Relations between Lagrangian models and synthetic random velocity fields

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The authors propose an alternative interpretation of Markovian transport models based on the well-mixed condition, in terms of the properties of a random velocity field with second order structure functions scaling linearly in the space-time increments. This interpretation allows direct association of the drift and noise terms entering the model, with the geometry of the turbulent fluctuations. In particular, the well-known nonuniqueness problem in the well-mixed approach is solved in terms of the antisymmetric part of the velocity correlations; its relation with the presence of nonzero mean helicity and other geometrical properties of the flow is elucidated. The well-mixed condition appears to be a special case of the relation between conditional velocity increments of the random field and the one-point Eulerian velocity distribution, allowing generalization of the approach to the transport of nontracer quantities. Application to solid particle transport leads to a model satisfying, in the homogeneous isotropic turbulence case, all the conditions on the behavior of the correlation times for the fluid velocity sampled by the particles. In particular, correlation times in the gravity and in the inertia dominated case, respectively, longer and shorter than in the passive tracer case; in the gravity dominated case, correlation times longer for velocity components along gravity, than for the perpendicular ones. The model produces, in channel flow geometry, particle deposition rates in agreement with experiments.

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I. INTRODUCTION

In Lagrangian models, the concentration of a quantity transported by a turbulent field is reconstructed from the trajectories of the individual particles advected by the flow [1–4]. Since each trajectory is associated with an independent realization of the turbulent flow, what is obtained is actually the mean concentration profile, given a certain distribution of sinks and sources for the transported quantity.

One keeps into account the fact that the turbulent length and time scales are nonzero, by assuming that the particle velocity obeys a Langevin equation; this results in the system of equations, in terms of the particle Lagrangian velocity $\mathbf{v}$ and coordinate $\mathbf{x}$,

$$\mathrm{d}\mathbf{x} = \mathbf{v}\ \mathrm{d}t,$$

$$d\mathbf{v} = \mathbf{a}(\mathbf{v},\mathbf{x},t)\mathrm{d}t + \mathrm{d}\mathbf{w},$$

$$\langle dw^i dw^j\rangle = \mathcal{B}^{ij}(\mathbf{v},\mathbf{x},t)\mathrm{d}t. \tag{1}$$

A strong physical motivation for a Lagrangian model in the form of Eq. (1) is the linear scaling of the Lagrangian velocity structure functions for inertial range time separations [5]. In the case of passive tracers, we have

$$\langle[v^i(t)-v^i(0)][v^j(t)-v^j(0)]\rangle \simeq \delta^{ij}C_0\overline{\epsilon}t \tag{2}$$

with $\overline{\epsilon}$ the mean viscous dissipation and C_0 a universal constant; Eq. (1) will then result from assuming a Markovian behavior for $\mathbf{v}$, i.e., that a^i and $\mathcal{B}^{ij}$ depend solely on the current values of $\mathbf{v}$ and $\mathbf{x}$ and not on their previous history.

The well-mixed condition, introduced in Ref. [4], lead to a great advance in Lagrangian models, providing a simple technique for expressing the drift coefficient $\mathbf{a}$ in Eq. (1) in terms of observed properties of the flow.

In the current approach, it is assumed that the noise term $\mathrm{d}\mathbf{w}$ in Eq. (1) accurately represents, in high Reynolds number turbulent regimes, the inertial range scaling of the Lagrangian velocity increments,

$$\mathcal{B}^{ij}(\mathbf{v},\mathbf{x},t) = \delta^{ij}C_0\overline{\epsilon}(\mathbf{x},t). \tag{3}$$

Then, at least when the flow is incompressible, the well-mixed criterion allows us to determine the drift coefficient $\mathbf{a}$ in terms of $\overline{\epsilon}$ and the Eulerian probability density function (PDF) for the fluid velocity $\mathbf{u}(\mathbf{x},t)$: $\rho_E(\mathbf{u},t,\mathbf{x})\equiv\rho(\mathbf{u}(\mathbf{x},t))$. (In the general compressible case, the Eulerian PDF must be weighed on the fluctuating fluid density, which is tantamount to substitute ρ_E with the Lagrangian PDF for the fluid parcels velocity and position [4]).

The great advantage of the well-mixed approach, coupled with Eq. (3), is that no knowledge of the spatiotemporal structure of the turbulent fluctuations is required, rather, it is the outcome, encoded in the drift coefficient, of the well mixed condition. This strength of the model, however, turns into weakness, whenever the turbulent structure plays a relevant role. A first hint that this, actually, is always the case, is the nonuniqueness of the solution for $\mathbf{a}$ given $\overline{\epsilon}$ and $\rho_E(\mathbf{u})$ [4,6]. In the well mixed approach, the drift coefficient $\mathbf{a}$ is determined only up to a velocity curl, and interpretation of this freedom in terms of the properties of the flow is awkward.

There are situations in which the structure of turbulence plays an explicit role. A first example is produced when coherent structures dominate the flow, a rather common occurrence in turbulence, which takes a dramatic form in near wall regions. These regions become relevant in many situations of practical interest in industrial flows, but also, to name a few, in the study of transport in tree canopies and in indoor pollution. These flows are often characterized by moderate Reynolds number, and, for this reason, not only the viscous scale may be not negligible, but a well developed inertial range may even be absent, so that the conditions justifying Eq. (3) cease to be valid. Now, standard techniques exist which allow for the inclusion of non-Gaussianity [7] and anisotropy [6,8] in Lagrangian models, as well as for the effect of finite Reynolds numbers [9]. However, these techniques do not take into account the geometric structure of the turbulent fluctuations, which calls for information about space correlation.

In the range of scales we are considering, another issue will come into play, if we are interested in modeling solid particle transport. In this case, nontracer behaviors associated with inertia and gravity will begin to be felt (inertia and trajectory crossing effects [10–12]). This is especially true for atmospheric aerosol, characterized by heavy particles with relative density of the order of 1000 and particle diameters in the range 10^{-2}–10^{2} μm. Inertia effects are generally negligible in ABL (atmospheric boundary layer) mesoscale modeling, but can become relevant in the near wall regions of wall bounded flows [13], which is relevant for problems of air conditioning and abatement of indoor pollution. Trajectory crossing effects related to gravity can have deep implications not only in wall bounded flows, but also for particulate transport in the ABL [10].

These effects reflect heavily on the possibility of using Eqs. (2) and (3) to model the Lagrangian velocity increments, and call back for the need of information on the spatiotemporal structure of the turbulent fluctuations.

Some attempts in this direction were carried out in Refs. [14,15], but disregard of correlations between fluid velocity increments along solid particle trajectories lead to difficulties in the fluid particle limit [16] and in the implementation of the well-mixed condition.

In order to be able to understand the constraints imposed by the turbulent structure on the form of a Lagrangian model, one may try a derivation from a velocity field $\mathbf{u}(\mathbf{x},t)$ of prescribed statistics. If the structure function $\langle[u^i(\mathbf{x},t)-u^i(0,0)][u^j(\mathbf{x},t)-u^j(0,0)]\rangle$ scaled linearly for small space-time separations, the velocity increment between two points lying on a trajectory would be given by

$$\mathrm{d}\mathbf{u} = \langle[\partial_t + \mathbf{u}(\mathbf{x},t)\cdot\nabla]\,\mathbf{u}(\mathbf{x},t)|\mathbf{u}(\mathbf{x},t)\rangle \mathrm{d}t + \mathrm{d}\mathbf{w} \qquad (4)$$

with $\langle d\mathbf{w}\rangle=0$ and $\mathrm{d}w^2=O(\mathrm{d}t)$. We could then introduce a Lagrangian model obeying Eq. (1), with

$$\mathbf{a}(\mathbf{v},\mathbf{x},t) = \langle[\partial_t + \mathbf{u}(\mathbf{x},t)\cdot\nabla]\mathbf{u}(\mathbf{x},t)|\mathbf{u}(\mathbf{x},t) = \mathbf{v}\rangle,$$

$$\mathcal{B}^{ij}(\mathbf{v},\mathbf{x},t)\mathrm{d}t = \langle \mathrm{d}w^i\ \mathrm{d}w^j|\mathbf{u}(\mathbf{x},t) = \mathbf{v}\rangle. \qquad (5)$$

With the coefficients given in this equation, Eq. (1) would provide a Markovianized version of the dynamics of a particle moving in the random velocity field $\mathbf{u}(\mathbf{x},t)$.

It turns out that, provided the structure functions for $\mathbf{u}$ scale linearly, the well-mixed technique could be imposed directly on the random field $\mathbf{u}(\mathbf{x},t)$, before any trajectory is defined. This means expressing the form of the conditional averages of the velocity derivatives: $\langle\partial_t\mathbf{u}(\mathbf{x},t)|\mathbf{u}(\mathbf{x},t)\rangle$ and $\langle\nabla\mathbf{u}(\mathbf{x},t)|\mathbf{u}(\mathbf{x},t)\rangle$ in terms of the Eulerian PDF $\rho_E(\mathbf{u},\mathbf{x},t)$. This has the important consequence that a passive tracer advected by an incompressible flow, will satisfy the ergodic property

$$\rho_L(\mathbf{v}|\mathbf{x},t) = \rho_E(\mathbf{v},\mathbf{x},t), \qquad (6)$$

where $\rho_L(\mathbf{v}|\mathbf{x},t)$ is the Lagrangian PDF for a tracer passing at the point $\mathbf{x}$ at time t, to have velocity $\mathbf{v}$. Thus, the well-mixed condition imposed on the random field extends naturally to the Lagrangian model defined by Eqs. (1) and (5). This is advantageous in the compressible case, where it will be shown that, contrary to the Thomson-87 approach, knowledge of ρ_E is sufficient for the determination of a well-mixed model.

Clearly, linear scaling at small separations does not correspond to the properties of a real turbulent field, which, at high Reynolds numbers is more rough, and whose time correlations, due to the sweep effect, have Lagrangian nature at short time scales [5]. This is compensated, however, by control over the large scale structure of the correlations, which is the relevant aspect for the determination of turbulent transport.

A related issue, concerning solid particle transport, is that anomalous scaling of the fluid velocity increments sampled by a solid particle, are known to occur at sufficiently short time scales [17]. Analysis of the different ranges characterizing solid particle motion was carried on in Ref. [16], and Lagrangian models resolving the anomalous scaling range were presented in Refs. [18,16], based, respectively, on the use of fractional Brownian motion and synthetic turbulence algorithms. Again, consideration of these short-time effects is neglected in favor of control of large scale geometry.

Compared to the standard approach in Lagrangian modeling, the one proposed here has definite advantages. Spatiotemporal turbulent structures can be included in a relatively simple way. The nonuniqueness problem is solved in a simpler way, since only purely Eulerian properties of the flow are invoked (helicity is one example). The advection of passive tracers and solid particles are treated exactly on the same footing, hence, extension of the model to solid particle transport is automatic and does not need introducing additional assumptions.

This paper is organized as follows. In Sec. II, a local characterization of a random velocity field will be given, introducing generalized "four-dimensional" Langevin and Fokker-Planck equations, and providing local and global existence conditions. The condition of local existence appears to take the form of a generalized form of the well-mixed condition, which will be discussed in Sec. III. This will be used to calculate conditional averages in the form $\langle\nabla\mathbf{u}|\mathbf{u}\rangle$ and $\langle\partial_t\mathbf{u}|\mathbf{u}\rangle$ from the property of $\mathrm{d}\mathbf{w}$ and the Eulerian velocity PDF, and the relation with the spatiotemporal structure of the random field will be discussed. In Sec. IV, expressions for the noise amplitude $\langle\mathrm{d}\mathbf{w}\ \mathrm{d}\mathbf{w}\rangle$ will be derived, and their relation with the symmetric sector of the velocity correlation will be discussed in terms of the SO(3) technique introduced in Ref. [19]. The antisymmetric sector of the velocity correlation will be discussed in Sec. V, illustrating how it relates to the problem of nonuniqueness in the well-mixed approach, and showing how helicity and other geometrical features could be included in the random field. Section VI will be devoted to the derivation of a Markovian Lagrangian model in the form of Eq. (5), and to presentation of its main properties. The relation with the Thomson-87 model [4] will be discussed. Sections VII and VIII will be devoted to analysis of the Markovian approximation in the Lagrangian model and to proof of the ergodic property given by Eq. (6). Sections IX and X will illustrate two applications of the Lagrangian model to solid particle transport, respectively, in homogeneous isotropic turbulence, and in a turbulent channel flow. Section XI contains the conclusions.

