Similarity solutions of nonlinear diffusion problems related to mathematical hydraulics and the Fokker-Planck equation

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Similarity solutions of the shallow-water equation with a generalized resistance term are studied for open channel flows when both inertial and gravity forces are negligible. The resulting model encompasses various particular cases that appear, in addition to mathematical hydraulics, in diverse physical phenomena, such as gravity currents, creeping flows of Newtonian and non-Newtonian fluids, thin films, and nonlinear Fokker-Planck equations. Solutions of both source-type and dam-break problems are analyzed. Closed-form solutions are discussed, when possible, along with a qualitative study of two phase-plane formulations based on two different variable transformations.

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I. INTRODUCTION

Strongly nonlinear parabolic partial differential equations arise in many problems of hydraulic engineering and hydrology, such as flood propagation in channels and rivers [1–3], groundwater flow [4], catchment and coastal hydrology [5–7], as well as infiltration and subsurface hydrology [8–12]. An investigation of this type of equation is also motivated by the multitude of physical problems having the same mathematical structure; we mention, for example, the applications of the porous-media equation to heat conduction, plasma physics, and non-Newtonian fluid mechanics [13–16], rock-blasting models [17], and gravity currents [18]. These equations also represent special forms of equations that govern many modern problems related to thin film flows [19–21], whose applications encompass different fields of engineering, biology, and chemistry [22–27]. Finally, in the theory of stochastic processes, similar equations are interpreted as nonlinear Fokker-Planck equations [28,29]; as will be seen, in this context self-similar source-type solutions represent important classes of distributions appearing in modern physics (e.g., Lévy-type and Tsallis-type distributions [30,31]).

The aim of this paper is to discuss some similarity solutions of a generalization of an equation that appears in mathematical hydraulics. Within the framework of the shallow-water approximation and in case of negligible inertial terms, the one-dimensional continuity and momentum equations for open-channel flows with an impermeable bed in the case of large cross sections are [1], respectively,

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(uh) = 0, \tag{1}$$

$$\frac{\partial h}{\partial x}\cos\theta = \sin\theta - j, \tag{2}$$

where t is time, x is the spatial coordinate along the channel, h is the water level, u is the velocity averaged over the water depth, $\sin\theta$ is the bed slope, and j (a function of h and u) is the friction slope, which accounts for the resistances to the flow.

In hydraulics, most of the open-channel flows fall in the dynamically rough regime and j is typically modeled using the empirical Chezy law, which for large cross sections reads

$$j = \frac{u|u|}{C^2 h} = \frac{u^2}{C^2 h}\mu, \tag{3}$$

where C is the Chezy coefficient and $\mu = |u|/u$ [1]. The coefficient C in general depends on the cross-section shape and on the roughness of the bed. For most practical applications it is mainly a function of h, so that $j \propto h^{-\beta}$. In particular, for large cross sections β is equal to 1 if C is assumed to be constant, while β is equal to $4/3$ if C is modeled according to the empirical Manning/Gauckler-Strickler formula $C \propto h^{1/6}$ [1].

The structure of Eq. (3) can then be extended to other problems by generalizing the relation between j and u, as

$$j = \frac{u|u|^{\alpha-1}}{k^\alpha h^\beta} = \frac{(\mu u)^\alpha}{k^\alpha h^\beta}\mu, \tag{4}$$

where $\alpha > 0$. As a result, Eqs. (1), (2), and (4) form a general system that, depending on the values of the parameters α and β, can represent very different physical processes. We notice in passing that when $\alpha = 0$, i.e., $j = \mu h^{-\beta}$, h does not depend on time and Eq. (2) admits the exact solution $h = [(\beta+1)(\mu x + C_1)]^{1/(\beta+1)}$, with C_1 integration constant. It is also interesting to notice that under uniform flow conditions (h=const), the relationship between h and $q = uh$, which is known as the rating curve, is a power law given by the bal-

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ance between gravity ($\sin\theta$) and the resistance to flow (j), i.e., $h \propto q^{\alpha/(\alpha+\beta)}$.

In the case of unsteady flow, with the further assumptions of negligible bed slope, $\theta \sim 0$, an equation for h can be obtained by introducing (4) in (2) and substituting the resulting expression for u in (1), i.e.,

$$\frac{\partial h}{\partial t} + \mu \frac{\partial}{\partial x}\left[k h^{(\alpha+\beta)/\alpha}\left(-\frac{\partial h}{\partial x}\mu\right)^{1/\alpha}\right] = 0. \qquad (5)$$

The previous relation includes in itself some classical equations, such as the heat equation ($\alpha=-\beta=1$) and the Boussinesq or porous-media equation ($\alpha=1$, $\beta=0$) [4]. In hydraulics, when $\alpha=2$ and $\beta=1$ or $\beta=4/3$, Eq. (4) corresponds to Chezy's law for very large cross sections. Viscous flows with different rheology can be described with $\beta=\alpha+1$ [26]. The cases with $\beta>-1$, which are typical of hydraulics, may show interesting effects such as fronts that propagate at finite speed or fronts that remain stationary for a finite time before beginning to move, i.e., waiting-time solutions [32]. Moreover, interpreting $h(x,t)$ and $q(x,t)$ as probability density functions (PDFs) and probability currents [33], respectively, Eq. (5) can be read as a nonlinear Fokker-Planck equation for $h(x,t)$.

