeneous Media* (Van Nostrand, New York, 1990).
[19] J. A. Kong, *Theory of Electromagnetic Waves* (Wiley, New York, 1986).
[20] W. Ren, Phys. Rev. E **47**, 664 (1993).
[21] D. Sarkar and N. J. Halas, Phys. Rev. E **56**, 1102 (1997).
[22] K. C. Yeh and C. H. Liu, *Theory of Ionosphere Waves* (Academic Press, New York, 1972).