II. CHARACTERIZATION OF THE RANDOM VELOCITY FIELD

Let us introduce a zero-mean, incompressible random velocity field $\mathbf{u}(\mathbf{x},t)$, with Second order structure functions scaling linearly in the increment at small space-time separations. We introduce 4-vector notation,

$$x^\mu = \{x^0, x^i\} \equiv \{t, \mathbf{x}\}, \quad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \tag{7}$$

and stick rigorously to the Einstein convention of summation over covariant-controvariant repeated indices. We have the following equation for the velocity increment:

$$\mathrm{d}u^i \equiv \mathrm{d}x^\mu\ \partial_\mu u^i = A_\mu{}^i(\mathbf{u}, x^\mu)\mathrm{d}x^\mu + \mathrm{d}w^i, \tag{8}$$

where

$$A_\mu{}^i(\mathbf{u}, x^\mu) = \langle \partial_\mu u^i(\mathbf{x},t) | \mathbf{u}(\mathbf{x},t)\rangle \tag{9}$$

and $\langle \mathrm{d}\mathbf{w} | \mathbf{u}\rangle = 0$. From linearity, the contribution to the velocity structure function is dominated, for small values of the increments, by the correlation for $\mathrm{d}\mathbf{w}$, and we have

$$\langle \mathrm{d}w^i\ \mathrm{d}w^j | \mathbf{u}(\mathbf{x},t)\rangle = \langle \mathrm{d}u^i(\mathbf{x},t)\mathrm{d}u^j(\mathbf{x},t)\rangle = O(|\mathrm{d}\mathbf{x}|, \mathrm{d}t). \tag{10}$$

We limit our analysis to random velocity fields where the statistics of the velocity increments is independent of that of the total velocity.

$$\langle \mathrm{d}w^i \mathrm{d}w^j | \mathbf{u}\rangle = \langle \mathrm{d}w^i\ \mathrm{d}w^j\rangle. \tag{11}$$

Incompressibility of the velocity field $\partial_i u^i = 0$, leads to the constraint, indicating $\Delta^\mu \equiv \Delta x^\mu$,

$$A_i^i = 0, \quad \frac{\partial \langle \Delta w^i \Delta w^j\rangle}{\partial \Delta^i} = 0. \tag{12}$$

From Eqs. (8) and (10), we obtain the following generalized Itô's lemma; for a generic smooth function $\phi(\mathbf{u})$:

$$\mathrm{d}\phi(\mathbf{u}) = (A_\mu{}^i\ \mathrm{d}x^\mu + \mathrm{d}w^i)\partial_{u^i}\phi(\mathbf{u}) + \frac{1}{2}\langle \mathrm{d}w^i\ \mathrm{d}w^j\rangle \partial_{u^i}\partial_{u^j}\phi(\mathbf{u}) \tag{13}$$

and from here, we can derive an equation for the change of the one-point PDF $\rho_E(\mathbf{u}, x^\mu) \equiv \rho(u(\mathbf{x},t))$, in passing from the point x^μ to to the point $x^\mu + \mathrm{d}x^\mu$,

$$\mathrm{d}\rho_E \equiv \mathrm{d}x^\mu\ \partial_\mu \rho_E = -\partial_{u^i}(A_\mu{}^i\ \mathrm{d}x^\mu\ \rho_E) + \frac{1}{2}\partial_{u^i}\partial_{u^j}(\langle \mathrm{d}w^i \mathrm{d}w^j\rangle \rho_E). \tag{14}$$

Notice that the form of the two equations (13) and (14) is independent of incompressibility and Eq. (12). The sequence leading from Eq. (8) to (14) is very suggestive, in that it generalizes the one from a Langevin to a Fokker-Planck equation [20]. However, contrary to the case of a standard Fokker-Planck equation, Eq. (14) does not admit in general solution for ρ_E. In fact, once the noise amplitude $\langle \mathrm{d}w^i\ \mathrm{d}w^j\rangle$ and the drift $A_\mu{}^i$ are given, Eq. (14) becomes a system of four partial differential equations for the single PDF ρ_E, and this system is generally overdetermined. In the next section, it will be shown how a generalized version of the well-mixed condition is able to take care of this local existence problem.

The ill-posedness of the problem is reflected at the global level, in the fact that a local definition for the "noise" increment amplitude $\langle \mathrm{d}w^i\ \mathrm{d}w^j\rangle$ is not sufficient to define a realization for $\mathbf{w}(\mathbf{x},t)$, and consequently for $\mathbf{u}(\mathbf{x},t)$. This in contrast with the case of the standard Langevin equation. In fact, if we integrate Eq. (8) along a closed curve in space-time, and consider uncorrelated increments $\mathrm{d}\mathbf{w}$ along the curve, we will obtain in general a nonzero total velocity increment in the closed loop. In other words, if we disregard these correlations for $\mathbf{w}$, the differential $\mathrm{d}\mathbf{u}$ entering Eq. (8) will not in general be exact.

The question becomes at this point the existence of a random velocity field with local structure described by Eq. (8). It turns out that such a velocity field can be constructed explicitly, although the construction described below is by no means unique.

Given a point x^μ and a direction in space-time defined by the versor r^μ ($r^\mu r_\mu = 1$), we can introduce the stochastic process $\hat{u}^i(s) \equiv \hat{u}^i(x^\mu, r^\mu; s)$ obeying the Langevin equation

$$\mathrm{d}\hat{u}^i(s) = r^\mu A_\mu{}^i(\hat{\mathbf{u}}, x^\mu)\mathrm{d}s + \mathrm{d}\hat{w}^i,$$

$$\langle \mathrm{d}\hat{w}^i\ \mathrm{d}\hat{w}^j\rangle = \frac{\mathrm{d}}{\mathrm{d}s}\langle [u^i(x^\mu + r^\mu s) - u^i(x^\mu)] \times [u^j(x^\mu + r^\mu s) - u^j(x^\mu)]\rangle_{s=0^+}\ \mathrm{d}s. \tag{15}$$

The correlation functions for the stochastic process $\hat{\mathbf{u}}(s)$, starting from the second order one $C^{ij}(x^\mu, sr^\mu) = \langle \hat{u}^i(x^\mu, r^\mu; -s/2)\hat{u}^j(x^\mu, r^\mu; s/2)\rangle$, will identify a random velocity field $\mathbf{u}(x^\mu)$ whose local statistical properties are those imposed by Eq. (8), and whose restriction to straight lines in space-time will be, by construction, Markovian. As with the correlation time of the solution of a standard Langevin equation, the correlation length in the direction r^μ will be encoded in the drift coefficient $r^\mu A_\mu{}^i(\hat{\mathbf{u}}, x^\mu)$. A random field realization is obtained, in the simpler Gaussian case, by first carrying on the principal orthogonal decomposition (POD) of $C^{ij}(x^\mu, \Delta^\mu)$, and then random superposing, with the appropriate amplitudes, the resulting POD modes [21].

III. DETERMINATION OF THE DRIFT

The real meaning of Eq. (14) is to provide a consistency condition for $\langle \mathrm{d}w^i \mathrm{d}w^j\rangle$ and $A_\mu{}^i$, that could be used to generalize the Thomson-87 technique and determine from $\langle \mathrm{d}w^i\, \mathrm{d}w^j\rangle$ and ρ_E, the expression for $A_\mu{}^i$. The difference with the standard case is that, instead of calculating the conditional mean $\langle \partial_t v^i | \mathbf{v}\rangle$ of the Lagrangian velocity time derivative, we seek here the conditional mean of all the derivatives of the Eulerian velocity, namely $\langle \partial_\mu u^i | \mathbf{u}\rangle$. These averages contain important information on the behavior of the velocity correlation $C^{ij}(x^\mu,\Delta^\mu)=\langle u^i(x^\mu-\Delta^\mu/2)u^j(x^\mu+\Delta^\mu/2)\rangle$

$$C^{ij}(x^\mu,\Delta^\mu) = \tfrac{1}{2}[R^{ij}(x^\mu-\Delta^\mu/2)+R^{ij}(x^\mu+\Delta^\mu/2)] - \tfrac{1}{2}\langle \Delta u^i \Delta u^j\rangle + C_A^{ij}. \tag{16}$$

Here, $R^{ij}(x^\mu)=C^{ij}(x^\mu,0)$ indicates the Reynolds tensor, while

$$C_A^{ij} = \tfrac{1}{2}[C^{ij}(x^\mu,\Delta^\mu) - C^{ij}(x^\mu,-\Delta^\mu)] \tag{17}$$

is the antisymmetric part of the velocity correlation. It is clear that the noise amplitude is associated with the symmetric sector of the velocity correlation, and for small Δ^μ: $\langle \Delta w^i \Delta w^j\rangle \simeq \langle \Delta u^i \Delta u^j\rangle$.

Let us try to generalize the Thomson-87 approach to calculate the drift $A_\mu{}^i$ from ρ_E. It is convenient to split the drift into three pieces,

$$A_\mu{}^i = \overline{A}_\mu{}^i + \frac{1}{\rho_E}\Phi_\mu{}^i + \frac{1}{\rho_E}\Psi_\mu{}^i, \tag{18}$$

where $\overline{A}_\mu{}^i$ is chosen to cancel the noise term in the Fokker-Planck equation (14). Exploiting independence of the noise amplitude from $\mathbf{u}$:

$$\overline{A}_\mu{}^i \mathrm{d}x^\mu = \frac{1}{2}\langle \mathrm{d}w^i \mathrm{d}w^j\rangle \partial_{u^j}\log \rho_E \tag{19}$$

and $\overline{A}_\mu{}^i$, from the second of Eq. (12), is automatically traceless. The term $\Phi_\mu{}^i$ is chosen to cancel the contributions to Eq. (14) from statistical nonuniformity and nonstationarity,

$$\partial_{u^i}\Phi_\mu{}^i = -\partial_\mu \rho_E. \tag{20}$$

The term Ψ is necessary to cancel the trace of Φ and must be divergenceless with respect to $\mathbf{u}$,

$$\partial_{u^i}\Psi_j{}^i = 0, \quad \Psi_0{}^i = 0, \quad \Psi_i^i = -\Phi_i^i. \tag{21}$$

As in the Thomson-87 approach [4], the drift is defined up to a nonunique term $(1/\rho_E)\Xi_\mu{}^i$ satisfying

$$\Xi_i^i = 0, \quad \partial_{u^i}\Xi_\mu{}^i = 0 \tag{22}$$

which, substituted into Eq. (14), will produce an identically zero contribution.

The drift $A_\mu{}^i$ is associated with the velocity correlation through the equation

$$\langle u^i \Delta u^j\rangle = \langle u^i \langle \Delta u^j | \mathbf{u}\rangle\rangle = \langle u^i A_\mu{}^j\rangle \Delta^\mu, \tag{23}$$

Let us analyze individually each of the terms in $A_\mu{}^i$. Substituting Eq. (19) into Eq. (23) we see that $\overline{A}_\mu{}^i$ gives just the symmetric piece of the correlation, i.e., the $-\frac{1}{2}\langle \Delta u^i \Delta u^j\rangle$ in Eq. (23). The Φ and Ψ terms are more easily analyzed in Fourier space, $\tilde{f}(\boldsymbol{\eta})=\int \mathrm{d}^3 u\ \mathrm{e}^{-\mathrm{i}\eta_i u^i} f(\mathbf{u})$. Using Eqs. (18), (19), and (23) will read

$$\langle u^i \Delta u^j\rangle = -\tfrac{1}{2}\langle \Delta w^i \Delta w^j\rangle - \mathrm{i}\Delta^\mu \partial_{\eta_j}(\tilde{\Phi}_\mu{}^i + \tilde{\Psi}_\mu{}^i)_{\eta=0}. \tag{24}$$

Using the fact that the generating function $\tilde{\rho}_E$ obeys $\tilde{\rho}_E=1-\frac{1}{2}R^{ij}\eta_i\eta_j+O(\eta^3)$, we can write, from Eq. (23),

$$\tilde{\Phi}_\mu{}^i = -\frac{\mathrm{i}}{2}\partial_\mu R^{ij}\eta_j + O(\eta^2) \tag{25}$$

so that the contribution from Φ to the correlation function is, from Eq. (24), $\frac{1}{2}\mathrm{d}x^\mu\, \partial_\mu R^{ij}$, which accounts for the spatial inhomogeneity of the correlation [the $\frac{1}{2}(R^{ij}+R^{ij})$ term on the right-hand side (RHS) of Eq. (16), which is centered at $\sim x^\mu$].

We see that $\overline{A}_\mu{}^i$ and $\Phi_\mu{}^i$ account for all of the contribution to the correlations, which either are symmetric, or come from inhomogeneity of the statistics. We know at this point that both $\Psi_\mu{}^i$ and $\Xi_\mu{}^i$ will be able to contribute only to the antisymmetric part of $A_\mu{}^i$. We give in explicit form the contribution from $\Psi_\mu{}^i$. Exploiting the first of Eq. (21), we utilize the ansatz

$$\tilde{\Psi}_j{}^i = \mathrm{i}(\delta_j^i \eta_k \tilde{\psi}^k - \eta_j \tilde{\psi}^i), \quad \tilde{\Psi}_0^i = 0 \tag{26}$$

and, from $\Psi_i^i=-\Phi_i^i$ and Eq. (25),

$$\tilde{\psi}^k = \frac{1}{4}(\partial_l R^{lk}) + \epsilon^{klm}\eta_l \tilde{f}_m + O(\eta^2) \tag{27}$$

with $\mathbf{f}$ arbitrary. The contribution to $\Psi_j{}^i$ from $\mathbf{f}$ is traceless and can be reabsorbed into the nonunique term Ξ; the final result is therefore

$$\tilde{\Psi}_j{}^i = \frac{\mathrm{i}}{4}[\delta_j^i(\partial_l R^{lk})\eta_k - (\partial_l R^{li})\eta_j] + O(\eta^3) \tag{28}$$

and the contribution to the correlation function is $\frac{1}{4}[\partial_l R^{lk}\, \mathrm{d}x^i - \partial_l R^{li}\, \mathrm{d}x^k]$, which is antisymmetric as required.

