

Waves in a reaction-transport system with memory, long-range interactions, and transmutations

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We develop a theory of wave propagation into an unstable state for a system of integral equations with memory, long-range interactions, and transmutations. In particular we use continuous-time random walk theory to describe the transport and transmutation processes. We use a hyperbolic scaling and Hamilton-Jacobi formalism to derive formulas for the speed of propagation of the traveling wave generated by the system in the long-time large-distance limit. Our theory is valid for arbitrary waiting-time, jump-length and, transmutation probability density functions and the propagation speed can generally be found numerically. However, we illustrate our theory by considering an example where analytic results are possible—that is, for a system of Markovian reaction-transport equations. We derive formulas to determine the propagation speed in both the so-called weakly coupled and strongly coupled cases.

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I. INTRODUCTION

Although in recent years there has been considerable progress in the modeling of complex biological, chemical, and physical systems in terms of interacting particle models, there are still many problems with the scaling limits of large systems [1], in particular the scaling problem for the long-time large-distance description of wave propagation into an unstable state of reaction-transport systems [2,3]. This problem has attracted considerable interest due to the large number of physical, chemical, and biological problems that can be treated in terms of wave propagation into an unstable state. A generic model, which describes these phenomena, is the Fisher-Kolmogorov-Petrovskii-Piskunov (FKPP) equation [4]. It was originally introduced to investigate the spread of advantageous genes. Since then, it has been widely used to describe combustion waves, population growth and dispersion, the spread of epidemics, propagation of a vortex front in an unstable fluid flow and magnetic fronts in disk dynamos, etc. [2–9]. Recently, there has been a tremendous amount of activity in extending the FKPP equation by introducing more realistic macroscopic descriptions of the transport processes [7–10]. It has been recognized that the deficiency of the FKPP equation is that it implicitly involves a long-time large-distance parabolic scaling, while as far as propagating fronts are concerned, the appropriate scaling is more likely to be hyperbolic [3]. The key point about unstable states is that they are very sensitive to small disturbances. While on average transport processes may behave diffusively, unstable media are more affected by the weak tails of transport processes which can behave quite differently.

The extensions mentioned, however, have only been concerned with a single integro-differential equation. One needs to develop the theory of wave propagation into an unstable state for a system of integro-differential equations, since this would allow a more realistic modeling of various phenomena in physics, chemistry, biology, etc. Most of the problems of real interest are described by systems of reaction-transport equations, rather than that of a single equation. Such systems of equations often have a far richer structure than their single

counterpart, but in general there is no analytical closed-form solution. The advantage in considering such systems is that it will allow us to take into account (i) realistic multicomponent cases, (ii) long-range interactions in space and in time, and (iii) transmutations. Long-range interactions are a significant feature in many areas of physics, chemistry, and biology, but may often be ignored through the difficulties of how to deal with them. General theory for the derivation of reaction-transport equations with distributed delay has been recently developed by Vlad and Ross [11]. They introduced the nonlinear age-dependent equations such that the transport is described by the continuous-time random walk, while the interactions between species are described by nonlinear transformation rates. It should be noted that the nonlocal evolution equations for multiple age variables were introduced in population dynamics in [12].

The primary objectives of this paper are (i) to develop a theory of wave propagation into an unstable state for the complex system of integral equations and (ii) to analyze stochastic transport involving non-Markovian random processes with long-range interactions and transmutations. We analyze the dynamics of fronts for these equations using a geometrical optics approach involving hyperbolic scaling and Hamilton-Jacobi techniques.

II. MESOSCOPIC EQUATIONS

The purpose of this section is to give a *mesoscopic* description of the complex reaction-transport system in terms of a system of integral equations incorporating memory effects, long-range interactions in space, and transmutations. The transport process is described by the continuous-time random walk (CTRW) model [10,11,13], while the reaction is assumed to be of KPP type [5].

Suppose that we have two different types of particles, 1 and 2, say. We introduce the concentrations of particles 1 and 2 at time t and position x : $n_1(t, x)$ and $n_2(t, x)$, respectively. We assume that for particle i the waiting time between jumps is random and the length of the jump is also random. Let us denote by $\psi_i(t)$ the probability density function (PDF) for the

waiting time and $\varphi_i(z)$ the PDF for the length of the jumps.

The mutation process is described as follows. Let us suppose that while the particles wait between successive jumps, particles of type 1 transmute into particles of type 2 after a random time given by the PDF $\beta_1(t)$. Similarly, the opposite transmutation of $2 \rightarrow 1$ occurs after a random time given by the PDF $\beta_2(t)$.

The concentration of the particles 1 and 2, $n_1(t, x)$ and $n_2(t, x)$, can then be described through the probabilistic-balance-type equations

$$\begin{aligned} n_1(t, x) = & n_1(0, x)\Psi_1(t)B_1(t) + \int_0^t \int_{-\infty}^{\infty} n_1(t-s, x-z) \\ & \times \varphi_1(z)\psi_1(s)B_1(s)dzds + \int_0^t f_1(n_1, n_2)n_1(t-s, x) \\ & \times \Psi_1(s)B_1(s)ds + \int_0^t n_2(t-s, x)\Psi_1(s)\beta_1(s)ds \quad (1) \end{aligned}$$

and

$$\begin{aligned} n_2(t, x) = & n_2(0, x)\Psi_2(t)B_2(t) + \int_0^t \int_{-\infty}^{\infty} n_2(t-s, x-z) \\ & \times \varphi_2(z)\psi_2(s)B_2(s)dzds + \int_0^t f_2(n_1, n_2)n_2(t-s, x) \\ & \times \Psi_2(s)B_2(s)ds + \int_0^t n_1(t-s, x)\Psi_2(s)\beta_2(s)ds, \quad (2) \end{aligned}$$

where we have introduced the new notations

$$\Psi_i(t) = \int_t^{\infty} \psi_i(s)ds, \quad i = 1, 2, \quad (3)$$

the probability that a particle i makes no jump over the interval $(0, t)$, and

$$B_i(t) = \int_t^{\infty} \beta_i(s)ds, \quad i = 1, 2, \quad (4)$$

the probability that a particle i does not transmute over the interval $(0, t)$. In what follows we assume that the local growth rate $f_i(n_1, n_2)$ is of KPP type:

$$U_i = \sup_{n_1, n_2 > 0} \{f_i(n_1, n_2)\} = f_i(0, 0). \quad (5)$$

Let us now discuss the meaning of Eqs. (1) and (2). Consider Eq. (1), which describes the balance of particles of type 1 at time t and position x . The first term on the right-hand side, $n_1(0, x)\Psi_1(t)B_1(t)$, represents the probability that the concentration of particles 1 at time t and position x is just the initial concentration, which can only happen provided that no jump has occurred and that no transmutation takes place. Due to the independence of the random waiting times and the transmutation process, this probability is given by $\Psi_1(t)B_1(t)$. The second term $\int_0^t \int_{-\infty}^{\infty} n_1(t-s, x-z)\varphi_1(z)\psi_1(s)B_1(s)dzds$ represents the probability that a par-

ticle of type 1 at time $t-s$ and position $x-z$ waits a time s before jumping a distance z and remains a particle of type 1 [i.e., not transmutating over the interval $(0, s)$]. The third term $\int_0^t f_1(n_1, n_2)n_1(t-s, x)\Psi_1(s)B_1(s)ds$ describes the growth rate of particle 1, which occurs provided that no jump takes place—i.e., no loss of the particles 1 and thus no transmutation from 1 to 2. The last term $\int_0^t n_2(t-s, x)\Psi_1(s)\beta_1(s)ds$ represents the probability that over the time interval $(0, s)$ particles of type 1 seeks to transmute to particles of type 2, which can only happen provided no jump takes place [hence the $\Psi_1(s)$ term]. It should be noted that the system (1) and (2) is derived by using probabilistic methods, but it is not a stochastic system. It does not take into account the random fluctuations of the species.

III. INTEGRO-DIFFERENTIAL EQUATIONS WITH MEMORY

Let us note that the system of equations (1) and (2) can be rewritten in terms of a system of integro-differential equations (see Appendix A):

$$\begin{aligned} \frac{\partial n_1(t, x)}{\partial t} = & \int_0^t \alpha_1(t-s) \int_{-\infty}^{\infty} [n_1(s, x-z) - n_1(s, x)]\varphi_1(z)dzds \\ & + \int_0^t \zeta_1(t-s)[n_2(s, x) - n_1(s, x)]ds + f_1(n_1, n_2)n_1, \quad (6) \end{aligned}$$

$$\begin{aligned} \frac{\partial n_2(t, x)}{\partial t} = & \int_0^t \alpha_2(t-s) \int_{-\infty}^{\infty} [n_2(s, x-z) - n_2(s, x)]\varphi_2(z)dzds \\ & + \int_0^t \zeta_2(t-s)[n_1(s, x) - n_2(s, x)]ds + f_2(n_1, n_2)n_2, \quad (7) \end{aligned}$$

where the “memory” kernels $\alpha_i(t)$ and $\beta_i(t)$ are defined in the following manner. If we let

$$f_i(t) = \psi_i(t)B_i(t), \quad g_i(t) = \Psi_i(t)\beta_i(t) \quad \text{for } i = 1, 2, \quad (8)$$

then

$$\tilde{\alpha}_i(u) = \frac{u\tilde{f}_i(u)}{1 - \tilde{f}_i(u) - \tilde{g}_i(u)}, \quad \tilde{\zeta}_i(u) = \frac{u\tilde{g}_i(u)}{1 - \tilde{f}_i(u) - \tilde{g}_i(u)} \quad \text{for } i = 1, 2, \quad (9)$$

where the Laplace transform of a function $k(t)$ is denoted by $\tilde{k}(u)$:

$$\tilde{k}(u) = \int_0^{\infty} k(t)e^{-ut}dt.$$

It is important to note that for arbitrary choices of waiting-time and transmutation PDFs, it may prove impossible to determine the inverse Laplace transform of the memory kernels (9). Our methodology will depend only upon the system of master equations (1) and (2) involving

the waiting-time and transmutation PDFs, without resort to any such memory kernels. The integro-differential approach is preferable only in the simplest of cases. Let us consider several nontrivial examples involving different assumptions on the waiting time and transmutation PDFs.

A. Markov random walk

Let us suppose that the waiting-time and mutation PDFs are exponentially distributed:

$$\psi_1(t) = \psi_2(t) = \lambda e^{-\lambda t}, \quad \beta_i(t) = \gamma_i e^{-\gamma_i t}, \quad i = 1, 2. \quad (10)$$

Then the appropriate substitutions of Eqs. (10) into Eqs. (1) and (2) or equivalently into Eqs. (6) and (7) give (see Appendix B)

$$\begin{aligned} \frac{\partial n_1}{\partial t} = & \lambda \int_{-\infty}^{\infty} [n_1(t, x-z) - n_1(t, x)] \varphi_1(z) dz + f_1(n_1, n_2) n_1 \\ & + \gamma_1 (n_2 - n_1), \end{aligned} \quad (11)$$

$$\begin{aligned} \frac{\partial n_2}{\partial t} = & \lambda \int_{-\infty}^{\infty} [n_2(t, x-z) - n_2(t, x)] \varphi_2(z) dz + f_2(n_1, n_2) n_2 \\ & + \gamma_2 (n_1 - n_2). \end{aligned} \quad (12)$$

In the diffusion limit, we can expand $n_i(t, x-z)$ by the Taylor series to get a classical reaction-diffusion system

$$\frac{\partial n_1}{\partial t} = \frac{D_1}{2} \frac{\partial^2 n_1}{\partial x^2} + f_1(n_1, n_2) n_1 + \gamma_1 (n_2 - n_1), \quad (13)$$

$$\frac{\partial n_2}{\partial t} = \frac{D_2}{2} \frac{\partial^2 n_2}{\partial x^2} + f_2(n_1, n_2) n_2 + \gamma_2 (n_1 - n_2), \quad (14)$$

where the diffusion coefficient D_i is determined as

$$D_i = \lim_{\lambda \rightarrow \infty} \lambda \sigma_i^2, \quad \sigma_i^2 = \int z^2 \varphi_i(z) dz. \\ \sigma_i^2 \rightarrow 0$$

Here it was assumed that $\int z \varphi_i(z) dz = 0$.