In what follows we will analyze similarity solutions of Eq. (5) and two possible phase-plane formalisms for their study. Two special cases, the source-type solution and the so-called dam-break problem, will be discussed in detail.

II. SIMILARITY SOLUTIONS

In this section we present a systematic approach to obtain similarity solutions of Eq. (5) for the general case $\alpha>0$ and $-\infty<\beta<+\infty$. As the dimensional quantities governing the space-time evolution of h are t, x, and k, and assuming that the boundary and initial conditions introduce only one additional parameter G, an appropriate class of units of measurement may be used to define all the dimensional quantities involved in the system, consisting of a characteristic scale for h, L_h, a horizontal length L_x, and a time scale T. The scale L_h has different interpretations depending on the physical meaning of h; so, for example, L_h can be a characteristic level in problems related to fluid flows, a characteristic scale of the probability distribution when Eq. (5) is read as a nonlinear Fokker-Planck equation, or a temperature scale when Eq. (5) is the heat equation. We notice that in the problems of interest in the present work, L_h may always be considered to be independent of L_x, even when both of them are lengths, since their ratio L_h/L_x does not appear explicitly in the governing parameters [34].

With these assumptions, the dimensions of h and of the governing parameters can be expressed as

$$[h]=L_h, \quad [t]=T, \quad [x]=L_x,$$

$$[k]=L_x^{(\alpha+1)/\alpha}L_h^{-(\beta+1)/\alpha}T^{-1}, \quad [G]=L_x^{\gamma}L_h^{\delta}T^{\lambda}. \qquad (6)$$

Noticing that t, k, and G are dimensionally independent when $\chi=(\alpha+1)\delta+(\beta+1)\gamma\neq 0$, a relation between two dimensionless variables $\phi=\phi(\xi)$ can be obtained [34], with

$$\phi = \frac{h}{G^{(\alpha+1)/\chi}k^{-\alpha\gamma/\chi}t^{-[\alpha\gamma+\lambda(\alpha+1)]/\chi}},$$

$$\xi = \frac{x}{G^{(\beta+1)/\chi}k^{\alpha\delta/\chi}t^{[\alpha\delta-\lambda(\beta+1)]/\chi}}. \qquad (7)$$

Substituting Eq. (7) into Eq. (5) yields

$$\frac{\alpha\gamma+\lambda(\alpha+1)}{\chi}\phi + \frac{\alpha\delta-\lambda(\beta+1)}{\chi}\xi\phi' - \mu[\phi^{(\alpha+\beta)/\alpha}(-\phi'\mu)^{1/\alpha}]' = 0, \qquad (8)$$

where primes denote derivatives with respect to ξ. In general, the previous relation is not solvable in closed form, but it can be reduced by the substitution [17]

$$X = \xi\frac{(-\phi'\mu)}{\phi}, \quad Y = \xi^2\frac{\alpha\delta-\lambda(\beta+1)}{\chi}\frac{(-\phi'\mu)^{(\alpha-1)/\alpha}}{\phi^{(\alpha+\beta)/\alpha}}, \qquad (9)$$

which maintains the same signs of the original system (ξ,ϕ). Inserting then Eq. (9) into Eq. (8), we obtain the autonomous equation

$$\frac{dY}{dX} = \frac{Y[2X+\sigma\mu(\alpha-1)Y+(1-\alpha)XY+\mu(\alpha+\beta)X^2]}{X[X+\mu\alpha\sigma Y-\alpha XY+\mu(\alpha+\beta+1)X^2]}, \qquad (10)$$

together with

$$ds = \frac{d\xi}{\xi} = \frac{dX}{X+\mu\alpha\sigma Y-\alpha XY+\mu(\alpha+\beta+1)X^2} \qquad (11)$$

and

$$ds = \frac{X\,dY}{Y[2X+\sigma\mu(\alpha-1)Y+(1-\alpha)XY+\mu(\alpha+\beta)X^2]}, \qquad (12)$$

where $s=\ln\xi$ and $\sigma=[\alpha\gamma+\lambda(\alpha+1)]/[\alpha\delta-\lambda(\beta+1)]$, with $\alpha\delta-\lambda(\beta+1)\neq 0$.

Each solution of Eq. (10) represents a particular self-similar current. The solutions of self-similar problems defined by specific initial and boundary conditions are given by one or more curves on the plane (X,Y); to determine which integral curve corresponds to the given problem, it is necessary to study the relation between Y and X about the singular points of the plane (X,Y) and the approximate behavior of $\phi(\xi)$ about them. The complete solution $\phi(\xi)$ can then be derived through one of the relations (9), once the function relating Y and X and one of either Eq. (11) or Eq. (12) are known.