Explicit expressions for the drift terms are promptly obtained in the case of Gaussian statistics (expressions for the case of a symmetric ρ_E with kurtosis larger than three are given in the Appendix A). The velocity PDF reads therefore

$$\rho_E(\mathbf{u},x^\mu) \equiv \rho_G(\mathbf{u},x^\mu) = (8\pi^3\|R\|)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}S_{ij}u^i u^j\right) \tag{29}$$

with $S_{ij}=(R^{-1})_{ij}$ the Reynolds tensor inverse. In this case, the higher order terms in η_i entering Eqs. (25)–(28) disappear and we are left with

$$\overline{A}_\mu{}^i\, \mathrm{d}x^\mu = -\frac{1}{2}\langle dw^i\, dw^j\rangle S_{jk}u^k, \tag{30}$$

$$\Phi_\mu{}^i = \frac{1}{2}(\partial_\mu R^{ik})S_{kl}u^l \rho_E, \tag{31}$$

$$\Psi_j{}^i = \frac{1}{4}[-\delta_j^i(\partial_l R^{lk})S_{km} + (\partial_l R^{li})S_{jm}]u^m\rho_E. \qquad (32)$$

We can use these explicit expressions to obtain more information on the nature of the various contributions to the drift. In analogy to the case of the standard Langevin equation, we see that $\overline{A_\mu{}^i}$ must be discontinuous at $\Delta^\mu=0$. From Eq. (19), discontinuity of the correlation function derivative at $\Delta^\mu=0$ is necessary to balance the linear scaling of $\overline{\mathrm{d}w^2}$. In the coordinate system where, for the given Δ^μ, $\overline{A_j{}^i}$ is diagonal, we shall then have

$$\langle \Delta u^i|\mathbf{u}\rangle \sim -|\Delta^i|. \qquad (33)$$

As regards $\Phi_\mu{}^i$, we see from Eq. (31) that it produces an amplification of $\mathbf{u}$ when $\mathrm{d}x^\mu$ is directed to a region in space-time where the turbulence is stronger (this is easy to see when $R^{ij}\propto\delta^{ij}$).

Finally, the Ψ term, turns out to produce a complicated mixture of rotations and amplification of the velocity vector. Indicating $\tilde{u}_i=S_{ik}u^k$, and choosing the coordinate system so that $\partial_i R^{il}=4\tilde{c}\delta_1^l$, and $\Delta^3=0$ we have from Eq. (32),

$$\langle \Delta u^1|\mathbf{u}\rangle = \tilde{c}\tilde{u}_2\Delta^2, \quad \langle \Delta u^2|\mathbf{u}\rangle = -\tilde{c}\tilde{u}_1\Delta^2, \quad \langle \Delta u^3|\mathbf{u}\rangle = 0. \qquad (34)$$

If $R^{ij}\propto\delta^{ij}$, we will have $\tilde{u}_i=u_i$ and the result of Eq. (34) will be a rotation of $\mathbf{u}$ in the plane 12 as one moves in the direction x_2.

IV. DETERMINATION OF THE NOISE TENSOR

In order to obtain the drift coefficients, which give the decay of the turbulent correlations in the various space-time directions, it is necessary first to determine the form of the noise tensor $\langle dw^i\, dw^j\rangle$. In fact, it is in the noise that all the information on the turbulent structure is encoded (at least that part relative to the symmetric sector of the correlations). In the case of a Gaussian random velocity field, the noise tensor can be determined directly from the turbulent correlations by means of a fit in terms of products of exponentials with sines and cosines (a common practice in turbulence theory; consider, e.g., the Frenkiel functions [22]). Indicating $\mathrm{d}x^\mu=r^\mu\,\mathrm{d}s$, $r^\mu r_\mu=1$, we fit the turbulent correlation by the expression

$$\frac{\partial}{\partial s}\langle u^i(x^\mu)u^j(x^\mu+r^\mu s)\rangle = c_k{}^j\langle u^i(x^\mu)u^k(x^\mu+r^\mu s)\rangle, \qquad (35)$$

where $c_i{}^j$ depends on the direction r^μ, the midpoint position $x^\mu+\Delta^\mu/2$, but not on $\mathrm{d}s$. This imposes linear dependence of the random field drift on the velocity

$$\langle \mathrm{d}u^i|\mathbf{u}\rangle = \mathrm{d}x^\mu A_\mu{}^i = c_j{}^i u^j\,\mathrm{d}s \qquad (36)$$

(notice that Gaussian statistics, by itself, imposes linearity through the well mixed condition, only on the symmetric contribution to the drift $\overline{A_\mu{}^i}$). Using Eq. (36), Eq. (23) takes the form $\langle u^k du^i\rangle=c_j{}^i R^{jk}\,\mathrm{d}s$ and, from Eq. (17), we obtain

$$\langle dw^i\, dw^j\rangle = \langle du^i\, du^j\rangle = \tfrac{1}{2}(c_k{}^i R^{kj} + c_k{}^j R^{ki})\mathrm{d}s. \qquad (37)$$

We stress that, although Eq. (37) describes the behaviors of the random field correlations at small separations, the coefficients $c_j{}^i$ descend from a fit of turbulent correlations at finite separations.

A very general form for the noise tensor, satisfying the incompressibility condition $\partial\langle\Delta w^i\Delta w^j\rangle/\partial\Delta^i=0$, allowing association of Eq. (37) with geometric features of the flow, is a superposition of terms in the form

$$\langle \Delta w^i\Delta w^j\rangle = \frac{2u_T^2}{\tau_E}[B_t^{ij}(\Delta^0) + B^{ij}(\mathbf{\Delta}-\bar{\mathbf{u}}\Delta^0)],$$

$$\partial_{\Delta^i}B^{ij}(\mathbf{\Delta}) = 0. \qquad (38)$$

Here, τ_E fixes the time scales of the fluctuations and in the Gaussian case coincides with the Eulerian correlation time (see next), $u_T^2=\frac{1}{3}R_i^i$, and $B_t^{ij}=|\Delta^0|\delta^{ij}+\hat{B}_t^{ij}$, with $\hat{B}_t^{ij}$ symmetric and traceless. (For lighter notation we leave in this section the dependence from the space-time position unindicated.)

We see that the presence of mixed space-time increment contributions can account for situations in which the time correlations have Lagrangian nature. In this way, pure time decorrelation will take place in the reference system moving locally with the mean flow $\bar{\mathbf{u}}$. A situation with purely Eulerian time correlation will be realized by setting $\bar{\mathbf{u}}=0$.

In moderately anisotropic situations, it may be expedient to expand the space component B^{ij} in spherical tensors, following the SO(3) decomposition technique [19],

$$B^{ij}(\mathbf{\Delta}) = \sum_{J=0} B_J^{ij}(\mathbf{\Delta}), \qquad (39)$$

where B_J^{ij} indicates a combination of Jth order spherical tensors (see Appendix B). The symmetry of $\langle\Delta w^i\Delta w^j\rangle$ imposes selection rules on which spherical tensors may contribute; it turns out that to keep the lowest order anisotropic contribution, it is enough to consider spherical tensors of order $J=0$ and $J=2$. The incompressibility condition $\partial_{\Delta^i}B_J^{ij}=0$ gives then (the hats identify versors)

$$B^{ij}(\mathbf{\Delta}) = \frac{|\mathbf{\Delta}|}{u_T}\Bigg((a+4b^{lm}\hat{\Delta}_l\hat{\Delta}_m)\delta^{ij} + \frac{1}{3}[-a+(2b^{lm}-c^{lm})\times\hat{\Delta}_l\hat{\Delta}_m]\hat{\Delta}^i\hat{\Delta}^j - \hat{\Delta}_l[(2b^{li}+c^{li})\hat{\Delta}^j + (2b^{lj}+c^{lj})\hat{\Delta}^i] + 4c^{ij}\Bigg), \qquad (40)$$

where a gives the $J=0$ part, while the tensors b^{ij} and c^{ij}, which are symmetric and traceless, account for the $J=2$ part. We consider next some relevant limit cases.

A. Isotropic turbulence

In this case, all the spherical tensors with $J>0$ are zero. We are thus left with the simple expression

$$\langle \Delta w^i \Delta w^j \rangle = \frac{2u_T^2}{\tau_E}\left[|\Delta^0| \delta^{ij} + \frac{a|\boldsymbol{\Delta}|}{u_T}\left(\delta^{ij} - \frac{1}{3}\hat{\Delta}^i \hat{\Delta}^j \right) \right]. \quad (41)$$

The parameter a identifies a length-scale $l_u = u_T \tau_E / a$ for the fluctuations and has therefore the meaning of a ratio between the eddy lifetime τ_E and the eddy rotation time l_u/u_T.

B. Long axisymmetric vortices

Let us imagine that the correlation tensor is dominated by the effect of long axisymmetric vortices directed along x^1. Let us try to use this information to impose a structure to the space structure tensor B^{ij} defined in Eq. (40). Let us impose the condition that $B^{ij}(\boldsymbol{\Delta})=0$ for $\boldsymbol{\Delta}=\{\Delta,0,0\}$. For $\boldsymbol{\Delta}=\{\Delta,0,0\}$, we have from Eq. (40),

$$\frac{u_T B^{11}}{|\boldsymbol{\Delta}|} = \frac{1}{3}(2a + 2b^{11} + 5c^{11}), \quad \frac{u_T B^{22}}{|\boldsymbol{\Delta}|} = a + 4b^{11} + 4c^{22},$$

$$\frac{u_T B^{33}}{|\boldsymbol{\Delta}|} = a + 4b^{11} + 4c^{33},$$

$$\frac{u_T B^{12}}{|\boldsymbol{\Delta}|} = 3c^{12} - 2b^{12}, \quad \frac{u_T B^{13}}{|\boldsymbol{\Delta}|} = 3c^{13} - 2b^{13},$$

$$\frac{u_T B^{23}}{|\boldsymbol{\Delta}|} = 4c^{23}, \quad (42)$$

and we find immediately the result

$$c^{12} = c^{13} = \frac{2}{3}b^{12} = \frac{2}{3}b^{13}, \quad b^{23} = c^{23} = 0,$$

$$b^{11} = -\frac{3}{8}a, \quad b^{22} = b^{33} = \frac{3}{16}a, \quad c^{11} = -\frac{a}{4}, \quad c^{22} = c^{33} = \frac{a}{8}. \quad (43)$$

We are free to impose the condition $b^{1i}=c^{1i}=0$ for $i \neq 1$ and we reach the expression for generic $\boldsymbol{\Delta}$ of the B components along 11 and 22:

$$\frac{u_T B^{11}(\boldsymbol{\Delta})}{|\boldsymbol{\Delta}|} = \frac{1}{6}\hat{\Delta}_1^2 + \frac{3}{4}\hat{\Delta}_\perp^2 - \frac{1}{6}\hat{\Delta}_1^4 + \frac{1}{12}\hat{\Delta}_1^2\hat{\Delta}_\perp^2,$$

$$\frac{u_T B^{22}(\boldsymbol{\Delta})}{|\boldsymbol{\Delta}|} = \frac{3}{2} - \frac{3}{2}\hat{\Delta}_1^2 + \frac{3}{4}\hat{\Delta}_\perp^2 - \frac{4}{3}\hat{\Delta}_2^2 - \frac{1}{6}\hat{\Delta}_1^2\hat{\Delta}_2^2 + \frac{1}{12}\hat{\Delta}_\perp^2\hat{\Delta}_2^2, \quad (44)$$

where $\Delta_\perp^2 = \Delta_2^2 + \Delta_3^2$ and superscripts 2 indicate squares. Analyzing Eq. (44) in function of $\hat{\Delta}_1$ first for $\hat{\Delta}_2=0$ and then for $\hat{\Delta}_2=\hat{\Delta}_\perp$, it is possible to show that B^{ij} is always positive defined, as required.

C. Two-dimensional structures

Suppose that the flow is two-dimensional, say $\mathbf{u}=\{u_1,0,u_3\}$. In this case, the SO(3) decomposition reduces to an SO(2) one. Keeping again only the lowest order anisotropic correction, we find, for $\Delta=\{\Delta_1,0,\Delta_3\}$,

$$B^{ij}(\boldsymbol{\Delta}) = \frac{|\boldsymbol{\Delta}|}{u_T}\Big((a + 3b^{lm}\hat{\Delta}_l\hat{\Delta}_m)\delta^{ij} + \frac{1}{2}[-a + (b^{lm} - c^{lm}) \times \hat{\Delta}_l\hat{\Delta}_m]\hat{\Delta}^i\hat{\Delta}^j - \hat{\Delta}_l[(2b^{li} + c^{li})\hat{\Delta}^j + (2b^{lj} + c^{lj})\hat{\Delta}^i] + 3c^{ij} \Big), \quad (45)$$

where $b^{22}=b^{33}=0$ and the traceless condition imposes $b^{33}=-b^{11}$, $c^{33}=-c^{11}$.