B. Non-Markov random walk

Let us consider an example involving memory effects, in particular via the transport process. Let us again suppose that the transmutation PDF are exponential, but the waiting-time PDF are given by the following member of the Gamma family [14]:

$$\psi_1(t) = \psi_2(t) = \lambda^2 t e^{-\lambda t}, \quad \beta_i(t) = \gamma_i e^{-\gamma_i t}, \quad i = 1, 2. \quad (15)$$

Then the system of equations (1) and (2) can be rewritten as (see Appendix C)

$$\begin{aligned} \frac{\partial n_1}{\partial t} = & \lambda^2 \int_0^t e^{-(2\lambda + \gamma_1)s} \int_{-\infty}^{\infty} [n_1(t-s, x-z) \\ & - n_1(t-s, x)] \varphi_1(z) dz ds + f_1(n_1, n_2) n_1 + \gamma_1 (n_2 - n_1), \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\partial n_2}{\partial t} = & \lambda^2 \int_0^t e^{-(2\lambda + \gamma_2)s} \int_{-\infty}^{\infty} [n_2(t-s, x-z) \\ & - n_2(t-s, x)] \varphi_2(z) dz ds + f_2(n_1, n_2) n_2 + \gamma_2 (n_1 - n_2). \end{aligned} \quad (17)$$

It is clear from Eqs. (16) and (17) that unlike the previous example, the transport process is now dependent upon the past history of the concentration of particles. The above equations can be reduced further to the following system of coupled hyperbolic reaction-diffusion equations (see Appendix D):

$$\begin{aligned} \tau \frac{\partial^2 n_1}{\partial t^2} + \left(1 - f_1 \tau - n_1 \frac{\partial f_1}{\partial n_1} \tau + 2 \gamma_1 \tau \right) \frac{\partial n_1}{\partial t} - \tau \left(n_1 \frac{\partial f_1}{\partial n_2} + \gamma_1 \right) \frac{\partial n_2}{\partial t} \\ = \frac{D_1}{2} \frac{\partial^2 n_1}{\partial x^2} + (1 + \gamma_1 \tau) f_1(n_1, n_2) n_1 + \gamma_1^2 \tau (n_2 - n_1), \end{aligned} \quad (18)$$

$$\begin{aligned} \tau \frac{\partial^2 n_2}{\partial t^2} + \left(1 - f_2 \tau - n_2 \frac{\partial f_2}{\partial n_2} \tau + 2 \gamma_2 \tau \right) \frac{\partial n_2}{\partial t} - \tau \left(n_2 \frac{\partial f_2}{\partial n_1} + \gamma_2 \right) \frac{\partial n_1}{\partial t} \\ = \frac{D_2}{2} \frac{\partial^2 n_2}{\partial x^2} + (1 + \gamma_2 \tau) f_2(n_1, n_2) n_2 + \gamma_2^2 \tau (n_1 - n_2), \end{aligned} \quad (19)$$

where

$$\tau = \frac{1}{2\lambda}$$

is often termed as the relaxation time. The diffusion coefficient D_i is determined by

$$D_i = \frac{\lambda}{2} \int_{-\infty}^{\infty} z^2 \varphi_i(z) dz.$$

IV. WAVE PROPAGATION, HAMILTON-JACOBI THEORY

Of particular value is the problem of the dependence of the propagation rate of traveling waves on the statistical characteristics of the random walk model. This still remains an unsettled controversial problem [3]. While other schemes require integro-differential equations to be established for mean-field scalars, we focus our attention on the balance equations (1) and (2) and the corresponding Hamiltonian functions. We can expect that under appropriate conditions, the asymptotic solution of the system of equations (1) and (2) will behave as a traveling wave with some velocity u common to both components. The objective is to derive effective equations governing the large-scale dynamics of fronts, varying only upon length scales larger than the characteristic thickness of the traveling waves. The idea is that in the long-time large-distance *macroscopic* limit, the detailed shape of the traveling wave is not important and therefore the problem of wave propagation is that of the dynamics of a traveling front [3,5]. The technique to be used in this paper involves a hyperbolic scaling $x \rightarrow x/\varepsilon$, $t \rightarrow t/\varepsilon$, with the rescaled concentrations $n_i^\varepsilon(t, x) = n_i(t/\varepsilon, x/\varepsilon)$, the nonlinear transformation

$$n_i^\varepsilon(t, x) = A_i \exp\left(-\frac{G^\varepsilon(t, x)}{\varepsilon}\right), \quad (20)$$

and the Hamilton-Jacobi formalism. Positive parameters A_1 and A_2 represent the asymptotic stable equilibrium points of the concentrations n_1^ε and n_2^ε , respectively. For simplicity we suppose the initial conditions

$$n_i(0, x) = \begin{cases} A_i, & x < 0, \\ 0, & x \geq 0, \end{cases} \quad (21)$$

to ensure the minimal propagation speed [4]. The main problem is to derive an eikonal equation from Eqs. (1) and (2) generally of the form

$$F\left[\frac{\partial G}{\partial t}, \frac{\partial G}{\partial x}, x\right] = 0, \quad (22)$$

where $G(t, x) = \lim_{\varepsilon \rightarrow 0} G^\varepsilon(t, x)$ and F is the integral operator. This equation allows us to find the action functional $G(t, x)$ and, thereby, the reaction front position $x(t)$ in the long-time large-distance limit, from the equation $G(t, x(t)) = 0$ [3].

We are now in a position to determine Eq. (22) for the function $G(t, x)$. Let us make the substitution of Eq. (20) for the rescaled concentration field $n_i^\varepsilon(t, x)$ into Eqs. (1) and (2); then,

$$\begin{aligned} A_1 = & A_1 \int_0^{t/\varepsilon} \int_{-\infty}^{\infty} \exp\left[\frac{G^\varepsilon(t, x) - G^\varepsilon(t - \varepsilon s, x - \varepsilon z)}{\varepsilon}\right] \\ & \times \varphi_1(z) \psi_1(s) B_1(s) dz ds + A_1 \int_0^{t/\varepsilon} f_1(A_1 e^{-G^\varepsilon/\varepsilon}, A_2 e^{-G^\varepsilon/\varepsilon}) \\ & \times \exp\left[\frac{G^\varepsilon(t, x) - G^\varepsilon(t - \varepsilon s, x)}{\varepsilon}\right] \Psi_1(s) B_1(s) ds \\ & + A_2 \int_0^{t/\varepsilon} \exp\left[\frac{G^\varepsilon(t, x) - G^\varepsilon(t - \varepsilon s, x)}{\varepsilon}\right] \Psi_1(s) \beta_1(s) ds, \end{aligned} \quad (23)$$

$$\begin{aligned} A_2 = & A_2 \int_0^{t/\varepsilon} \int_{-\infty}^{\infty} \exp\left[\frac{G^\varepsilon(t, x) - G^\varepsilon(t - \varepsilon s, x - \varepsilon z)}{\varepsilon}\right] \\ & \times \varphi_2(z) \psi_2(s) B_2(s) dz ds + A_2 \int_0^{t/\varepsilon} f_2(A_1 e^{-G^\varepsilon/\varepsilon}, A_2 e^{-G^\varepsilon/\varepsilon}) \\ & \times \exp\left[\frac{G^\varepsilon(t, x) - G^\varepsilon(t - \varepsilon s, x)}{\varepsilon}\right] \Psi_2(s) B_2(s) ds \\ & + A_1 \int_0^{t/\varepsilon} \exp\left[\frac{G^\varepsilon(t, x) - G^\varepsilon(t - \varepsilon s, x)}{\varepsilon}\right] \Psi_2(s) \beta_2(s) ds. \end{aligned} \quad (24)$$