An alternative phase-plane description of the problem, similar to that of [25] (see also [26,35]), can be obtained by choosing a different set of dimensionally independent quantities and studying separately Eqs. (1) and (2). In particular, using x, t, and k, which implies that $\beta\neq-1$, dimensional analysis leads to

$$h = \left[\frac{|x|^{(\alpha+1)/\alpha}}{kt} Z(\eta) \right]^{\alpha/(\beta+1)}, \quad u = \frac{x}{t} V(\eta), \tag{13}$$

where

$$\eta = \frac{|x|^{\chi/(\beta+1)}}{Gk^{\alpha\delta/(\beta+1)} t^{\alpha\delta/(\beta+1)-\lambda}}. \tag{14}$$

Z, V, and η are dimensionless variables; Z and η are always positive, while V has the same sign of xu. If $\chi=0$ and $\lambda=\alpha\delta/(\beta+1)$, η is simply a parameter and Z and V are two constants, determined by the boundary and initial conditions. In the general case where η is a variable dependent on x and t, the expressions corresponding to the continuity and momentum equations can be obtained introducing Eq. (13) in Eqs. (1) and (2), respectively, as

$$\eta \frac{\chi}{\alpha} \frac{dV}{d\eta} = 1 - \frac{\alpha+\beta+2}{\alpha} V - \left(\frac{\chi}{\beta+1} V + \lambda - \frac{\alpha\delta}{\beta+1} \right) \frac{\eta}{Z} \frac{dZ}{d\eta}, \tag{15}$$

$$\mu' \left(\mu' \frac{V}{Z} \right)^{\alpha} + \frac{\alpha}{\beta+1} \left(\frac{\alpha+1}{\alpha} + \frac{\chi}{\beta+1} \frac{\eta}{Z} \frac{dZ}{d\eta} \right) = 0, \tag{16}$$

where $\mu' = |V|/V$, with $\mu\mu' = |x|/x$. Substituting Eq. (16) into Eq. (15), one obtains

$$\frac{dV}{dZ} = \frac{N(Z,V)}{D(Z,V)}, \quad \frac{d\eta}{\eta} = \frac{\chi}{\alpha D(Z,V)} dZ \tag{17}$$

where

$$N(Z,V) = 1 - \frac{\alpha+\beta+2}{\alpha} V - D(V,Z) \left[\frac{\chi}{\beta+1} V - \frac{\alpha\delta}{\beta+1} + \lambda \right] \frac{\alpha}{\chi Z} \tag{18}$$

and

$$D(Z,V) = -Z \frac{\chi}{\alpha} \left[\mu' \left(\mu' \frac{V}{Z} \right)^{\alpha} \frac{(1+\beta)^2}{\chi\alpha} + \frac{(1+\alpha)(1+\beta)}{\chi\alpha} \right]. \tag{19}$$

The solution of Eq. (17), usually numerical, employed in Eq. (13) allows one to find $Z(\eta)$ and thus $h(x,t)$.

Notice that the qualitative study of the equation in the phase plane (Z,V) can be more complex than that of Eq. (10). On the other hand, besides being more directly connected to the physical variables of the physical problem, the phase plane (Z,V) has the advantage of requiring only one integration, instead of two, to go back to the original coordinates.

In what follows the source-type and the dam-break problems will be analyzed. The values of γ, δ, and λ for the source-type solutions lead to an expression of Eq. (8) solvable in closed form for any α and β. The dam-break problem, instead, has analytical solutions for particular values of the parameters α and β. In the other cases, the phase-plane analysis is useful to find the approximate behavior of the solutions about $\xi=0$ and $\xi=\pm\infty$.

III. SELF-SIMILAR SOURCE-TYPE SOLUTIONS

In this section we study the spreading of a finite volume initially concentrated at a point, thus generalizing the classical solutions of the heat and the porous medium equations [16,34,36], as well as the solutions for viscous fluids with power-law rheology [26].

We assume that a given volume per unit width, G, is initially released in the section $x=0$. With the further hypothesis of symmetrical evolution with respect to $x=0$, the analysis can be limited to the semi-plane $x>0$ with $\mu=+1$. Since the dimensions of G are $[G]=L_x L_h$ (i.e., $\gamma=\delta=1$ and $\lambda=0$), Eq. (8) becomes

$$\frac{\alpha}{\alpha+\beta+2} (\xi\phi)' = [\phi^{(\alpha+\beta)/\alpha} (-\phi')^{1/\alpha}]', \tag{20}$$

where

$$\phi = \frac{h}{G^{(\alpha+1)/(\alpha+\beta+2)} (kt)^{-\alpha/(\alpha+\beta+2)}},$$

$$\xi = \frac{x}{G^{(\beta+1)/(\alpha+\beta+2)} (kt)^{\alpha/(\alpha+\beta+2)}}. \tag{21}$$

Equation (20) is an exact differential that can be integrated once to yield

$$\frac{\alpha}{\alpha+\beta+2} \xi\phi - \phi^{(\alpha+\beta)/\alpha} (-\phi')^{1/\alpha} = C_1, \tag{22}$$

where $C_1=0$ for symmetry. Moreover, since the second term of Eq. (22) is positive, the condition $\beta > -(2+\alpha)$ must be satisfied. With a second integration, an analytical solution for $\beta \neq -1$ can be obtained as

$$\phi = \left([C_3(\alpha+1) - C_2 \xi^{\alpha+1}] \frac{\beta+1}{\alpha+1} \right)^{1/(\beta+1)}, \tag{23}$$

with $C_2=[\alpha/(\alpha+\beta+2)]^{\alpha}$. C_3 is given by the condition of constant volume, that for symmetry can be written as $\int_0^{\xi_0} \phi(\xi) d\xi = 1/2$, where, when $\beta > -1$, $\xi_0=[C_3(\alpha+1)/C_2]^{1/(1+\alpha)}$ is the position of the front, while, when $\beta < -1$, ϕ decays with a power-law tail for ξ that tends to ∞ ($\xi_0 \to +\infty$). In both of the cases the expression for C_3 is