D. Streaks

Two-dimensional streaks along the direction of the mean flow appear to be one of the characteristic structures in the viscous sublayer of wall turbulence [5]. Contrary to the three-dimensional case, elongated structures cannot be accommodated at $J=2$ in an SO(2) decomposition: the resulting noise tensor would not be positive definite. Nonetheless, a noise expression accounting for such structures can still be determined. For instance, it is easy to see that, if the streaks are oriented along x_1 and the flow is two-dimensional in the x_1x_3 plane, an appropriate expression for the noise tensor will be

$$B^{ij}(\boldsymbol{\Delta}) = a\frac{|\boldsymbol{\Delta}_3|}{u_T}\delta_1^i\delta_1^j. \quad (46)$$

In the non-Gaussian case, Eq. (36) ceases to be valid, and the random field correlation profile ceases to be in general a simple product of exponentials and sines or cosines. Even if we fit the turbulent correlation with an equation like (35), the random field correlations will not obey that equation, rather, one involving higher order correlations. This is because of the relation, imposed by the well-mixed condition, between non-Gaussian ρ_E and nonlinear $A_\mu{}^i$. For instance, if we used a bi-Gaussian distribution to model a high kurtosis PDF [7,23,24], a double exponential decay of correlations would ensue, with the slower decay associated with the intermittent bursts (see the end of Appendix A) [25].

For large kurtosis, the noise amplitude determines only the correlation times and lengths of the fast decaying exponential. The simplest approach, in this case, is to renormalize the noise amplitude, with respect to the Gaussian case, in order to correct for the longer correlations produced by the slowly decaying exponential. At the end of Appendix A, it is shown that, in order to have the desired space and time scales for the bursts, it is necessary to renormalize the noise amplitude by a factor $\beta=(2/3)k-1$ with k the kurtosis [see Eq. (A2)].

V. NONUNIQUENESS AND THE ANTISYMMETRIC SECTOR

Once the noise tensor and the PDF ρ_E are fixed and the well-mixed condition is imposed, the symmetric sector of the velocity correlation is completely determined. The nonunique term $\Xi_\mu{}^i$ can be used to fix the structure of the anisotropic sector. We consider for simplicity the homogeneous case $C^{ij}(x^\mu,\Delta^\mu)=C^{ij}(\Delta^\mu)$.

We discover immediately the following important fact: not all expressions for the nonunique term $\Xi_\mu{}^i$, and consequently for $A_\mu{}^i$, lead to a statistically realizable $C^{ij}(\Delta^\mu)$. This is a different face of the problem of local existence for the solutions of Eq. (14). Consider as a first example, $\Xi_j{}^i=\epsilon^{i2}{}_j u^2$. A contribution $\Delta C^{11}=\epsilon^{12}{}_3 R^{12}\,\mathrm{d}x^3$ is then added to $C^{11}(\mathrm{d}\mathbf{x})$, with $\mathrm{d}\mathbf{x}=\{0,0,\mathrm{d}x^3\}$, that has the inadmissible symmetry $\Delta C^{ij}(\mathrm{d}\mathbf{x})=-\Delta C^{ji}(-\mathrm{d}\mathbf{x})$. The second example is $\Xi_0{}^1=u^2$, $\Xi_0{}^2=-u^1$; in this case we find a contribution $R^{12}\,\mathrm{d}t$ to $C^{12}(\mathrm{d}t)$ with the inadmissible symmetry $\Delta C^{12}(\mathrm{d}t)=-\Delta C^{21}(\mathrm{d}t)$.

We seek a form of $\Xi_\mu{}^i$ satisfying all the required symmetries, but still sufficiently general to describe most geometric structures one may think of. In analogy with the case of the noise tensor, this can be done in the frame of an SO(3) expansion starting from C_A, the antisymmetric component of the correlation [see Eq. (17)].

As with the noise, the nonunique term $\Xi_\mu{}^i$ can be determined in a unique way from C_A in the case of Gaussian $\mathbf{u}$, fitting the turbulent correlations with exponentials multiplying sines or cosines; in this case, $\Xi_\mu{}^i$ will depend linearly on $\mathbf{u}$. Repeating with C_A the steps followed to obtain the noise tensor in Eq. (37), we obtain

$$C_A^{ki}=\tfrac{1}{2}[c_j{}^i R^{jk}-c_j{}^k R^{ji}]\mathrm{d}s. \qquad (47)$$

It turns out that the appropriate quantity on which to carry on the SO(3) expansion is not C_A^{ij}, rather

$$S_{ik}S_{jl}C_A^{kl}=\tfrac{1}{2}[c_i{}^k S_{kj}-c_j{}^k S_{ki}]\mathrm{d}s=r^\mu \xi^l{}_\mu \epsilon_{lij}\mathrm{d}s+\cdots, \qquad (48)$$

where $S_{ij}=(R^{-1})_{ij}$ and the terms in the expansion indicated in the formula are antisymmetric spherical tensors of order $J=0,1,2$ [see Eq. (B6)]. Isolating in Eq. (36) the contribution from $\Xi_\mu{}^i$ and using Eqs. (47) and (48), we obtain therefore for the nonunique term,

$$\Xi_\mu{}^i=\rho_E \xi^l{}_\mu \epsilon_{lmj}R^{mi}u^j+\cdots \qquad (49)$$

and it is immediate to check that the divergence free condition $\partial_{u^i}\Xi_\mu{}^i$ [the second of Eq. (22)] is satisfied.

We notice that, had we carried on the expansion directly on C_A^{ij}, the tensor R^{ij} entering Eq. (49), and consequently the inverse S_{ij} entering ρ_E in that formula would have been substituted by the identity matrix δ_{ij}. The contribution $\Xi_\mu{}^i/\rho_E$ to the drift (here ρ_E contains the right S_{ij}!) would have produced therefore explosive behaviors ($\mathrm{e}^{\alpha|\mathbf{u}|^2}$, $\alpha>0$) in some direction of $\mathbf{u}$. What happens is that linearity of $A_\mu{}^i$ together with antisymmetry of C_A^{ij}, impose the property $C_A^{ij}=\tilde{c}_{lm}R^{li}R^{mj}$ with $\tilde{c}_{lm}$ antisymmetric, and, keeping only the first terms in the SO(3) expansion for C_A^{ij} would cause losing this property.

We still need to enforce incompressibility, i.e., the zero trace condition $\Xi_i^i=0$ [the first of Eq. (22)]. The fact that the SO(3) expansion is carried on $S_{ik}S_{jl}C_A^{kl}$, rather than on C_A^{ij}, will lead to mixing of harmonics of different order J [compare with the case of the noise tensor and Eqs. (B3) and (B4)]. It is convenient to separate the antisymmetric part of $\xi^\mu{}_l$:

$$\xi^\mu{}_l=\overline{\xi^\mu}{}_l+\epsilon^l_{mk}\,\zeta^k. \qquad (50)$$

The zero trace condition becomes

$$\Xi_i^i=\rho_E\{\overline{\xi_i^k}R^{im}\epsilon_{klm}u^l+[R_i^i\zeta_l-R_l^i\zeta_i]u^l\}=0 \qquad (51)$$

which must be satisfied for any u^l. This leads to the relation between ζ^k and $\overline{\xi_m^l}$,

$$(R_l^j-R_k^k\delta_l^j)\zeta_j=\frac{1}{2}\epsilon_{klm}[\overline{\xi}R-R\overline{\xi}]^{km}. \qquad (52)$$

The presence of the commutator $[\overline{\xi}R-R\overline{\xi}]^{km}$ suggests that we should work in the diagonal system for R^{ij}. It becomes easy in this way to separate the part of $\overline{\xi_j^i}$ which anticommutes with R^{ij}, which is simply the part out of diagonal. Solution of Eq. (52) gives then in the diagonal coordinate system, after little algebra,

$$\zeta_1=\overline{\xi}_{23}\frac{R_{22}-R_{33}}{R_{22}+R_{33}},\quad \zeta_2=\overline{\xi}_{13}\frac{R_{33}-R_{11}}{R_{11}+R_{33}},\quad \zeta_3=\overline{\xi}_{12}\frac{R_{11}-R_{22}}{R_{11}+R_{22}}. \qquad (53)$$

We are now in the position to determine the effect of the various components of $\Xi_i{}^j$, on $A_i{}^j$ and on the velocity correlations.

A. Time component

It turns out that the $\mu=0$ component of Eq. (49) is associated with a combination of rotation and strain of the velocity, as time passes, at any given position $\mathbf{x}$; this is the Eulerian version of the mean velocity rotation along Lagrangian trajectories discussed in Ref. [6]. Working in the diagonal coordinate system for R^{ij}, the contribution from $\xi^0{}_3$ will be, for instance,

$$\Xi_0{}^1=\rho_E\xi^0{}_3R^{11}u^2,\quad \Xi_0{}^2=-\rho_E\xi^0{}_3\,R^{22}u^1. \qquad (54)$$

This turns into a pure rotation if $R^{ij}\propto\delta^{ij}$.

From the point of view of SO(3), this is the $J=1$ contribution to the second of Eq. (B6), which is trivially symmetric in space and antisymmetric in time.

B. Space component: diagonal part

Also this component is associated with a combination of rotation and strain of the velocity, this time, as one moves at fixed time from one space point to another. Focusing, e.g., on the contribution from $\overline{\xi_2^2}$, we find in the diagonal system for R^{ij},

$$\Xi_2{}^1=-\rho_E\overline{\xi_2^2}R^{11}u^3,\quad \Xi_2{}^3=\rho_E\overline{\xi_2^2}R^{33}u^1. \qquad (55)$$

In the case $R^{ij}\propto\delta^{ij}$, this becomes a pure rotation in the plane x^1x^3 as one moves in the x^2 direction. The diagonal component of the nonunique spatial term is the one associated with

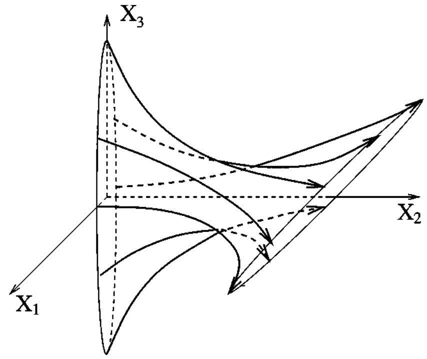

FIG. 1. Sketch of the velocity lines in a coherent structure characterized by a nonzero value of $\overline{\xi_1^3}$ (part out of diagonal of the nonunique spatial term). The velocity components in the 13 plane are arranged along strain line with expanding and compressing directions, respectively, along x_1 and x_3.

the presence of helicity H in the turbulent field. Indicating with $\omega^i = \epsilon^{ij}{}_k \partial_j u^k$ the vorticity, we can write

$$H = \langle u_i \omega^i \rangle = \langle u_i \langle \omega^i | \mathbf{u} \rangle \rangle = \epsilon^{ij}{}_k \langle u_i A_j{}^k \rangle. \tag{56}$$

Substituting the various contributions to $A_j{}^k$ in the above formula, we see that the only terms giving nonzero result are the diagonal ones in $\overline{\xi}$. Working in the diagonal coordinate system, we obtain then

$$H = 2[\overline{\xi_{11}} R_{22} R_{33} + \overline{\xi_{22}} R_{11} R_{33} + \overline{\xi_{33}} R_{11} R_{22}]. \tag{57}$$

From the point of view of SO(3), this is a combination of $J=2$ contributions from the second of Eq. (B6) (zero trace part of ξ) and $J=0$ contributions from the first of the same equation.

C. Space component: part out of diagonal

The effect of this component is illustrated, for the contribution from $\overline{\xi_1^3}$, in Fig. 1. This effect consists of a strain of the velocity components as one moves in the direction 2. We give in the equation below the nonzero matrix elements of $\Xi_k{}^i$ corresponding to $\overline{\xi_1^3}$ (components still evaluated in the diagonal coordinate system),

$$\Xi_1{}^1 = 2\rho_E \overline{\xi_1^3} \frac{R_{11} R_{33}}{R_{11} + R_{33}} u_2, \quad \Xi_1{}^2 = -\rho_E \overline{\xi_1^3} R_{22} u_1,$$

$$\Xi_2{}^1 = 2\rho_E \overline{\xi_1^3} \frac{R_{11}(R_{11} - R_{33})}{R_{11} + R_{33}} u_1,$$

$$\Xi_2{}^3 = 2\rho_E \overline{\xi_1^3} \frac{R_{33}(R_{11} - R_{33})}{R_{11} + R_{33}} u_3,$$

$$\Xi_3{}^2 = \rho_E \overline{\xi_1^3} R_{22} u_3, \quad \Xi_3{}^3 = -2\rho_E \overline{\xi_1^3} \frac{R_{11} R_{33}}{R_{11} + R_{33}} u_2. \tag{58}$$

From the point of view of SO(3), this is combination of $J=2$ components from the Eq. (B6) (the ζ piece) and $J=2$ components from the second of the same equation (the out of diagonal ξ piece). This is the case in which the incompressibility condition needs, in order to be enforced, consideration of spherical tensors of different order J.