We derive the equation for $G(t, x)$ by taking the limit $\varepsilon \rightarrow 0$ in the above equations. It follows that

$$\begin{aligned} A_1 = & A_1 \int_0^\infty \int_{-\infty}^\infty e^{\partial G/\partial t s + \partial G/\partial x z} \varphi_1(z) \psi_1(s) B_1(s) dz ds \\ & + A_1 U_1 \int_0^\infty e^{\partial G/\partial t s} \Psi_1(s) B_1(s) ds \\ & + A_2 \int_0^\infty e^{-\partial G/\partial t s} \Psi_1(s) \beta_1(s) ds, \end{aligned} \quad (25)$$

$$\begin{aligned} A_2 = & A_2 \int_0^\infty \int_{-\infty}^\infty e^{\partial G/\partial t s + \partial G/\partial x z} \varphi_2(z) \psi_2(s) B_2(s) dz ds \\ & + A_2 U_2 \int_0^\infty e^{\partial G/\partial t s} \Psi_2(s) B_2(s) ds \\ & + A_1 \int_0^\infty e^{\partial G/\partial t s} \Psi_2(s) \beta_2(s) ds. \end{aligned} \quad (26)$$

Recall that the growth rate parameter U_i is determined in Eq. (5).

It turns out that the system of equations (25) and (26) can be rewritten in a very useful form. Let us introduce the following notations—namely, the Hamiltonian function H and the generalized momentum p :

$$H = -\frac{\partial G}{\partial t}, \quad p = \frac{\partial G}{\partial x}, \quad (27)$$

and the moment generating function

$$\hat{\varphi}_i(p) = \int_{-\infty}^\infty \varphi_i(z) e^{pz} dz.$$

Then Eqs. (25) and (26) become

$$\begin{aligned} A_1 \left[1 - \int_0^\infty e^{-Hs} [\hat{\varphi}_1(p) \psi_1(s) + U_1 \Psi_1(s)] B_1(s) ds \right] \\ - A_2 \int_0^\infty e^{-Hs} \Psi_1(s) \beta_1(s) ds = 0, \\ A_2 \left[1 - \int_0^\infty e^{-Hs} [\hat{\varphi}_2(p) \psi_2(s) + U_2 \Psi_2(s)] B_2(s) ds \right] \\ - A_1 \int_0^\infty e^{-Hs} \Psi_2(s) \beta_2(s) ds = 0. \end{aligned}$$

The above system of linear algebraic equations for A_1 and A_2 has a nontrivial solution when the corresponding determinant is equal to zero. This gives us the equation for $G(t, x)$:

$$\begin{aligned} & \left[1 - \int_0^\infty e^{-Hs} [\hat{\varphi}_1(p)\psi_1(s) + U_1\Psi_1(s)]B_1(s)ds \right] \\ & \times \left[1 - \int_0^\infty e^{-Hs} [\hat{\varphi}_2(p)\psi_2(s) + U_2\Psi_2(s)]B_2(s)ds \right] \\ & - \int_0^\infty e^{-Hs}\Psi_1(s)\beta_1(s)ds \int_0^\infty e^{-Hs}\Psi_2(s)\beta_2(s)ds = 0. \end{aligned} \quad (28)$$

Solving Eq. (28) for $H(p)$, one can find from Eq. (27) that $G(t,x)=px-H(p)t$ which we recognize as the action functional of a free particle. An expression for the propagation rate u can then be found from $G(t,ut)=0$ [3]:

$$u = \frac{H}{p}, \quad p \frac{\partial H}{\partial p} = H(p). \quad (29)$$

We may equivalently write

$$u = \min_p \left\{ \frac{H(p)}{p} \right\},$$

provided $d^2H/dp^2 > 0$. Note that Eq. (28) is valid for arbitrary waiting-time, jump-length, and transmutation PDF and generally can only be solved numerically. However, let us illustrate the use of the above theory through the following tractable example, where the waiting-time and transmutation PDF are exponential and where analytic results are possible.

V. MARKOVIAN-COUPLED REACTION-TRANSPORT EQUATIONS

Let us consider the case of reaction-transport equations with transmutations when the waiting-time and mutation PDF are exponential, Eq. (10):

$$\psi_1(t) = \psi_2(t) = \lambda e^{-\lambda t}, \quad \beta_i(t) = \gamma_i e^{-\gamma_i t}, \quad i = 1, 2, \quad (30)$$

corresponding to the system of equations

$$\begin{aligned} \frac{\partial n_1}{\partial t} &= \lambda \int_{-\infty}^\infty [n_1(t, x-z) - n_1(t, x)] \varphi_1(z) dz \\ &+ f_1(n_1, n_2) n_1 + \gamma_1 (n_2 - n_1), \end{aligned}$$

$$\begin{aligned} \frac{\partial n_2}{\partial t} &= \lambda \int_{-\infty}^\infty [n_2(t, x-z) - n_2(t, x)] \varphi_2(z) dz \\ &+ f_2(n_1, n_2) n_2 + \gamma_2 (n_1 - n_2). \end{aligned}$$

Let us make the appropriate substitutions of Eq. (30) into Eq. (28). One can get

$$\begin{aligned} & \left[1 - [\lambda \hat{\varphi}_1(p) + U_1] \int_0^\infty e^{-(H+\lambda+\gamma_1)s} ds \right] \\ & \times \left[1 - [\lambda \hat{\varphi}_2(p) + U_2] \int_0^\infty e^{-(H+\lambda+\gamma_2)s} ds \right] \\ & - \gamma_1 \gamma_2 \int_0^\infty e^{-(H+\lambda+\gamma_1)s} ds \int_0^\infty e^{-(H+\lambda+\gamma_2)s} ds = 0. \end{aligned}$$

After a simple integration and rearrangement we get the following quadratic equation for the Hamiltonian H :

$$(H - \{\lambda[\hat{\varphi}_1(p) - 1] + U_1 - \gamma_1\})(H - \{\lambda[\hat{\varphi}_2(p) - 1] + U_2 - \gamma_2\}) - \gamma_1 \gamma_2 = 0. \quad (31)$$