$$C_3 = \left[\frac{\Gamma\left(1 + \frac{1}{1+\alpha} + \frac{1}{1+\beta}\right)}{2\Gamma\left(1+\frac{1}{1+\alpha}\right)\Gamma\left(1+\frac{1}{1+\beta}\right)} \right]^{(1+\alpha)(1+\beta)/(2+\alpha+\beta)} \times \left[(1+\beta)^{-1/(1+\beta)} \left(\frac{1+\alpha}{C_2} \right)^{-1/(1+\alpha)} \right]^{(1+\alpha)(1+\beta)/(2+\alpha+\beta)}, \tag{24}$$

where $\Gamma(\cdot)$ is the Gamma function [37].

In the limiting case of $\beta=-1$, the solution of Eq. (22) reads

$$\phi = \frac{1}{2}\frac{\alpha^{1/(1+\alpha)}}{\Gamma\left(\frac{1}{1+\alpha}\right)}\exp\left[-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha}\frac{\xi^{\alpha+1}}{\alpha+1}\right] \tag{25}$$

and contains the Gaussian distribution for $\alpha=1$ (heat equation).

The passage from solutions with fronts to solutions without fronts as β decreases is shown in Fig. 1(a) for a fixed value of α. Figure 1(b) shows the role of α: as α increases, the resistances to the flow are higher and the velocity is reduced. As a consequence, when α is higher the product $(\phi)^{(\alpha+\beta)/\alpha}(-\phi')^{1/\alpha}$, which is proportional to the volumetric flux $q=uh$, is lower and thus the curves in Fig. 1(b) tend to flatten.

When Eq. (5) is interpreted as a nonlinear Fokker-Planck equation, the source-type solutions (23) and (25) represent the transient evolution of the PDF of a process that starts with probability equal to 1 from the same initial condition. Accordingly, for $\alpha=1$ and for different values of the parameter β, $h(x,t)$ corresponds to Tsallis-type distributions, that appear in nonextensive statistical mechanics [28,30]. The support for $h(x,t)$ is compact when $\beta>-1$ [Eq. (23)], while $h(x,t)$ has exponential tails for $\beta=-1$ [Eq. (25)] and power-law tails for $\beta<-1$ [Eq. (23)]. In particular, with an appropriate rescaling, $h(x,t)$ is a Cauchy (or Lorentz) distribution when $\beta=-2$ and a Student's t distribution of degree ω when $\beta=-(\omega+3)/(\omega+1)$, with $-3<\beta<-1$.

In closing this section, we notice that solutions formally equal to Eq. (23) and (25), but with different values of constants and parameters, can be obtained whenever the left-hand side of Eq. (8) is an exact differential, i.e., when $\lambda=\alpha(\delta-\gamma)/(\alpha+\beta+2)$. In these cases the value of the integration constants is related to the nature of the parameter G, which in general may also be time dependent.

IV. DAM-BREAK PROBLEM

The classical dam-break problem is the study of a flow in plane geometry generated by the removal of a wall separating two pools of different depth h_1 and h_2 [3,38–40]. Without loss of generality, the dam may be assumed to be at $x=0$.

In this problem the boundary conditions are formally defined by two external parameters G and G^*, which can be chosen to be, for example, h_1 and h_2. The external parameter G is thus a constant height scale ($[G]=L_h$), so that γ and λ are zero, while $\delta=1$. In general, $h=h(t,x,k,G,G^*)$ so that dimensional analysis leads to a relation $\phi=\phi(\xi,\Pi^*)$, where

$$\phi=\frac{h}{G}, \quad \xi=\frac{x}{G^{(\beta+1)/(\alpha+1)}(kt)^{\alpha/(\alpha+1)}}, \quad \Pi^*=\frac{G^*}{G}. \tag{26}$$

Π^* is a parameter that does not enter explicitly in the equation, but defines only the boundary conditions of the similarity process. Accordingly, for the dam-break problem, Eq. (8) becomes

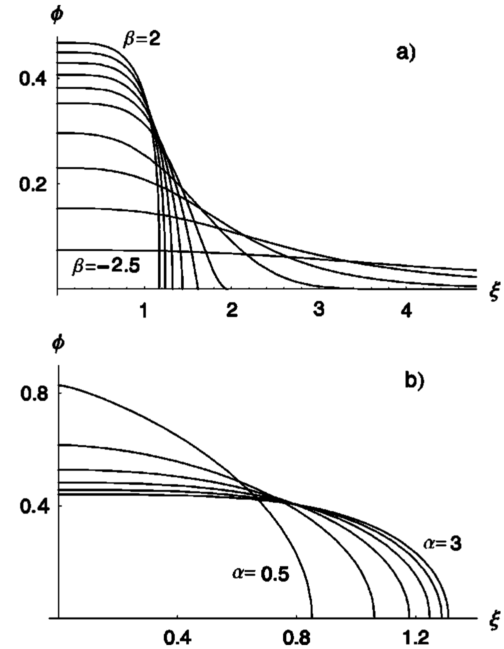

FIG. 1. Profiles of source-type solutions, Eq. (23), with (a) $\alpha=2, -5/2\leqslant\beta\leqslant 2$ and (b) $\beta=1, 1/2\leqslant\alpha\leqslant 3$.

$$\frac{\alpha}{\alpha+1}\xi\phi' = \mu[\phi^{(\alpha+\beta)/\alpha}(-\phi'\mu)^{1/\alpha}]'. \tag{27}$$

Analytical solutions of Eq. (27) exist only for particular values of the parameters. In the other cases numerical procedures are necessary to have quantitative results, while a qualitative behavior of the solutions can be obtained by studying the two phase planes (X,Y) and (Z,V). In the following some analytical solutions for special values of α and β are presented along with the study of both phase planes for the particular case of the Chezy law ($\alpha=2$, $\beta=1$).