It is possible to see that, in the non-Gaussian case, all the results obtained starting from Eq. (49) can be recovered substituting in that equation ρ_E with a Gaussian PDF ρ_G with identical S^{ij}. Notice that, if a bi-Gaussian is used to model ρ_E, the ratio ρ_G/ρ_E entering the contribution to the drift will decay like a Gaussian for large u, when the slowly decaying Gaussian entering ρ_E, which decays slower than ρ_G as well [see Eqs. (A1), (A2), and (A4)] become dominant. However, as in the case of the symmetric sector (see discussion at the end of the preceding section), the correspondence between drift and second order correlations ceases to be unique as Eq. (36) becomes nonlinear and Eq. (35) begins to involve higher order velocity correlations.

VI. DERIVATION OF MARKOVIAN LAGRANGIAN MODELS

Knowing the form of the tensors $A_\mu{}^i$ and $\langle \mathrm{d}w^i\, \mathrm{d}w^j \rangle$, allows the derivation of Lagrangian stochastic models. This is done most naturally setting in Eq. (8) $\mathrm{d}x^\mu = \{\mathrm{d}t, \mathbf{v}\, \mathrm{d}t\}$, where $\mathbf{v}$ is the particle velocity. This entails a Markovian assumption on the Lagrangian statistics, whose validity will be checked in the next two sections, although it is not very different from the one used in standard Lagrangian models.

A. Passive tracers

Let us write explicitly the Langevin and Fokker-Planck equations associated with our Lagrangian model, considering first the simpler case of a passive tracer $dx^\mu = \{\mathrm{d}t, \mathbf{u}\, \mathrm{d}t\}$ where $\mathbf{u}(t)$ identifies the fluid velocity sampled by the moving particle,

$$\mathrm{d}u^i \equiv \dot{u}^i\, \mathrm{d}t = (\mathbf{u} \cdot \mathbf{A}^i + A_0{}^i)\mathrm{d}t + \mathrm{d}w^i, \tag{59}$$

$$(\partial_t + \mathbf{u} \cdot \nabla)\rho_L + \partial_{u^i}[(\mathbf{u} \cdot \mathbf{A}^i + A_0{}^i)\rho_L] = \tfrac{1}{2} \partial_{u^i} \partial_{u^j} (\mathcal{B}^{ij} \rho_L), \tag{60}$$

where

$$\mathcal{B}^{ij} = \frac{\mathrm{d}}{\mathrm{d}\Delta^0} \langle \Delta w^i \Delta w^j \rangle = \frac{u_T^2}{\tau_E} [B_t^{ij}(1) + B^{ij}(\mathbf{u})]$$

[see Eq. (38)] and $\rho_L(\mathbf{u}, \mathbf{x}, t)$ is the PDF of finding a Lagrangian tracer at $\mathbf{x}$ with velocity $\mathbf{u}$.

Exactly as in the Thomson-87 approach, in the Gaussian case, the contribution to the drift $\mathbf{u} \cdot \mathbf{A}^i + A_0{}^i$ from turbulence nonhomogeneity is at most quadratic in $\mathbf{u}$, with the quadratic terms produced by $\mathbf{u} \cdot (\mathbf{\Phi}^i + \mathbf{\Psi}^i)$. However, disregarding the nonunique terms, the form of this contributions differs from the one discussed in Refs. [4,6].

Also the nonunique contribution $\mathbf{u} \cdot \mathbf{\Xi}^i$, to lowest order in the SO(3) expansion, is at most quadratic in $\mathbf{u}$, with the quadratic piece associated with the space component $\mathbf{u} \cdot \mathbf{\Xi}^i$. The observation in Ref. [26] that helical contributions in Lagrangian stochastic models must be quadratic in the velocity is thus confirmed.

Notice that, from the relation $\mathbf{\Delta}=\mathbf{u}\Delta^0$, higher orders in the SO(3) expansion correspond in the Lagrangian model to higher order polynomials in $\mathbf{u}$ contributing to $\mathcal{B}^{ij}$, $\mathbf{u}\cdot\overline{\mathbf{A}}^i+\overline{A}_0{}^i$ and $\mathbf{u}\cdot\Xi^i+\Xi_0{}^i$. Conversely, at the random field level, independently of the order in the SO(3) expansion (and for Gaussian statistics), the drift terms are at most linear in $\mathbf{u}$.

The important feature of the model described by Eqs. (59) and (60) is that the well-mixed condition imposed on the random field, encoded in Eqs. (18), (20), and (28) [Eqs. (30) and (32) in the Gaussian case], translates automatically into an identical condition on the trajectories. In the incompressible case considered here, this condition is equivalent to the ergodic property $\rho_L(\mathbf{u}|\mathbf{x},t)=\rho_E(\mathbf{u},\mathbf{x},t)$, which will be shown to hold, in the next section, right thanks to the condition $A_i^i=0$. This property implies trivially that Eulerian averages $\langle\rangle$ and averages along trajectories $\langle\rangle_L$ coincide.

At this point, the model described by Eqs. (59) and (60), is undistinguishable from a model derived through the Thomson-87 technique starting from the same PDF, the only difference being in the form of the noise term. In its simplest form, the noise tensor $\mathcal{B}^{ij}$ is isotropic, and is obtained by setting $\mathbf{\Delta}=\mathbf{u}\Delta^0$ in Eq. (41) and deriving with respect to Δ^0,

$$\mathcal{B}^{ij}=\frac{2u_T^2}{\tau_E}\left[\delta^{ij}+\frac{a|\mathbf{u}|}{u_T}\left(\delta^{ij}-\frac{u^iu^j}{3|\mathbf{u}|^2}\right)\right]. \tag{61}$$

This expression must be compared with the one in the Thomson-87 approach: $\mathcal{B}^{ij}=\delta^{ij}C_0\overline{\epsilon}$. (As a technical aside, notice that we started in Sec. II with an additive noise and we have arrived here at a multiplicative noise term, which is automatically intended, in the approach that we have followed, in the Itô sense [20].)

The difference in the analytical expressions underlies a difference in physical interpretation: while in the Thomson-87 technique, $\mathcal{B}^{ij}\,\mathrm{d}t$ is precisely the Lagrangian time structure function for inertial time separation, in our approach, it is a nonuniversal quantity whose form is determined in function of the large scale turbulence geometry. In the Thomson-87 approach, the time scale is fixed by the viscous dissipation $\overline{\epsilon}$, which fixes the expression for the Lagrangian correlation time

$$\tau_L=\frac{2u_T^2}{C_0\overline{\epsilon}}. \tag{62}$$

In our approach, the time scale is fixed directly by τ_E. To be precise, the association between τ_L and $\overline{\epsilon}$ in the Thomson-87 model, is strictly valid only in the Gaussian case [25]. Also in our approach, however, non-Gaussian statistics leads to τ_E not being directly associated with the Eulerian time scales, but only with the fast part of the correlation decay (see the end of Appendix A).

The two approaches depend on dimensionless constants, C_0 and a, which can be related in a semiquantitative way. As discussed in correspondence to Eq. (41), the parameter a identifies a characteristic length for the random field $l_u=u_T\tau_E/a$, which, at least in the Gaussian case, corresponds to the integral length of the turbulence. Substituting the estimate for the viscous dissipation from our model $\overline{\epsilon}\sim u_T^3/l_u$ into Eq. (62) and setting $\tau_E\sim\tau_L$, we obtain

$$a\sim C_0^{-1}.$$

This tells us that the Thomson-87 model cannot be recovered from Eq. (61), for finite C_0, by setting simultaneously $a=0$ and $2u_T^2/\tau_E=C_0\overline{\epsilon}$. The equivalent limits $a\to 0$ and $C_0\to\infty$ correspond to the regime of τ_E much shorter than the eddy rotation time corresponding to the Kraichnan model [28]. The way in which this limit is carried out, however, is different in the two approaches: in ours, it is the turbulence integral scale l_u that is sent to infinity [see comment after Eq. (41)]; in the standard approach, the diffusive limit $C_0\to\infty$ corresponds directly to the one $\tau_E\to 0$.

B. Solid particles

Let us pass to the analysis of the solid particle case. The solid particle dynamics obeys an equation, which in quite general form (neglecting memory effects associated with the Basset force) can be written as

$$\dot{\mathbf{v}}=\mathbf{F}(\mathbf{v},\mathbf{u}). \tag{63}$$

A general form was derived in Ref. [30], where neglect lift effects Ref. [31]. These equations are all derived in the limit of particle diameter d small with respect to the scales of the flow, and low particle Reynolds number $\mathrm{Re}_p=d|\mathbf{u}-\mathbf{v}|/\nu$, where ν is the kinematic viscosity. We will consider Eq. (63) in simplified form by accounting only for linear Stokes drag and gravity,

$$\dot{\mathbf{v}}=\frac{\mathbf{u}-\mathbf{v}+\mathbf{v}_G}{\tau_S}, \tag{64}$$

where $\tau_S=1/18\gamma d^2/\nu$, with γ the density of the particle relative to that of the fluid, is the Stokes time, and $\mathbf{v}_G$ is the particle terminal velocity in a uniform force field and a quiescent fluid. In the case of gravity, $\mathbf{v}_G=\tau_S\mathbf{g}$ with $\mathbf{g}$ the gravitational acceleration. More in general, $\mathbf{v}_G$, may account for body forces like the effect of the Saffman lift [31].

The analogue of Eq. (59), in the solid particle case will be obtained setting in Eq. (8) $\mathrm{d}x^\mu=\{\mathrm{d}t,\mathbf{v}\,\mathrm{d}t\}$, with $\mathbf{v}$ the solid particle velocity,

$$\mathrm{d}u^i\equiv\dot{u}^i\,\mathrm{d}t=(\mathbf{v}\cdot\mathbf{A}^i+A_0{}^i)\mathrm{d}t+\mathrm{d}w^i, \tag{65}$$

where now $\mathbf{u}(t)$ is the fluid velocity sampled by the solid particle and $\langle\mathrm{d}w^i\mathrm{d}w^j\rangle=\mathcal{B}^{ij}\mathrm{d}t=(u_T^2/\tau_E)[B_t^{ij}(1)+B^{ij}(\mathbf{v})]\mathrm{d}t$. Notice that the drift tensor $A_\mu{}^i$ still depends on $\mathbf{u}$, while $\langle dw^i\,dw^j\rangle$ depends only on $\mathbf{v}$. The Lagrangian PDF $\rho_L(\mathbf{u},\mathbf{v},\mathbf{x},t)$ will obey the Fokker-Planck equation,

$$(\partial_t+\mathbf{v}\cdot\nabla)\rho_L+\partial_{v^i}(F^i\rho_L)+\partial_{u^i}[(\mathbf{v}\cdot\mathbf{A}^i+A_0{}^i)\rho_L]$$
$$=\frac{1}{2}\partial_{u^i}\partial_{u^j}(\mathcal{B}^{ij}\rho_L). \tag{66}$$

Equations (63), (65), and (66) are in the standard form for a "two-fluid" Lagrangian model for solid particle transport, i.e., a model in which the fluid and solid phase are taken into account at the same time and are treated on the same footing. All problems in the fluid limit, present in models in which the separation of fluid and solid particle trajectories was considered without accounting for the geometry of the process [16] are clearly avoided: when inertia and gravity are sent to zero, the fluid case described by Eqs. (59) and (60) is automatically recovered.

We can easily estimate the turbophoretic drift, i.e., the component of particle transport due to the turbulence intensity gradient [32,33]. Multiplying Eq. (66) by v^i and integrating in $\mathbf{v}$ and $\mathbf{u}$ we obtain at stationary state and for uniform concentration $\partial_j\langle v^i v^j\rangle_{\mathrm{L}}=\langle F^i(\mathbf{v},\mathbf{u})\rangle_{\mathrm{L}}$, which, for linear F^i, can be inverted to obtain $\langle v^i\rangle_{\mathrm{L}}$ [subscript L indicates that we are averaging over the Lagrangian PDF $\rho_L(\mathbf{u},\mathbf{v},\mathbf{x},t)$]. In the Stokesian case described by Eq. (66), and small Stokes number $\mathrm{St}=\tau_S/\tau_E$, we can approximate $\langle v^i v^j\rangle_L=R^{ij}$, and we obtain

$$\langle v^i\rangle_{\mathrm{L}} = -\tau_S\partial_j R^{ij} + v^i_G. \tag{67}$$

We can try to understand Eqs. (65) and (67) from the point of view of a model satisfying the well mixed condition. The particle flow, due to the effect of inertia, is compressible and preferential concentration phenomena are known to occur [29]. Therefore, we do not expect in general the ergodic property $\rho_L(\mathbf{u},\mathbf{v}|\mathbf{x},t)=\rho_E(\mathbf{v},\mathbf{u},\mathbf{x},t)$ to be satisfied. Turbophoresis provides the simplest illustration of this phenomenon. Averaging Eq. (64) over $\rho_L(\mathbf{u},\mathbf{v},\mathbf{x},t)$ and combining with Eq. (67), we obtain in fact the relation

$$\langle u^i\rangle_{\mathrm{L}} = -\tau_S\partial_j R^{ij} \neq \langle u^i\rangle = 0,$$

i.e., in inhomogeneous turbulence conditions, Eulerian and Lagrangian averages give different results.