In fact, this is a characteristic equation of the eigenvalue problem

$$\begin{pmatrix} \lambda[\hat{\varphi}_1(p) - 1] + U_1 - \gamma_1 & \gamma_1 \\ \gamma_2 & \lambda[\hat{\varphi}_2(p) - 1] + U_2 - \gamma_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = H \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \quad (32)$$

To ensure the positivity of A_1 and A_2 we need to choose the largest eigenvalue $H(p)$:

$$\begin{aligned} H(p) &= \frac{1}{2} \lambda [\hat{\varphi}_1(p) - 1] + \frac{1}{2} \lambda [\hat{\varphi}_2(p) - 1] + \frac{U_1 + U_2}{2} - \frac{(\gamma_1 + \gamma_2)}{2} \\ &+ \sqrt{\left[\frac{\lambda}{2} [\hat{\varphi}_1(p) - 1] - \frac{\lambda}{2} [\hat{\varphi}_2(p) - 1] + \frac{U_1 - U_2}{2} + \frac{\gamma_2 - \gamma_1}{2} \right]^2 + \gamma_1 \gamma_2}. \end{aligned} \quad (33)$$

Note that this expression holds for any jump PDF $\varphi_i(z)$. For example, for Gaussian-distributed jumps with variance σ_i^2 ,

$$\varphi_i(z) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-z^2/2\sigma_i^2},$$

we have, for the function $\hat{\varphi}_i(p)$,

$$\hat{\varphi}_i(p) = e^{\sigma_i^2 p^2/2}.$$

For the discrete jumps distribution

$$\varphi_i(z) = \frac{\delta(z - a_i)}{2} + \frac{\delta(z + a_i)}{2},$$

one can get

$$\hat{\varphi}_i(p) = \frac{e^{\alpha_i p} + e^{-\alpha_i p}}{2}.$$

The speed of propagation, u , can then be determined from Eqs. (29) and (33).

Let us consider weakly coupled and strongly coupled cases.

A. Weakly coupled case

Let us consider the so-called weakly coupled case when the transmutation rates tend to zero: $\gamma_1 \rightarrow 0$, $\gamma_2 \rightarrow 0$. For simplicity we assume that $\gamma_1 = \gamma_2 = \gamma$ and consider $\gamma \rightarrow 0$. If we define

$$\theta_i(p) = \frac{1}{2} \lambda [\hat{\varphi}_i(p) - 1] + \frac{1}{2} U_i, \quad i = 1, 2, \quad (34)$$

the Hamiltonian (33) becomes

$$H_\gamma(p) = \theta_1(p) + \theta_2(p) - \gamma + \sqrt{[\theta_1(p) - \theta_2(p)]^2 + \gamma^2}. \quad (35)$$

We may consider the case $\gamma=0$; the Hamiltonian (35) is then

$$H_0(p) = \theta_1(p) + \theta_2(p) + |\theta_1(p) - \theta_2(p)|. \quad (36)$$

There are three solutions for the momentum p which gives the minimum to H/p . They can be found from

$$\frac{d\theta_1}{dp} = \frac{\theta_1}{p}, \quad \frac{d\theta_2}{dp} = \frac{\theta_2}{p}, \quad \theta_1 = \theta_2. \quad (37)$$

It turns out that for $\gamma \rightarrow 0$, the propagation speed may be larger than in the decoupled case $\gamma=0$. This will be discussed further in the section concerning coupled reaction-diffusion equations. It is important to note that the unique solution will depend explicitly upon U_1 , U_2 , λ and the variance of the jump PDF.

B. Strongly coupled case

Let us suppose again that the transmutations rates are the same $\gamma_1 = \gamma_2 = \gamma$; then, the Hamiltonian (33) takes the form of Eq. (35). Now consider the limit $\gamma \rightarrow \infty$; then,

$$H_\infty = \theta_1(p) + \theta_2(p).$$

For an arbitrary choice of jump PDF, the propagation speed

$$u_\infty = \min_p \left\{ \frac{H_\infty(p)}{p} \right\}$$

can only be found numerically. In the following section we consider the diffusion limit where some analytic results are possible.

1. Coupled reaction-diffusion equations

Let us consider the system of equations (6) and (7) in the diffusion limit:

$$\frac{\partial n_1}{\partial t} = \frac{D_1}{2} \frac{\partial^2 n_1}{\partial x^2} + f_1(n_1, n_2) n_1 + \gamma_1 (n_2 - n_1), \quad (38)$$

$$\frac{\partial n_2}{\partial t} = \frac{D_2}{2} \frac{\partial^2 n_2}{\partial x^2} + f_2(n_1, n_2) n_2 + \gamma_2 (n_1 - n_2). \quad (39)$$

The Hamiltonian in Eq. (33) becomes

$$H = \left(\frac{D_1 + D_2}{2} \right) \frac{p^2}{2} + \frac{U_1 + U_2}{2} - \frac{\gamma_1 + \gamma_2}{2} + \sqrt{\left[\left(\frac{D_1 - D_2}{2} \right) \frac{p^2}{2} + \frac{U_1 - U_2}{2} + \frac{\gamma_2 - \gamma_1}{2} \right]^2 + \gamma_1 \gamma_2}. \quad (40)$$

Then Eq. (40) together with Eq. (29) allows us to determine the propagation speed u which is identical to the result obtained by Freidlin [15] (see Appendix E).

Even in such ‘‘simple’’ reaction-diffusion equations like the above, the behavior of the traveling waves is often far richer than their singular counterparts. As an example, let us analyze the behavior of the propagation speed determined from Eq. (40) in relation to the transmutation rates γ_1 and γ_2 . In what follows we assume without loss of generality that $U_1 > U_2$.