Again we notice that the results obtained for the dam break can be extended to all the situations in which $\sigma=0$, i.e., $\lambda=-\alpha\gamma/(\alpha+1)$, since the structure of the equation describing the problem remains the same of Eq. (27). In these cases the boundary and initial conditions may also depend on time according to the dimensions of G.

A. Analytical solutions

Analytical solutions of Eq. (27) can be obtained when $\beta=-\alpha$, in which case

$$\frac{\alpha}{\alpha+1}\xi\phi' = \mu[(-\phi'\mu)^{1/\alpha}]' = -\frac{1}{\alpha}(-\phi'\mu)^{(1-\alpha)/\alpha}\phi''. \tag{28}$$

When $\alpha=-\beta=1$, Eq. (5) is the well known heat equation and the solution of Eq. (28) is given by $\phi=C_1\text{erf}(\xi^2/4)+C_2$, where C_1 and C_2 are constants obtained imposing the value of ϕ when ξ goes to $\pm\infty$, and $\text{erf}(\cdot)$ is the error function [37], i.e., the integral of the Gaussian distribution.

When $\alpha\neq 1$, integrating Eq. (28) once leads to

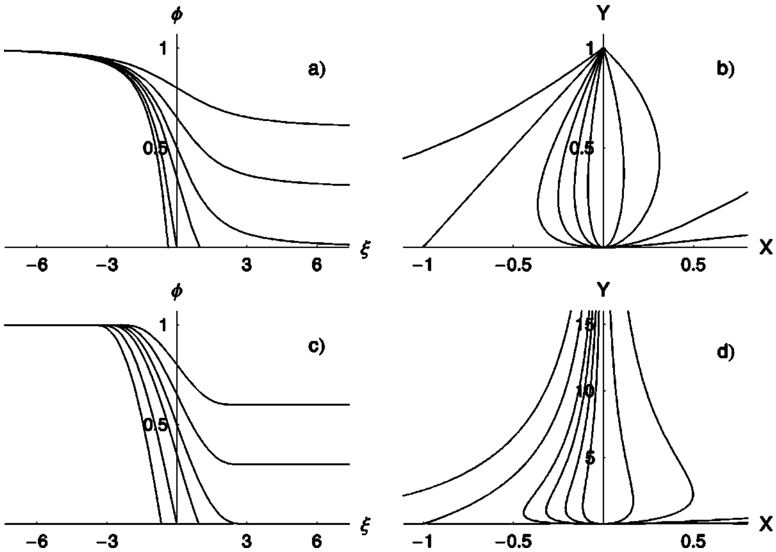

FIG. 2. Profiles and relative phase plane of the dam-break problem when $\beta=-\alpha$ [Eq. (30)]; (a),(b) $\alpha=3$ and (c),(d) $\alpha=2/3$.

$$\phi' = -\mu\left[\frac{1-\alpha}{\alpha}\left(C_3 - \frac{\alpha^2}{\alpha+1}\frac{\xi^2}{2}\right)\right]^{\alpha/(1-\alpha)}, \tag{29}$$

C_3 being an integration constant. When $\alpha>1$, $\alpha/(1-\alpha)<0$ and ϕ' tends to 0 for $\xi=\pm\infty$; the constant sign of ϕ' assures that the flow follows the same direction along the entire x axis. When $\alpha<1$, $\phi'=0$ for $\xi=\pm\xi_0=\pm\sqrt{2(\alpha+1)C_3/\alpha}$. In this situation the flow is undisturbed upstream of $-\xi_0$ and downstream of $+\xi_0$ and the free surface passes from the level h_1 to h_2 in the interval limited by $-\xi_0$ and $+\xi_0$, maintaining at those points the same derivative $\phi'(\pm\xi_0)=0$, but having a discontinuity in $\phi''(\pm\xi_0)$.

A second integration offers the general solution of Eq. (28) as

$$\phi = C_4 - \mu\xi\left(C_3\frac{1-\alpha}{\alpha}\right)^{\alpha/(1-\alpha)} \times {}_2F_1\left(\frac{1}{2},\frac{\alpha}{1-\alpha};\frac{3}{2};\frac{\alpha^2\xi^2}{2C_3(1+\alpha)}\right), \tag{30}$$

where C_3 and C_4 are integration constants, which can be obtained by imposing the value of ϕ at $\pm\infty$ or at $\pm\xi_0$ when, respectively, $\alpha>1$ and $\alpha<1$; ${}_2F_1(\cdot,\cdot;\cdot;\cdot)$ is the hypergeometric function [37].