For these reasons, in the Thomson-87 approach, a two-fluid solid particle transport model, would require knowledge, in some reference situation, of the Lagrangian PDF $\rho_L(\mathbf{u},\mathbf{v},\mathbf{x},t)$, meaning that information must be available on both the mean particle concentration $\theta(\mathbf{x},t)=\int \mathrm{d}^3u\, \mathrm{d}^3v\, \rho_L(\mathbf{u},\mathbf{v},\mathbf{x},t)$ and the conditional PDF (the PDF along a single particle trajectory) $\rho_L(\mathbf{u},\mathbf{v}|\mathbf{x},t)$. Notice that this may imply substituting Eq. (63) and (64) with a model equation whose coefficient are determined by the well mixed condition. Actually, one-fluid models exist, in which only the particle phase is considered and only the PDF $\rho_L(\mathbf{v}|\mathbf{x},t)$ must be known, while θ is obtained from the continuity equation $\partial_t\theta+\nabla\cdot(\langle\mathbf{v}\rangle_{\mathrm{L}}\theta)=0$ [39].

In our approach, knowledge of ρ_E is sufficient to determine the form of the equation for $\mathbf{u}$ (65) (the one for $\mathbf{v}$ is unchanged) without any assumption on the form of $\rho_L(\mathbf{u},\mathbf{v},\mathbf{x},t)$. This will be shown in Sec. IX to produce automatically the correct form of the Lagrangian correlations for the fluid velocity along solid particle trajectories, accounting for the effect of inertia and trajectory crossing [11]. Notice that, imposing the well mixed condition on $\overline{\rho_L}(\mathbf{u}|\mathbf{x},t)=\int \mathrm{d}^3v\, \rho_L(\mathbf{u},\mathbf{v}|\mathbf{x},t)$, within a simplifying ergodic hypothesis $\overline{\rho_L}=\rho_E$, is not sufficient to obtain these correct behaviors; the anisotropic renormalization of the correlation times may be accounted for only by *ad hoc* modification of the expression for the noise tensor Eq. (3) [12].

VII. LAGRANGIAN ONE-TIME STATISTICS AND ERGODIC PROPERTIES

As discussed at the end of Sec. II, we can imagine Eq. (8) as giving the local behavior of a random velocity field whose restriction to straight lines in space-time are Markovian processes. This allowed to have a random velocity field with Eulerian correlations in time and space, which are both well defined and easy to calculate. Unfortunately, unless the $a\to 0$ limit of the Kraichnan model is considered [28], it is not possible to hypothesize at the same time a Markovian behavior along trajectories. In consequence of this, the Lagrangian statistics becomes a complicated business.

However, it turns out that different statistical quantities are affected by the presence of memory in qualitative different ways and there are situations in which Markovianization of the trajectories becomes appropriate. Let us try to understand what happens in detail.

The central quantity one needs for a description of Lagrangian statistics are conditional probabilities in the form

$$\rho_L(\mathbf{X}_0|\mathbf{X}_1\ldots\mathbf{X}_n), \tag{68}$$

where $\mathbf{X}_k\equiv\mathbf{X}(t_k)\equiv\{\mathbf{u}(\mathbf{x}(t_k),t_k),\mathbf{x}(t_k)\}$, $k=0,\ldots n$. Let us consider for now the simplest case of a passive tracer. Such conditional probabilities could be obtained ideally by carrying on a Monte Carlo of trajectories originating from $\{t_n,\mathbf{X}_n\}$ and sampling the particle positions and velocities at $t=t_k$, $k=n-1,\ldots 1$. Let us focus on the case $n=1$, which presents already all the difficulties due to memory. Actually, the transition probability $\rho(\mathbf{X}_0|\mathbf{X}_1)$ gives precisely the evolution of a cloud of tracers from an instantaneous release, i.e., a puff; from the same transition probability, also the Lagrangian correlation time could be determined and the calculation will be illustrated in the next section. Suppose we have a set of trajectories starting at time t_1 with initial condition $\mathbf{X}_1$, whose form is known up to time t. This allows us to reconstruct $\rho_L(\mathbf{X}(t)|\mathbf{X}_1)$. The conditional probability at the instant $t+\mathrm{d}t$ will be given by the formula

$$\rho_L(\mathbf{X}(t+\mathrm{d}t)|\mathbf{X}_1) = \int \mathrm{d}^6X(t)\rho_L(\mathbf{X}(t+\mathrm{d}t)|\mathbf{X}(t),\mathbf{X}_1)\rho_L(\mathbf{X}(t)|\mathbf{X}_1) \tag{69}$$

which corresponds to first summing all the trajectories going from $\{t_1,\mathbf{X}_1\}$ to $\{t+\mathrm{d}t,\mathbf{X}(t+\mathrm{d}t)\}$ passing through $\{t,\mathbf{X}(t)\}$, and then summing over $\mathbf{X}(t)$. Now, to determine $\rho(\mathbf{X}(t+\mathrm{d}t)|\mathbf{X}(t),\mathbf{X}_1)$, we could average first on the part of the trajectories going from $\{t_1,\mathbf{X}_1\}$ to $\{t,\mathbf{X}(t)\}$ and then on that going from $\{t,\mathbf{X}(t)\}$ to $\{t+\mathrm{d}t,\mathbf{X}(t+\mathrm{d}t)\}$. From the point of view of a Monte Carlo, this means that we can consider an ensemble of fictitious trajectories whose dynamics are only conditioned to the initial condition $\{t_1,\mathbf{X}_1\}$ and to the current position $\{t,\mathbf{X}(t)\}$.

We thus reach the not so obvious conclusion that, to determine the evolution of a PDF with conditions at n previous instants, we need to study a dynamics conditioned to $n+1$ instants, but we do not need the whole trajectory history.

Because of this, if we are interested in a one-time PDF, Markovianization of the dynamics will be an appropriate procedure.

This fact allows us to verify analytically that the one-point velocity PDF sampled by a passive tracer coincides with the Eulerian PDF; in other words the ergodic property one expects from incompressibility is satisfied.

In the case of a passive tracer, $\mathrm{d}x^\mu=\{\mathrm{d}t,\mathbf{u}\,\mathrm{d}t\}$ where $\mathbf{u}(t)$ identifies the fluid velocity sampled by the moving particle, and Eqs. (59) and (60) will be the Langevin and Fokker-Planck equations associated with the Markovianized dynamics. From incompressibility [see Eq. (12)], and from the properties of the drift components $\underline{A}$, Φ, and Ψ [see Eqs. (18) and (21)], we can write

$$(\partial_t+\mathbf{u}\cdot\nabla)\rho_L+\partial_{u^i}\Phi_0^i+u^j\partial_{u^i}\Phi_j^i = -\partial_{u^i}\Big[\Big(\overline{A_j}^i u^j+\overline{A_0}^i-\tfrac{1}{2}\mathcal{B}^{ik}\partial_{u^k}\Big)\rho_L\Big]. \tag{70}$$

Setting $\rho_L=\rho_E$, from Eqs. (19) and the time component of Eq. (20), Eq. (70) reduces to $u^i\partial_i\rho_E=-u^j\partial_{u^i}\Phi_j^i$, which, from Eq. (20), is an identity. The ergodic property is thus satisfied.

This is not a trivial property. We can easily construct a counter-example in which the incompressibility property $\partial_{\Delta^i}\langle\Delta w^i\Delta w^j\rangle=0$ [the second of Eq. (12)] is not satisfied and ergodicity is violated. Considering for simplicity stationary homogeneous turbulence (hence $\Phi_\mu{}^i=\Psi_\mu{}^i=0$) and choosing $\Xi_\mu{}^i=0$, we have in fact, setting in Eq. (19) $\mathrm{d}x^\mu=\{\mathrm{d}t,\mathbf{u}\,\mathrm{d}t\}$,

$$\mathbf{u}\cdot\mathbf{A}^i+A_0{}^i=\tfrac{1}{2}\mathcal{B}^{ij}\partial_{u^j}\log\rho_E \tag{71}$$

while Eq. (60) dictates

$$\mathbf{u}\cdot\mathbf{A}^i+A_0{}^i=\tfrac{1}{2}\mathcal{B}^{ij}\partial_{u^j}\log\rho_L+\tfrac{1}{2}\partial_{u^j}\mathcal{B}^{ij}. \tag{72}$$

Combining Eqs. (72) and (73), leads to a differential equation for ρ_L which, due to homogeneity of $\mathcal{B}$ in $|\mathbf{u}|$, can be integrated along the direction $\hat{\mathbf{u}}=\mathbf{u}/|\mathbf{u}|$,

$$\rho_L(\mathbf{u})=\text{const}\ \rho_E(\mathbf{u})\exp\{-\hat{u}^j\int \mathrm{d}s[(\mathcal{B}^{-1})_{jl}\partial_{u^k}\mathcal{B}^{lk}]_{\mathbf{u}=\hat{\mathbf{u}}s}\}. \tag{73}$$

Taking a noise term not satisfying incompressibility, e.g., $B^{ij}(\mathbf{\Delta})=2u_T/\tau_E|\mathbf{\Delta}|\delta^{ij}$, we would obtain $\rho_L(\mathbf{u})=\text{const}|\mathbf{u}|^{-1}\rho_E(\mathbf{u})$ and ergodicity violation.

We can repeat the calculation to check for departures from ergodicity in the solid particle case. Ergodicity means in this case that the fluid velocity distribution sampled by the solid particle

$$\overline{\rho_L}(\mathbf{u}|\mathbf{x})=\int\rho_L(\mathbf{u},\mathbf{v}|\mathbf{x})d\mathbf{v} \tag{74}$$

coincides with $\rho_E(\mathbf{u},\mathbf{x})$. In all the Monte Carlo simulations that we have carried on, described in detail in Sec. IX, we have found that, despite compressibility of the solid particle flow, ergodicity was satisfied in isotropic homogeneous conditions. The mechanism seems to be the following.

In homogeneous stationary conditions, the Fokker-Planck equation for the distribution $\rho_L(\mathbf{u},\mathbf{v})$ will read, from Eq. (66),

$$\partial_{v^i}(F^i\rho_L)+\partial_{u^i}[(\mathbf{v}\cdot\mathbf{A}^i+A_0{}^i)\rho_L]=\tfrac{1}{2}\partial_{u^i}\partial_{u^j}\mathcal{B}^{ij}\rho_L, \tag{75}$$

where $\mathcal{B}^{ij}=\mathcal{B}^{ij}(\mathbf{v})$. Exploiting well-mixed [see Eq. (19)] and setting from isotropy $\Xi=0$, this equation can be rewritten in the form

$$\partial_{v^i}(F^i\rho_L)+\partial_{u^i}\Big[\tfrac{1}{2}\mathcal{B}^{ij}(\partial_{u^j}\rho_L-\rho_L\partial_{u^j}\log\rho_E)\Big]=0 \tag{76}$$

and, integrating in d^3v, we reach the following equation for the deviation from ergodicity,

$$\partial_{u^i}[\overline{\rho_L}(\langle\mathcal{B}^{ij}|\mathbf{u}\rangle\partial_{u^j}\log\overline{\rho_L}/\rho_E+\partial_{u^j}\langle\mathcal{B}^{ij}|\mathbf{u}\rangle)]=0. \tag{77}$$

We see that nonergodic behaviors are associated with the divergence of the average over solid particle trajectories of the velocity structure function,

$$\partial_{u^j}\langle\mathcal{B}^{ij}|\mathbf{u}\rangle=\int\mathrm{d}^3v\,\partial_{u^j}\rho_L(\mathbf{v}|\mathbf{u})\mathcal{B}^{ij}(\mathbf{v}). \tag{78}$$

In the case of solid particles, for which $\langle\mathcal{B}^{ij}|\mathbf{u}\rangle\neq\mathcal{B}^{ij}(\mathbf{u})$, we would expect in general $\partial_{u^j}\langle\mathcal{B}^{ij}|\mathbf{u}\rangle\neq 0$. This turns out not to be true, however, when turbulence is isotropic. Let us show how this happens.