2. Weak coupling

For $\gamma_1 \rightarrow 0$, $\gamma_2 \rightarrow 0$ we have from Eq. (37) that there are three possible values of p satisfying, $\min_p \{H/p\}$:

$$p = \sqrt{\frac{2U_1}{D_1}}, \sqrt{\frac{2U_2}{D_2}}, \sqrt{\frac{2(U_1 - U_2)}{D_2 - D_1}}.$$

Let us assume that $U_1 > U_2$. Now we are in a position to find $u = \min_p \{H/p\}$; it turns out that u depends on the constants U_1 , U_2 , D_1 , and D_2 as follows:

$$u = \begin{cases} \sqrt{2U_1 D_1}, & \text{if } D_1 \geq D_2, \\ \sqrt{2U_1 D_1}, & \text{if } D_1 < D_2, 2U_1 D_1 \geq U_1 D_2 + U_2 D_1, \\ \sqrt{2U_2 D_2}, & \text{if } D_1 < D_2, 2U_2 D_2 \geq U_1 D_2 + U_2 D_1, \\ \frac{U_1 D_2 - U_2 D_1}{\sqrt{2(U_1 - U_2)(D_2 - D_1)}} & \text{if } D_1 < D_2, \max\{2U_1 D_1, 2U_2 D_2\} < U_1 D_2 + U_2 D_1. \end{cases} \quad (41)$$

We note that the above result was first derived by Freidlin [15]. If we suppose $D_1 \rightarrow 0$, $U_2 \rightarrow 0$, then the system of reaction-diffusion equations reduces to

$$\frac{\partial n_1}{\partial t} = \frac{D_1}{2} \frac{\partial^2 n_1}{\partial x^2} + \gamma(n_2 - n_1),$$

$$\frac{\partial n_2}{\partial t} = U_2 n_2(1 - n_2) + \gamma(n_1 - n_2).$$

For $\gamma=0$, it is clear that no traveling front will be established; however, in the limit $\gamma \rightarrow 0$, the propagation speed is given as [15]

$$u = \sqrt{\frac{U_1 D_2}{2}}.$$

The physical interpretation is that of a ‘‘piggyback’’-type effect; one component of the system provides the diffusion and the other the growth.

3. Strong coupling

Let us suppose again that $\gamma_1 = \gamma_2 = \gamma$ and consider the limit $\gamma \rightarrow \infty$; then, we obtain

$$H_\infty = \left(\frac{D_1 + D_2}{2} \right) \frac{p^2}{2} + \frac{U_1 + U_2}{2}. \quad (42)$$

It clearly follows that the momentum p , which gives the minimum to H/p is

$$p = \sqrt{\frac{U_1 + U_2}{D_1 + D_2}}$$

and corresponding propagation speed is

$$u_\infty = \sqrt{\frac{(U_1 + U_2)(D_1 + D_2)}{2}}. \quad (43)$$

As in the weakly coupled case, it is possible that the wave speed is greater than in the decoupled case. Consider again the case $D_1 \rightarrow 0$, $U_2 \rightarrow 0$; then, u_∞ becomes

$$u_\infty = \sqrt{\frac{U_1 D_2}{2}}.$$

One can see that propagation speeds in the weakly coupled and strongly coupled cases are equal.

We have already mentioned that in general, for arbitrary waiting-time, jump length, and transmutation PDFs, one needs to proceed numerically; however, as demonstrated in the Markovian weakly coupled reaction-diffusion case, results are critical upon the relationship between the growth and diffusion constants U_1 , U_2 , D_1 , and D_2 . In the more complex systems of reaction-transport equations, where one may introduce non-Markovian waiting-time PDFs and jump-length PDFs with long-range behavior, the introduction of extra parameters will mean that one will have to take great care in order to capture the correct structure of the solutions.

VI. CONCLUSION

In this paper we have presented a model for a *system* of reaction-transport equations with transmutations incorporating long-memory and long-range interactions. In particular, we use a probabilistic approach based upon the CTRW theory, which is valid for *arbitrary* waiting-time, jump-length, and transmutation PDFs. We primarily consider probabilistic-balance-type equations, but also show their equivalence to a system of generalized master equations involving memory kernels for the transport and transmutation processes. By using a hyperbolic scaling and Hamilton-Jacobi formalism we derive formulas which allow us to determine the propagation speed of the traveling front generated by such systems of equations. In general, each choice of PDF for the random processes will result in equations which have to be solved numerically and need to be investigated in order to determine the structure of the solutions. We illustrate our model by considering the more tractable interacting systems of Markovian reaction-transport equations, including deriving formulas for the special weakly and strongly coupled cases.

APPENDIX A: DERIVATION OF INTEGRO-DIFFERENTIAL EQUATIONS

For brevity, let us consider only Eq. (1):

$$\begin{aligned} n_1(t, x) = & n_1(0, x)\Psi_1(t)B_1(t) + \int_0^t \int_{-\infty}^{\infty} n_1(t-s, x \\ & - z)\varphi_1(z)\psi_1(s)B_1(s)dzds + \int_0^t f_1(n_1, n_2)n_1(t \\ & - s, x)\Psi_1(s)B_1(s)ds + \int_0^t n_2(t-s, x)\Psi_1(s)\beta_1(s)ds. \end{aligned} \quad (A1)$$

We define the Laplace and Fourier transforms as

$$\tilde{f}(u) = \int_0^\infty f(t)e^{-ut}dt, \quad \hat{\varphi}(k) = \int_{-\infty}^\infty \varphi(x)e^{ikx}dx \quad (A2)$$

and the Fourier-Laplace transform as

$$\hat{\tilde{n}}(u, k) = \int_{-\infty}^\infty \int_0^\infty n(t, x)e^{-ut+ikx}dt dx. \quad (A3)$$

For ease of notation, we introduce two functions $f_1(t)$ and $g_1(t)$:

$$f_1(t) = \psi_1(t)B_1(t), \quad g_1(t) = \Psi_1(t)\beta_1(t). \quad (A4)$$

The term $\Psi_1(t)B_1(t)$ which appears in Eq. (A1) is related to $f_1(t)$ and $g_1(t)$ in the following manner:

$$\Psi_1(t)B_1(t) = 1 - \int_0^t [f_1(s) + g_1(s)]ds.$$

This follows from the $\Psi_1(0)B_1(0)=1$ and

$$\frac{d}{dt}\{\Psi_1(t)B_1(t)\} = -\psi_1(t)B_1(t) - \Psi_1(t)\beta_1(t).$$

If we take the Fourier-Laplace transform of Eq. (A1), then

$$\begin{aligned} \hat{n}_1(u, k) &= \hat{n}_1(0, k) \frac{1 - \tilde{f}_1(u) - \tilde{g}_1(u)}{u} + \hat{n}_1(u, k) \hat{\phi}_1(k) \tilde{f}_1(u) \\ &+ \hat{F}_1(u, k) \frac{1 - \tilde{f}_1(u) - \tilde{g}_1(u)}{u} + \hat{n}_2(u, k) \tilde{g}_1(u), \end{aligned} \quad (\text{A5})$$

where $\hat{F}_1(u, k)$ is the Fourier-Laplace transform of $f_1(n_1, n_2)n_1(t, x)$. Here we used the convolution property

$$\begin{aligned} \hat{n}_1(u, k) \hat{\phi}_1(k) \tilde{f}_1(u) &= \int_{-\infty}^{\infty} \int_0^{\infty} \left[\int_0^t \int_{-\infty}^{\infty} n_1(t-s, x-z) \right. \\ &\left. \times \varphi_1(z) f_1(t) dz ds \right] e^{-ut+ikx} dt dx. \end{aligned} \quad (\text{A6})$$