Figures 2(a) and 2(c) show different forms of the solution for $\mu=+1$, i.e., for flow in the ξ direction. G is assumed to be the level h_1 at $-\infty$, so that ϕ tends to 1 when ξ either goes to $-\infty(\alpha>1)$ or goes to $-\xi_0(\alpha<1)$, while ϕ tends to h_2/h_1 when ξ goes either to $+\infty(\alpha>1)$ or to $+\xi_0(\alpha<1)$. Lowering the downstream level from 1 to 0 gives different profiles connecting smoothly the two levels; as already mentioned, when $\alpha<1$ the second derivative of the profile in $\pm\xi_0$ is not continuous. Other self-similar profiles may be obtained by imposing $\phi=0$ at a certain ξ, which physically correspond to the case of a drain at a fixed ξ. Interestingly, in groundwater hydraulics, when Darcy's law is employed $(\alpha=1,\beta=0)$, the problem of a drain at $\xi=0$, i.e., fixed in the physical coordinates (x,t), is mathematically equivalent to the boundary-layer problem studied by Blasius [38,41].

Since the structure of the phase plane (X,Y) remains similar for other values of α and β, it is useful to analyze the phase plane in the case $\beta=-\alpha$, which can be obtained exactly. In the plane of Fig. 2(b), where $\alpha=3$, the origin and the point $(-1,0)$ correspond to $\xi=0$, while $(X,Y)=(0,1)$ is reached when ξ tends to $\pm\infty$. When $X<0$, the lines connecting the point (0, 1) to the origin are profiles from $\phi=1$ at $-\infty$ to different values of ϕ at $\xi=0$; for $X>0$, instead, the curves from (0, 0) to (0, 1) correspond to free surfaces from a certain value at $\xi=0$ to $\phi=h_2/h_1$ for $\xi=+\infty$. When $\phi=0$ for $\xi=+\infty$, the corresponding curve in the phase plane is that from (0, 0) to a saddle at (2, 1)—not shown in the figure. The flow from $\phi=1$ when $\xi=-\infty$ to $\phi=0$ at $\xi=0$ is represented by the curve joining the points (0, 1) and $(-1,0)$ in the plane (X,Y).

For $\alpha<1$ the phase-plane structure changes [Fig. 2(d)]. When $\phi(+\xi_0)$ is positive, the position of the interfaces at $+\xi_0(X>0)$ and $-\xi_0(X<0)$ moves to the point $(0,+\infty)$ in the plane (X,Y). The point $\phi(+\xi_0)=0$ is located at $(+\infty,+\infty)$ in the plane (X,Y). Finally, the profile starting from $\phi=1$ at $\xi=-\xi_0$ and reaching $\phi=0$ at $\xi=0$ is represented by the curve starting from $(-1,0)$ in the plane (X,Y).

A different class of analytical solutions can be obtained solving in a closed form the equation defining the phase plane. For the dam-break problem this corresponds to [see Eq. (10)]

$$\frac{dY}{dX} = \frac{Y[2+(1-\alpha)Y+\mu(\alpha+\beta)X]}{X[1-\alpha Y+\mu(\alpha+\beta+1)X]}. \tag{31}$$

Following [42] (see also [15]), analytical solutions of Eq. (31) can be found when $\beta=-(3+\alpha)/2$ and $\alpha\neq 1$ as

$$X^2Y^{-1}\left[1+\frac{1-\alpha}{2}Y+\mu\frac{\alpha-1}{2}X\right]^{(\alpha+1)/(1-\alpha)} = \text{const}. \tag{32}$$

When $\alpha>1$, the phase planes (not shown) have similar behaviors to that analyzed in Fig. 2(b); in particular, when $\alpha=-\beta=3$, Eq. (32) describes exactly the phase plane of Fig.

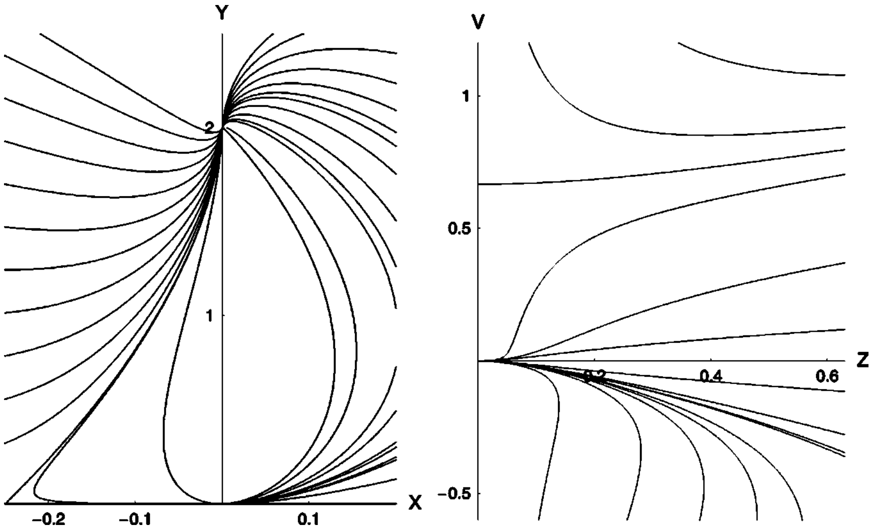

FIG. 3. Comparison of the two planes (X,Y) and (Z,V), for Chezy's law $(\alpha=2,\beta=1)$, in the case of the dam-break problem.