We can decompose $\nabla_u\rho_L(\mathbf{v}|\mathbf{u})$ in spherical vectors depending on $\mathbf{v}$ [see Eqs. (B7) and (B8)]

$$\nabla_u\rho_L(\mathbf{v}|\mathbf{u})=\rho_{01}\mathbf{v}+\rho_{11}(\mathbf{u}\cdot\mathbf{v})\mathbf{v}+\rho_{12}\mathbf{u}+\cdots, \tag{79}$$

where, from isotropy, $\rho_{lk}=\rho_{lk}(|\mathbf{v}|,|\mathbf{u}|)$. Higher harmonics (not indicated) are by construction orthogonal (see Appendix B). From Eq. (41), we see that only the term

$$\mathbf{h}=\rho_{11}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}+\rho_{12}\mathbf{u} \tag{80}$$

can contribute to $\partial_{u^j}\langle\mathcal{B}^{ij}|\mathbf{u}\rangle$. In order for this contribution to be zero, it is sufficient that the curl with respect to $\mathbf{v}$ of $\mathbf{h}$ be identically zero,

$$[|\mathbf{v}|^{-1}\partial_{|\mathbf{v}|}\rho_{12}-\rho_{11}]\mathbf{v}\times\mathbf{u}=0 \tag{81}$$

so that $\mathbf{h}$ can be written in the form of a potential term $\nabla_v g(\mathbf{u},\mathbf{v})$, and, substituting into Eq. (78) and integrating by parts,

$$\partial_{u^j}\langle\mathcal{B}^{ij}|\mathbf{u}\rangle=-\int\mathrm{d}^3v\ g(\mathbf{u},\mathbf{v})\partial_{v^j}\mathcal{B}^{ij}(\mathbf{v})=0.$$

Now, we can obtain Eq. (81) simply by imposing the condition, from isotropy,

$$\partial_{u^i}\langle|\mathbf{v}|^n v^j v^k|\mathbf{u}\rangle=\partial_{u^j}\langle|\mathbf{v}|^n v^i v^k|\mathbf{u}\rangle, \tag{82}$$

with $i\neq j\neq k$ and $n\geqslant 0$. In fact, writing the averages in explicit form, Eq. (82) can be shown to be equivalent to

$$\int\mathrm{d}^3v|\mathbf{v}|^{n+1}v^k[\partial_{v^i}\partial_{u^j}-\partial_{v^i}\partial_{u^j}]\rho_L(\mathbf{v}|\mathbf{u}) \tag{83}$$

and again, from orthogonality of the decomposition, only the ρ_{11} and ρ_{12} terms in $\nabla_u\rho_L$ could contribute. Hence, exploiting the fact that ρ_{ij} depends only on $|\mathbf{u}|$ and $|\mathbf{v}|$, Eq. (83) becomes equivalent to

$$\int d^3v[|\mathbf{v}|^{-1}\partial_{|\mathbf{v}|}\rho_{12}-\rho_{11}]|\mathbf{v}|^{n+1}=0$$

which implies Eq. (81) and satisfaction of the ergodic property.

VIII. TWO-TIME STATISTICS AND THE LAGRANGIAN CORRELATION TIME

Explicit determination of the Lagrangian dynamics taking into account memory of an initial condition is possible when the $\mathbf{u}(\mathbf{x},t)$ is isotropic, homogeneous, and Gaussian. It thus becomes possible to estimate the error implied in the Markovianization of the trajectories. The simplest estimator is the Lagrangian correlation time

$$\tau_L=\frac{1}{3u_T^2}\int_0^\infty dt\langle\mathbf{u}(t)\cdot\mathbf{u}(0)\rangle, \qquad (84)$$

where $\mathbf{u}(t)\equiv\mathbf{u}(\mathbf{x}(t),t)$ and we are considering passive tracers. As discussed at the start of the preceding section, we need an evolution equation for the trajectory $\{\mathbf{u}(t),\mathbf{x}(t)\}$, given an initial condition at $t=0$ [for simplicity, fix $\mathbf{x}(0)=0$]. The starting point is the following decomposition for the tracer velocity:

$$\mathbf{u}(t+\Delta)=\langle\mathbf{u}(t+\Delta)|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle+\Delta\mathbf{w} \qquad (85)$$

plus knowledge of the conditional averages,

$$\langle\mathbf{u}(t+\Delta)|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle$$

and

$$\langle\Delta\mathbf{w}\Delta\mathbf{w}|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle. \qquad (86)$$

As discussed in detail in Appendix C, these averages can be obtained from the correlation between velocities at $\{0,0\}$, $\{t,\mathbf{x}(t)\}$, and $\{t+\Delta,\mathbf{x}(t+\Delta)\}$. We identify correlations between points on a trajectory by

$$\hat{C}^{ij}(\Delta)=\langle u^i(t)u^j(t+\Delta)\rangle. \qquad (87)$$

If $\mathbf{u}$ is Gaussian, homogeneous, and isotropic, these correlations can be expressed in analytical form. The mean rate of fluid velocity change along a generic space-time direction $\{1,\mathbf{V}\}$ will in this case take the form

$$A_0^i+V^jA_j^i=-\frac{1}{\tau_E}\left[\left(1+a\frac{|\mathbf{V}|}{u_T}\right)u_\perp^i+\left(1+a\frac{2|\mathbf{V}|}{3u_T}\right)u_\parallel^i\right], \qquad (88)$$

where $\mathbf{u}_\perp$ and $\mathbf{u}_\parallel$ are the components of $\mathbf{u}$ perpendicular and parallel to the fixed direction $\mathbf{V}$, and we have used Eqs. (29), (30), and (41), and the expression $R^{ij}=u_T^2\delta^{ij}$.

Solving the equations for the correlation function along $\{1,\mathbf{V}\}$,

$$\frac{d}{dt}\langle u^i(\mathbf{V}t,t)u^j(0,0)\rangle=\langle[A_0^i+V^jA_j^i]u^j(0,0)\rangle \qquad (89)$$

and introducing longitudinal and transverse projectors

$$\Pi_\parallel^{ij}(\mathbf{V})=\frac{V^iV^j}{|\mathbf{V}|^2},\quad \Pi_\perp^{ij}(\mathbf{V})=\delta^{ij}-\Pi_\parallel^{ij}(\mathbf{V}), \qquad (90)$$

we obtain

$$\langle u^i(\mathbf{V}t,t)u^j(0,0)\rangle=\Pi_\perp^{ij}(\mathbf{V})C_{V\perp}(t)+\Pi_\parallel^{ij}(\mathbf{V})C_{V\parallel}(t), \qquad (91)$$

where

$$C_{V\perp}(t)=u_T^2\exp\left(-\left(1+\frac{a|\mathbf{V}|}{u_T}\right)\frac{t}{\tau_E}\right) \qquad (92)$$

and

$$C_{V\parallel}(t)=u_T^2\exp\left(-\left(1+a\frac{2|\mathbf{V}|}{3u_T}\right)\frac{t}{\tau_E}\right). \qquad (93)$$

We shall need also the inverse $D_{ij}(t_l-t_m)$ of the correlation matrix $\langle u^i(t_l)u^j(t_m)\rangle$, $l,m=1,2$; $t_1=0$, $t_2=t$, defined by the relation

$$\sum_m D_{ij}(t_l-t_m)\langle u^j(t_m)u^k(t_n)\rangle=\sum_m\langle u^k(l_n)u^j(t_m)\rangle D_{ji}(t_m-t_l)=\delta_i^k\delta_{ln}. \qquad (94)$$

From Eqs. (91)–(93), we find

$$D_{ij}(t_l-t_m)=\Pi_{ij}^\perp(\mathbf{U})D_\perp(t_l-t_m)+\Pi_{ij}^\parallel(\mathbf{U})D_\parallel(t_l-t_m), \qquad (95)$$

where $\mathbf{U}=\mathbf{u}(t)-\mathbf{u}(0)$,

$$D_\perp=\frac{1}{C_{U\perp}^2(0)-C_{U\perp}^2(t)}\begin{pmatrix}C_{U\perp}(0) & -C_{U\perp}(t)\\ -C_{U\perp}(t) & C_{U\perp}(0)\end{pmatrix} \qquad (96)$$

and we have similar expression for $D_\parallel$. At this point, we can obtain from Eq. (C7) the expression for the average evolution of the velocity along a trajectory, conditioned to an initial condition at time zero,

$$\langle u^i(t+\Delta)|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle=[\hat{C}^{ij}(\Delta)D_{jk}(t)+\hat{C}^{ij}(t+\Delta)\times D_{jk}(0)]u^k(0)+[C^{ij}(\Delta)D_{jk}(0)+C^{ij}(t+\Delta)D_{jk}(t)]u^k(t). \qquad (97)$$

As obvious, memory of the initial condition at time zero is lost when $t\to\infty$. Notice that, if all points $\{0,0\}$, $\{t,\mathbf{x}(t)\}$, and $\{t+\Delta,\mathbf{x}(t+\Delta)\}$ are aligned along the same space-time direction $\{1,\mathbf{V}\}$, all the $\Pi_\parallel$ and $\Pi_\perp$ entering the correlation functions in Eq. (95) will project along or perpendicular to the same vector $\mathbf{V}$. In this case the components of $\mathbf{u}$ parallel and perpendicular to $\mathbf{V}$ decouple and the indices disappear; for instance,

$$\langle u_\parallel(t+\Delta)|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle=[C_{V\parallel}(\Delta)D_{V\parallel}(t)+C_{V\parallel}(t+\Delta)\times D_{V\parallel}(0)]u_\parallel(0)+[C_{V\parallel}(\Delta)D_{V\parallel}(0)+C_{V\parallel}(t+\Delta)D_{V\parallel}(t)]u_\parallel(t). \qquad (98)$$

and all the correlations have the same decay rate fixed by Eq.

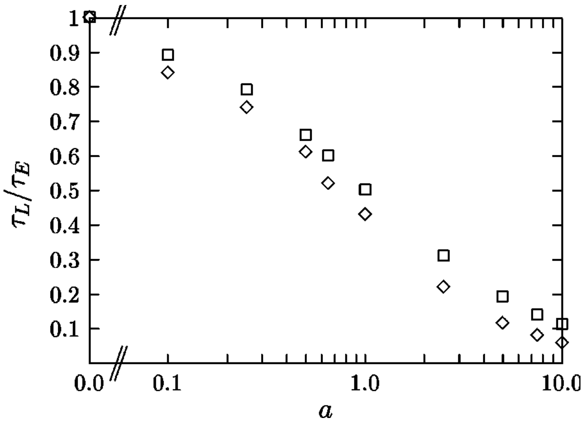

FIG. 2. Dependence of the Lagrangian correlation time on the ratio a between eddy lifetime and eddy rotation time; □, exact; ◇, Markovian approximation.

(93). It is then easy to show that the first term on the RHS of Eq. (98) disappears and we have

$$\langle u_{\parallel}(t+\Delta)|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle = u_{\parallel}(t)\exp\left(-\left(1+a\frac{2|\mathbf{V}|}{3u_T}\right)\frac{\Delta}{\tau_E}\right). \tag{99}$$

Hence $\langle u_{\parallel}(t+\Delta)|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle = \langle u_{\parallel}(t+\Delta)|\mathbf{u}(t),\mathbf{x}(t)\rangle$. If the trajectory is developing along a straight line, we will recover Markovian statistics as required.

Expressing $\Delta\mathbf{w}$ as the difference between $\langle \mathbf{u}(t+\Delta)|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle$ and $\mathbf{u}(t+\Delta)$, and substituting into Eq. (C8), we obtain instead, for the fluctuation term

$$\begin{aligned}\langle \Delta w^i \Delta w^j|\mathbf{u}(t),\mathbf{x}(t);\mathbf{u}(0),0\rangle &= C^{ij}(0) - D_{kl}(0)[C^{li}(\Delta)C^{kj}(\Delta) \\ &\quad + C^{li}(t+\Delta)C^{kj}(t+\Delta)] - D_{kl}(t) \\ &\quad \times[C^{li}(\Delta)C^{kj}(t+\Delta) + C^{li}(t+\Delta) \\ &\quad \times C^{kj}(\Delta)].\end{aligned} \tag{100}$$

In Fig. 2 we compare the result of a Monte Carlo for the Lagrangian correlation time Eq. (84) using the exact dynamics described by Eqs. (85), (97), and (100), with that obtained from the Markovianized version given by Eq. (59). As could be guessed, the Markovian approximation becomes exact in the $a=0$ limit, when the trajectory, in a correlation time, remains close to the time line $\{1,0\}$. At least in this case, the choice given by Eq. (15), of Markovian statistics along rectilinear cuts in space-time, is the most appropriate.

IX. SOLID PARTICLE TRANSPORT IN HOMOGENEOUS ISOTROPIC TURBULENCE

Inertia and crossing trajectory effects determine a substantial change in the statistics of fluid velocities sampled by the solid particle with respect to that of passive tracer velocities.

Several authors reserved particular attention to the long time behavior of correlation functions of sampled fluid velocities and long-time particle diffusion coefficients [10,11,34,35].

Following Csanady [10], Sawford and Guest [12] kept into account the effect of gravity produced trajectory crossing by a suitable assumption on the renormalization of the correlation time of the fluid velocity sampled by the falling particle. Their model was applied to grid-generated turbulence and their results were found to agree with experimental wind tunnel data. It is not clear, however, how much this approach can be extended to generic nonhomogeneous and nonstationary turbulent flows, especially in the case of strong turbulence gradients [36].

Free-flight models [13] (see also Ref. [37] for a brief review) are known to make unphysical assumptions about the velocity that particles assume when they are projected towards the wall from the buffer and logarithmic regions. As regards the eddy-interaction model of Kallio and Reeks [37], this has been shown in Ref. [38] not to satisfy the well-mixed condition. The model described in Ref. [38] improved this aspect, but without reproducing the build-up of concentration. The issue, to be discussed in the next section, is the difficulty in isolating near wall solid particle accumulation effects from spurious concentration build-up from unproper treatment of the well-mixed condition.