Rearranging Eq. (A5) as

$$\begin{aligned} \frac{\hat{n}_1(u, k)u}{1 - \tilde{f}_1(u) - \tilde{g}_1(u)} &= \hat{n}_1(0, k) + u \hat{\phi}_1(k) \frac{\hat{n}_1(u, k) \tilde{f}_1(u)}{1 - \tilde{f}_1(u) - \tilde{g}_1(u)} \\ &+ \hat{F}_1(n_1(u, k), n_2(u, k)) \\ &+ u \frac{\hat{n}_2(u, k) \tilde{g}_1(u)}{1 - \tilde{f}_1(u) - \tilde{g}_1(u)} \end{aligned} \quad (\text{A7})$$

and introducing the auxiliary functions

$$\tilde{\alpha}_1(u) = \frac{\tilde{f}_1(u)u}{1 - \tilde{f}_1(u) - \tilde{g}_1(u)}, \quad \tilde{\zeta}_1(u) = \frac{\tilde{g}_1(u)u}{1 - \tilde{f}_1(u) - \tilde{g}_1(u)}, \quad (\text{A8})$$

we find, then,

$$\begin{aligned} u \hat{n}_1(u, k) &= \hat{n}_1(0, k) + \tilde{\alpha}_1(u) \hat{n}_1(u, k) [\hat{\phi}_1(k) - 1] + \hat{F}_1(u, k) \\ &+ \tilde{\gamma}_1(u) [\hat{n}_2(u, k) - \hat{n}_1(u, k)]. \end{aligned} \quad (\text{A9})$$

By applying the inverse Laplace-Fourier transform to Eq. (A9), we get the equation

$$\begin{aligned} \frac{\partial n_1(t, x)}{\partial t} &= \int_0^t \alpha_1(t-s) \int_{-\infty}^{\infty} [n_1(s, x-z) - n_1(s, x)] \varphi_1(z) dz ds \\ &+ \int_0^t \gamma_1(t-s) [n_2(s, x) - n_1(s, x)] ds \\ &+ f(n_1, n_2)n_1(t, x). \end{aligned}$$

APPENDIX B: MARKOV RANDOM WALKS

For brevity we consider only the derivation of Eq. (11). There are two ways to proceed: either make the appropriate

substitutions of Eq. (10) into Eq. (1) and differentiate both sides directly, or alternatively, we can make use of the integro-differential master equation (6). We follow the latter. Let us first determine the functions (8):

$$\begin{aligned} f_1(t) &= \psi(s) \int_t^{\infty} \beta_1(s) ds = \lambda e^{-(\lambda+\gamma_1)t}, \\ g_1(t) &= \beta_1(t) \int_t^{\infty} \psi(s) ds = \gamma_1 e^{-(\lambda+\gamma_1)t}. \end{aligned}$$

The respective Laplace transforms are

$$\tilde{f}_1(u) = \frac{\lambda}{u + \lambda + \gamma_1}, \quad \tilde{g}_1(u) = \frac{\gamma_1}{u + \lambda + \gamma_1}.$$

If we substitute these expressions into

$$\tilde{\alpha}_i(u) = \frac{u \tilde{f}_i(u)}{1 - \tilde{f}_i(u) - \tilde{g}_i(u)}, \quad \tilde{\zeta}_i(u) = \frac{u \tilde{g}_i(u)}{1 - \tilde{f}_i(u) - \tilde{g}_i(u)}, \quad (\text{B1})$$

then the Laplace transforms (9) can be found to be

$$\tilde{\alpha}_1(u) = \lambda, \quad \tilde{\zeta}_1(u) = \gamma_1.$$

This corresponds to the δ functions for the inverse Laplace transforms:

$$\alpha_1(t) = \lambda \delta(t), \quad \zeta_1(t) = \gamma_1 \delta(t). \quad (\text{B2})$$

Substituting Eq. (B2) into Eq. (6) gives Eqs. (11) and (12).

APPENDIX C: NON-MARKOV RANDOM WALKS

Similarly to Appendix B, we find

$$\begin{aligned} f_1(t) &= \psi(s) \int_t^{\infty} \beta_1(s) ds = \lambda^2 t e^{-\lambda t} \int_t^{\infty} \gamma_1 e^{-\gamma_1 s} ds = \lambda^2 t e^{-(\lambda+\gamma_1)t}, \\ g_1(t) &= \beta_1(t) \int_t^{\infty} \psi(s) ds = \gamma_1 e^{-\gamma_1 t} \int_t^{\infty} \lambda^2 s e^{-\lambda s} ds \\ &= \gamma_1 (1 + \lambda t) e^{-(\lambda+\gamma_1)t}, \end{aligned}$$

with the respective Laplace transforms

$$\tilde{f}_1(u) = \left(\frac{\lambda}{u + \lambda + \gamma_1} \right)^2,$$

$$\tilde{g}_1(u) = \gamma_1 \left[\frac{1}{u + \lambda + \gamma_1} + \frac{\lambda}{(u + \lambda + \gamma_1)^2} \right].$$

The memory kernels defined in Eq. (9) can be found as

$$\tilde{\alpha}_1(u) = \frac{1}{u + 2\lambda + \gamma_1}, \quad \tilde{\zeta}_1(u) = \gamma_1.$$

This corresponds to the inverse Laplace transforms

$$\alpha_1(t) = e^{-(2\lambda+\gamma_1)t}, \quad \zeta_1(t) = \gamma_1 \delta(t). \quad (\text{C1})$$

Substituting Eq. (C1) into Eq. (6) gives Eq. (16).