2(b). For $\alpha<1$, the phase planes are similar to that of Fig. 2(d), with the only difference that now there is a saddle in $(2\mu,1)$, which corresponds to a free surface that goes to $\phi=0$ when $\xi=+\infty$.

When $\alpha=1$ and $\beta=-(3+\alpha)/2=-2$, Eq. (31) can be solved with separation of variables and the solution, in implicit form, is

$$Y\exp(-Y)=C_1X^2\exp(-\mu X), \tag{33}$$

with C_1 an integration constant. The behavior of the system in this last particular situation is analogous to the cases when $\alpha<1$ [Fig. 2(d)].

B. Phase-plane analysis for Chezy's law

Because of its frequent appearance in physical applications concerning diffusion problems, Eq. (31) has been studied in various contexts [15,17,42,43]. An extended qualitative study of the nature of each singularity for different parameter values can be found in [44] and references therein. Here we will analyze the phase plane (X,Y) for the dam-break problem for the case in which the resistances follow Chezy's law (i.e., $\alpha=2$ and $\beta=1$). The results of the phase plane (X,Y) will be then compared to those obtained from the plane (Z,V).

In this case and with $\mu=1$, Eq. (31) and Eq. (11) are

$$\frac{dY}{dX}=\frac{Y(2-Y+3X)}{X(1-2Y+4X)} \tag{34}$$

and

$$\frac{d\xi}{\xi}=\frac{dX}{X(1-2Y+4X)}, \tag{35}$$

with

$$X=\xi\frac{(-\phi')}{\phi},\quad Y=\frac{2}{3}\xi^2\frac{(-\phi')^{1/2}}{\phi^{3/2}}. \tag{36}$$

A full representation of the numerical solutions of Eq. (34) is shown in Fig. 3(a). From the analytical point of view, Eq. (34) presents seven singular points. Only four of them will be studied, since they are connected to specific problems of practical interest.

The origin is an unstable node; the study of Eq. (34) linearized about it leads to $Y\sim A_0X^2$, where A_0 is an arbitrary parameter. Linearizing Eq. (35) about $(X,Y)=(0,0)$ and integrating gives $\xi\sim A_1X$, where A_1 is another parameter. It follows that, for $\xi=0$, $X=Y=0$. Substituting the previous results into Eq. (36), we obtain that, about $\xi=0$, ϕ is approximately given by $\phi=A_2\exp(-\xi/A_1)$, where A_2 is the integration constant. As a result, ϕ tends to the finite value A_2 as ξ goes to zero.

The behavior of the system about the stable node $(X,Y)=(0,2)$ can be studied by shifting the origin to this point and proceedings as before. Linearizing then about the origin of the new coordinate system, we get $Y\sim-6X+A_3X^{2/3}+2$ and $\xi\sim A_4X^{-1/3}$, with A_3 and A_4 arbitrary parameters. So, when ξ tends to $-\infty$ and $+\infty$, X goes to zero from negative and positive values, respectively, while Y tends to 2. Furthermore, from Eq. (36) it follows that $\phi=A_5\exp(A_4^3/3\xi^3)$, which means that ϕ tends to a finite value when ξ goes to infinity.

Thus, the curves joining the two points (0, 2) and (0, 0) in the plane (X,Y) correspond to self-similar profiles from $\xi=\pm\infty$ to $\xi=0$.

The point $(X,Y)=(-1/4,0)$ is a saddle, about which $Y\sim9(X+1/4)/2$, so that $\xi\sim A_6(X+1/4)^{4/5}$ and $\phi=A_7\exp[4(-\xi/A_6)^{5/4}](\xi/A_6)^{1/4}$, where $A_6<0$ and $A_7>0$ are integration constants. The curve from (0, 2) to $(-1/4,0)$ represents a self-similar flow going from $\phi=1$ at $\xi=-\infty$ to $\phi=0$ at $\xi=0$.

The last point analyzed is the saddle $(X,Y)=(+\infty,+\infty)$ in the direction $Y/X=1$. Its analysis is more complex than the other points. To this regard, the substitution $X=1/x$ and $Y=1/y$ leads to an equation whose dominant terms are quadratic in both x and y. Following [45], the behavior of the system about this point is found to be $(7X-3Y)=A_8X^{21/2}Y$. From Eq. (35) one obtains $\xi=X/(A_8X+1)$ and, with Eq. (36), $\phi=A_9(A_8\xi-1)$. This point corresponds to a front in the coordinates (ϕ,ξ), where the depth of the current ϕ goes to zero at a finite value of ξ. Accordingly, the curve connecting the origin of the plane (X,Y) to $(+\infty,+\infty)$ is a current from a particular value of ϕ at $\xi=0$ to $\phi=0$ at $\xi=1/A_8$.

It is instructive to compare the previous analysis in the plane (X,Y) with that obtained using the phase plane (Z,V) defined by Eq. (17) [Fig. 3(b)]. With the parameters corresponding to Chezy's law, Eq. (17) is

$$\frac{dV}{dZ}=\frac{2V[3Z^2+\mu'(2-3V)V]}{3Z(2\mu' V^2+3Z^2)}. \qquad (37)$$

A qualitative study of the previous equation results more complex than that of Eq. (34), since the degree of the dominant terms of both numerator and denominator is higher; moreover, given the presence of μ', Eq. (37) has two different formulations for V respectively higher and lower than zero. This makes it more difficult to find conditions that may be useful in the numerical integration of the equation. On the other hand, Z and V are more directly related to the physical variables h and u and therefore their interpretation is somehow easier. In particular, Z is always positive, while the sign of V is equal to that of η. When η goes to ∞, Z tends to zero and vice versa, so that the curves going from the origin to the point $(+\infty,0)$ correspond to profiles from $\eta=\pm\infty$ to $\eta=0$. The curve that in the semiplane $V<0$ separates the curves that reach $(+\infty,0)$ from those that go to $(0,-\infty)$ is the profile that, starting from $\phi=1$ at $\eta=-\infty$, reaches 0 when $\eta=0$ [25]. The point $(0,2/3)$ has a finite velocity when $h=0$ and it thus represents a front in the physical coordinates.

It should be noticed that, while in general the approach using the phase plane (Z,V) is more complicated than the one using (X,Y), in those cases where the degree of the dominant terms is equal for both the phase planes, the analysis in the plane (Z,V) may be more efficient. This is the case of $\alpha=1$, which finds applications in groundwater hydrology and Newtonian viscous fluids [25,43]. On the other hand, for the classic case of the heat equation the description with the plane (Z,V) is not possible.

V. CONCLUSIONS

We have studied self-similar solutions of a nonlinear differential equation obtained by generalizing the relation usually employed in hydraulics to model the resistances in open channel flows. The resulting equation admits a wide range of self-similar solutions, which, for certain values of the two parameters α and β, include already known solutions. Figure 4 summarizes the relation between the particular physical processes and the couples of α and β for which analytical solutions of the source-type and the dam-break problems have been found.

Solutions of the source-type problem exist for values of α and β satisfying the condition $\beta>-(2+\alpha)$, that is for couples of values in the region of the semiplane $\alpha>0$ above the line d reported in Fig. 4. When $\beta\leqslant-1$, the finite volume of the concentrated source spreads immediately to infinite distance with tails that decay exponentially, when $\beta=-1$, and algebraically, when $\beta<-1$. The point H reported in Fig. 4 corresponds to the heat equation, while the vertical segment e corresponds to Student's t distributions of different degrees

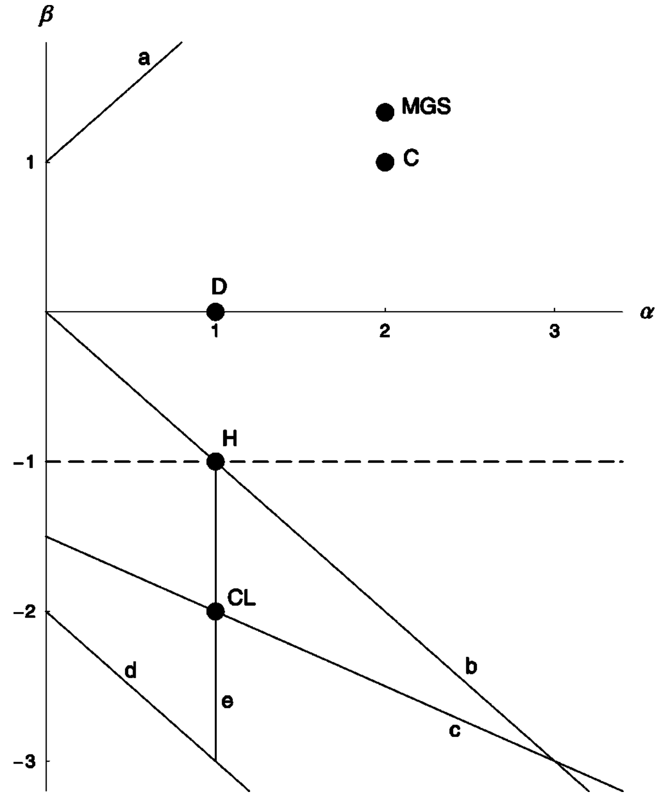

FIG. 4. Relation between different couples of α and β and particular analytical solutions of the source-type and the dam-break problems (see text for details).

of freedom; in particular the Cauchy (or Lorentz) distribution is the solution of the source-type problem when $\alpha=1$ and $\beta=-2$ (point CL). When $\beta>-1$, fronts occur and the solution has compact support. In such a region, various couples of the parameters can describe different problems, such as the porous-media equation (point D in Fig. 4) [34], the motion of a finite volume of water following Chezy's law (point C in Fig. 4) or the Manning–Gauckler-Strickler law (point MGS in Fig. 4), and the flows of non-Newtonian viscous fluids (line a in Fig. 4) [26].

Similarly to the source-type solutions, also the dam-break problem has solutions with an infinite celerity of the perturbation when $\beta\leqslant-1$, while fronts or interfaces arise for $\beta>-1$. Analytical solutions of the problem have been found for couples of values of the two parameters lying on the lines b and c in Fig. 4 [see Eqs. (30), (32), and (33)]. When the governing equation does not have exact solutions, the study of the problem can be carried out qualitatively through the phase-plane analysis employing some suitable variable transformation. The approximate behavior of the system about critical points is useful to determine the boundary conditions for the numerical integration and to obtain an overall understanding of the families of self-similar solutions for different parameter values.

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