APPENDIX D: HYPERBOLIC REACTION-TRANSPORT EQUATIONS

For brevity let us consider only Eq. (16):

$$\begin{aligned} \frac{\partial n_1}{\partial t} = & \int_0^t \lambda^2 e^{-(2\lambda+\gamma_1)s} \int_{-\infty}^{\infty} [n_1(t-s, x-z) \\ & - n_1(t-s, x)] \varphi_1(z) dz ds + n_1 f_1(n_1, n_2) + \gamma_1(n_2 - n_1). \end{aligned} \quad (D1)$$

If we make the change of variable $r=t-s$,

$$\begin{aligned} \frac{\partial n_1}{\partial t} = & e^{-(2\lambda+\gamma_1)t} \int_0^t \lambda^2 e^{(2\lambda+\gamma_1)r} \int_{-\infty}^{\infty} [n_1(r, x-z) \\ & - n_1(r, x)] \varphi_1(z) dz dr + f_1(n_1, n_2) n_1 + \gamma_1(n_2 - n_1), \end{aligned} \quad (D2)$$

then differentiating both sides with respect to t one can get

$$\begin{aligned} \frac{\partial^2 n_1}{\partial t^2} = & -(2\lambda + \gamma_1) e^{-(2\lambda+\gamma_1)t} \int_0^t \lambda^2 e^{(2\lambda+\gamma_1)r} \int_{-\infty}^{\infty} [n_1(r, x-z) \\ & - n_1(r, x)] \varphi_1(z) dz dr + \lambda^2 \int_{-\infty}^{\infty} [n_1(t, x-z) \\ & - n_1(t, x)] \varphi_1(z) dz + \left(f_1 + n_1 \frac{\partial f_1}{\partial n_1} \right) \frac{\partial n_1}{\partial t} + n_1 \frac{\partial f_1}{\partial n_2} \frac{\partial n_2}{\partial t} \\ & + \gamma_1 \left(\frac{\partial n_2}{\partial t} - \frac{\partial n_1}{\partial t} \right), \end{aligned} \quad (D3)$$

Then, from Eqs. (D3) and (D2),

$$\begin{aligned} \frac{\partial^2 n_1}{\partial t^2} = & -(2\lambda + \gamma_1) \left(\frac{\partial n_1}{\partial t} - f_1(n_1, n_2) n_1 - \gamma_1(n_2 - n_1) \right) \\ & + \lambda^2 \int_{-\infty}^{\infty} [n_1(t, x-z) - n_1(t, x)] \varphi_1(z) dz \\ & + \left(f_1 + n_1 \frac{\partial f_1}{\partial n_1} \right) \frac{\partial n_1}{\partial t} + n_1 \frac{\partial f_1}{\partial n_2} \frac{\partial n_2}{\partial t} + \gamma_1 \left(\frac{\partial n_2}{\partial t} - \frac{\partial n_1}{\partial t} \right), \end{aligned} \quad (D4)$$

By rearranging Eq. (D4) and dividing both sides by

$$\lambda = \frac{1}{2\tau},$$

one can get

$$\begin{aligned} \tau \frac{\partial^2 n_1}{\partial t^2} + \left(1 - f_1 \tau - n_1 \frac{\partial f_1}{\partial n_1} \tau + 2\gamma_1 \tau \right) \frac{\partial n_1}{\partial t} - \tau \left(n_1 \frac{\partial f_1}{\partial n_2} + \gamma_1 \right) \frac{\partial n_2}{\partial t} \\ = \frac{1}{4\tau} \int_{-\infty}^{\infty} [n_1(t, x-z) - n_1(t, x)] \varphi_1(z) dz + (1 + \gamma_1 \tau) \\ \times [f_1(n_1, n_2) n_1 + \gamma_1(n_2 - n_1)]. \end{aligned} \quad (D5)$$

By expanding $n_1(t, x-z)$ into a Taylor series and taking the first three terms in the transport integral we arrive at Eq. 18.

APPENDIX E: REACTION-DIFFUSION EQUATIONS

In the following we show that the solution (40) and (29) is identical to that derived by Freidlin [15]. In particular, for the system of coupled reaction-diffusion equations (13) and (14),

$$\frac{\partial n_1}{\partial t} = \frac{D_1}{2} \frac{\partial^2 n_1}{\partial x^2} + f_1(n_1, n_2) n_1 + \gamma_1(n_2 - n_1),$$

$$\frac{\partial n_2}{\partial t} = \frac{D_2}{2} \frac{\partial^2 n_1}{\partial x^2} + f_2(n_1, n_2) n_2 + \gamma_2(n_1 - n_2),$$

Freidlin [15] showed that the propagation speed u is given by

$$u = \frac{|\lambda(\alpha^*) + B|}{\sqrt{2(\alpha^* - A)}},$$

where $\lambda(\alpha)$ is the maximal eigenvalue of the matrix

$$\begin{pmatrix} \alpha D_1 - \gamma_1 & \gamma_1 \\ \gamma_2 & \alpha D_2 - \gamma_1 \end{pmatrix}$$

and α^* is the root of the equation

$$2 \frac{d\lambda(\alpha)}{d\alpha} (\alpha - A) = \lambda(\alpha) + B,$$

where $A = (U_1 - U_2)/(D_1 - D_2)$, $B = (D_1 U_2 - D_2 U_1)/(D_1 - D_2)$. Recall Eq. (40):

$$\begin{aligned} H(p) = & \left(\frac{D_1 + D_2}{2} \right) \frac{p^2}{2} + \frac{U_1 + U_2}{2} - \frac{\gamma_1 + \gamma_2}{2} \\ & + \sqrt{\left[\left(\frac{D_1 - D_2}{2} \right) \frac{p^2}{2} + \frac{U_1 - U_2}{2} + \frac{\gamma_2 - \gamma_1}{2} \right]^2 + \gamma_1 \gamma_2}. \end{aligned} \quad (E1)$$

To find the speed of propagation u we first determine p from Eq. (29):

$$\frac{\partial H(p)}{\partial p} = \frac{H(p)}{p}. \quad (E2)$$

If we make the substitution $\alpha = p^2/2 + A$, where $A = (U_1 - U_2)/(D_1 - D_2)$, then Eq. (E2) becomes

$$\frac{\partial H[\sqrt{2(\alpha - A)}]}{\partial \alpha} \frac{\partial \alpha}{\partial p} = \frac{H[\sqrt{2(\alpha - A)}]}{p}. \quad (E3)$$

Clearly $\partial \alpha / \partial p = p = \sqrt{2(\alpha - A)}$: therefore,

$$2 \frac{\partial H[\sqrt{2(\alpha-A)}]}{\partial \alpha} (\alpha-A) = H[\sqrt{2(\alpha-A)}],$$

$$2 \frac{d\lambda(\alpha)}{d\alpha} (\alpha-A) = \lambda(\alpha) + B,$$

with propagation speed

and in terms of the notation used by Freidlin $H[\sqrt{2(\alpha-A)}] = \lambda(\alpha) + B$, where $B = (D_1 U_2 - D_2 U_1) / (D_1 - D_2)$: thus,

$$u = \frac{H(p)}{p} = \frac{|\lambda(\alpha) + B|}{\sqrt{2(\alpha-A)}}.$$

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