A recent advance was obtained in Ref. [39], but in this approach a turbophoretic force had to be introduced from the outside, whereas in our approach the turbophoretic flux results in self-consistent way from the dynamics [see Eq. (67)].

The central role in the solid particle dispersion is played by the correlation time τ_L', of the fluid velocity sampled by the solid particles. In particular, with $\tau_L'(\parallel)$ we indicate the longitudinal effective Lagrangian time, i.e., along the direction of gravity, and with $\tau_L'(\perp)$ the transverse effective Lagrangian time.

Let us briefly summarize the main properties of $\tau_L'(i)$. When gravity is dominant $(v_G \gg u_T)$, the correlation function of sampled fluid velocities decays faster than that of passive tracers and $\tau_L'(i) < \tau_L$, where i is referred to longitudinal $(\parallel)$ or transverse $(\perp)$ [10,11]. Taking $v_G = g\tau_S$ with g fixed, we see that $\tau_L'(i)$ decreases from the value τ_L, corresponding to $\tau_S=0$, to zero as τ_S increases. Furthermore, the correlation functions do not decay in the same way in all directions. Due to the continuity effect described in Ref. [10], the decay is slower in the direction of gravity, so that the longitudinal Lagrangian time $\tau_L'(\parallel)$ is longer than the transverse one $\tau_L'(\perp)$.

In the inertia dominated case $(v_G \ll u_T)$, the sampled correlation function decays slower than that of passive tracer velocities and $\tau_L' > \tau_L$. The limit $\tau_S \to 0$ is the same as above, but now τ_L' increases with τ_S and, in the limit $\tau_S \to \infty$, tends to the Eulerian time scale τ_E [11,34].

We will show shortly how all these effects are automatically reproduced in our approach.

We consider a Gaussian homogeneous and stationary isotropic zero-mean random velocity field. Thus, the drift term $\overline{A}_\mu^{\ i}$ is given by Eq. (30) with $S^{ij} = \delta^{ij}/u_T^2$, the PDF is given by Eq. (29) and the noise tensor is isotropic [see Eq. (41)]. The terms $\Phi_\mu^{\ i}$ and $\Psi_j^{\ i}$ are zero for homogeneity and the nonunique terms $\Xi_\mu^{\ i}$ are zero for isotropy.

From now on in this section, we rewrite the equations expressing the velocities $\mathbf{u}$ and $\mathbf{v}$ and time t in units of u_T and

τ_E, respectively. Note that, in this way, τ_S becomes equivalent to the Stokes number St [30], i.e., the ratio between the Stokes time and a flow time scale (τ_E in this case). As regards gravity, it results $v_G=\mathrm{St}/\mathrm{Fr}$, being $\mathrm{Fr}=g\tau_E/u_T$ the Froude number related to the magnitude of the gravity g with respect to turbulence scales [11].

Under these conditions, Eq. (65) for the sampled fluid velocity $\mathbf{u}(t)$ and the associated expression for the noise tensor $\mathcal{B}^{ij}$ will take the simplified form

$$du^i = -(1+a|v|)u^i\,dt + a/6v^j u_j \hat{v}^i dt + dw^i,$$

$$\langle dw^i\,dw^j\rangle = 2[(1+a|\mathbf{v}|)\delta^{ij} - a/3|\mathbf{v}|\hat{v}^i\hat{v}^j]dt \qquad (101)$$

with $\hat{\mathbf{v}}=\mathbf{v}/|\mathbf{v}|$ and $\mathbf{v}$ the particle velocity, whose dynamics is given by Eq. (64). Applying to Eq. (101) the projectors defined in Eq. (90), with $\mathbf{V}=\mathbf{v}$, it is easily seen that d$\mathbf{u}$ is split into a longitudinal and a transverse component, characterized by different values of the drift: $1+2/3a|\mathbf{v}|$ for the fluid velocity component parallel to the particle velocity and $1+a|\mathbf{v}|$ for the normal component. As the role of gravity increases, the symmetry breaking of particle motion due to the presence of a preferential direction, i.e., the direction of gravity, involves a separation between the longitudinal and transverse time scale (continuity effect). In the gravity dominated case, $v_G \gg 1$, we have $|\mathbf{v}| \simeq v_G$, with the result

$$\tau'_L(\Vert) \simeq \frac{3}{2av_G}, \quad \tau'_L(\perp) \simeq \frac{1}{av_G}, \quad \frac{\tau'_L(\Vert)}{\tau'_L(\perp)} = \frac{3}{2}. \qquad (102)$$

It is difficult to obtain an analytical solution for the PDF $\rho_L(\mathbf{u},\mathbf{v})$ and the velocity correlation functions $\langle u^i(t)u^j(0)\rangle$ because of the multiplicative noise. For this reason, numerical simulations by means of Monte Carlo technique have been performed to obtain solutions of Eqs. (64) and (101). Following Yeung and Pope [40] (see also Ref. [41]), we choose a value $\tau_L/\tau_E \approx 0.5$, which, from Fig. 2, corresponds to $a=0.65$.

As mentioned in Sec. VII, the ergodic property has been verified to hold also in the solid particle case. Both in the case of Gaussian statistics and of an isotropic kurtosis described by Eq. (A3) (see Appendix A), with and without gravity, the marginal Lagrangian PDF $\overline{\rho_L}(\mathbf{u})$ defined in Eq. (74) has been found to coincide, to within numerical error, with the Eulerian PDF $\rho_E(\mathbf{u})$.

In the presence of gravity, this means that the average of the sampled fluid velocity $\langle\mathbf{u}\rangle_L$ coincides with its Eulerian counterpart $\langle\mathbf{u}\rangle$, which is zero, and that, therefore, no renormalization is produced on the value of the terminal velocity $\mathbf{v}_G$.

Another nontrivial result from the numerical simulation is that the correlation functions of both passive tracers velocities ($\tau_S=0$ and $v_G=0$) and of sampled fluid velocities appears to decay exponentially as in the Gaussian case for standard Lagrangian models.

In Fig. 3 the effective transverse Lagrangian times $\tau'_L(\perp)$ have been plotted as function of τ_S (in units of τ_E) for different values of v_G (the Lagrangian time scale of passive tracer has been reported for a comparison). In the range $v_G<1$ (inertia dominant) the curves are increasing from $\tau_L/\tau_E=0.52$ and tend to about 1 as $\tau_S\to\infty$ (i.e., the Lagrangian time tends to the Eulerian time). For $v_G>1$ (gravity dominant) the curves lose their dependence on τ_S and the correlation time is approximately equal to the eddy crossing time given (always in dimensionless units) by v_G^{-1}. This is in agreement with the asymptotic formulas in Eqs. (102). Figure 4 shows the behavior of $\tau'_L(\Vert)$ and $\tau'_L(\perp)$ as function of v_G for fixed τ_S. As expected, each curve tends to a constant value in the limit $v_G\to 0$. The longitudinal times are always longer than the transverse ones and, in the limit $v_G\to\infty$, they collapse onto two different curves, whose ratio is 3/2 as predicted by Eq. (102).

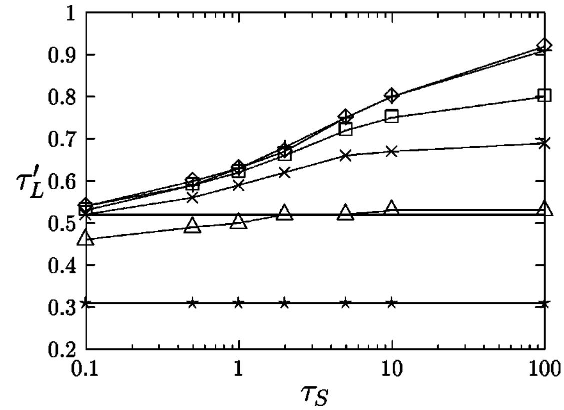

FIG. 3. Behavior of $\tau'_L(\perp)$ as a function of the Stokes time τ_S for different values of the adimensional terminal velocity v_G (dimensionless units). $\diamond$, $v_G=0$; $+v_G=0.1$; $\square$, $v_G=0.5$; $\times$, $v_G=1$; Δ, $v_G=2$; $\star$, $v_G=5$. Reference line at $\tau'_L=\tau_L=0.52$.

An exponential decay of the sampled fluid velocity correlation function allows easy analytical calculation of the correlation function for $\mathbf{v}$ [34,35,42]. The last one reads, for $i=\Vert,\perp$,

$$C_p^i(t) = \langle v^i(t)v^i(0)\rangle = C_p^i(0)\left[e^{-t/\tau_S} + \frac{e^{-t/\tau'_L(i)} - e^{-t/\tau_S}}{1-\tau_S/\tau'_L(i)}\right], \qquad (103)$$

where

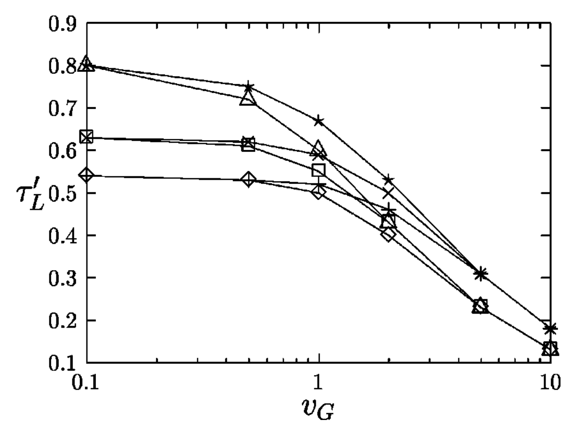

FIG. 4. Behavior of $\tau'_L(\Vert)$ and $\tau'_L(\perp)$ as a function of the terminal velocity $v_G=$ for different values of τ_S (dimensionless units). Longitudinal: $\star$, $\tau_S=10$; $\times$, $\tau_S=1$; $+$, $\tau_S=0.1$. Transverse: Δ, $\tau_S=10$; $\square$, $\tau_S=1$; $\diamond$, $\tau_S=0.1$.

$$C_p^i(0) = \frac{u_T^2}{1 + \tau_S/\tau_L'(i)}. \tag{104}$$

The particle correlation time $\tau_p(i)$ can then be calculated

$$\tau_p(i) = \int_0^\infty \frac{C_p^i(t)}{C_p^i(0)} dt = \tau_S + \tau_L'(i).$$

By using Taylor's theorem, the (long-time) diffusion coefficients

$$\kappa(i) = \frac{1}{2} \lim_{t\to\infty} \frac{1}{t} \langle [x_i(t) - x_i(0)]^2 \rangle, \quad i = \parallel, \perp$$

can be expressed in terms of the Lagrangian correlation times for **v** as follows:

$$t \gg \tau_p(i), \quad \kappa(i) = \tau_p(i) C_p^i(0) = \tau_L'(i).$$

Thus, the diffusion coefficients $\kappa(i)$ will behave exactly as the effective Lagrangian times $\tau_L'(i)$. The adimensional diffusion coefficient of passive tracers is simply given by $\kappa = \tau_L = 0.52$. Hence, in agreement with Ref. [11], $\kappa(i)$ will be larger than in the passive scalar case when inertia is dominant, smaller when gravity is dominant. Furthermore, when gravity is dominant, the longitudinal solid particle diffusion coefficient will be larger than the transverse one.

X. SOLID PARTICLE TRANSPORT IN TURBULENT CHANNEL FLOW

We focus in this section on phenomena of accumulation and deposition associated with the interaction of inertial particles with the inhomogeneity of the flow and the presence of solid boundaries. Starting from the work of McLaughlin [43], the reference situation that is typically considered, both to identify the main features of particle transport, and to test the functionality of transport models, is that of the turbulent channel flow.

We have tested our model in its simplest form, with a Gaussian PDF, an isotropic noise and nonunique term $\Xi_\mu{}^i$ set to zero. We recall that, in this form, the model is described by Eqs. (64) and (65), with the drift given by Eqs. (18) and (30)–(32), and the noise by Eq. (41) with $\mathbf{\Delta} = \mathbf{v}\Delta^0 \equiv \mathbf{v}\Delta t$. As in the homogeneous-isotropic turbulence case, we have set the free parameter $a = 0.65$, corresponding to the value of the ratio between Lagrangian and Eulerian correlation time $\tau_L/\tau_E = 0.52$ [40,41]. We adopt standard wall variables identified where necessary with +, normalized with the friction velocity u_*, and the reference length and time scales $x_2^* = \nu/u_*$ and $\tau_* = x_2^*/u_*$, where ν is the kinematic viscosity.

For the Lagrangian correlation time we have used the interpolation formula

$$\tau_L = 7.122 + 0.5731 x_2^+ - 0.001\,29 (x_2^+)^2, \quad x_2^+ < 140,$$

$$\tau_L = -19.902 + 0.959 x_2^+ - 0.002\,67 (x_2^+)^2, \quad 140 < x_2^+ < 180, \tag{105}$$

where x_2 identifies the cross-stream direction (we take x_1 and x_3, respectively, in the streamwise and spanwise direction).

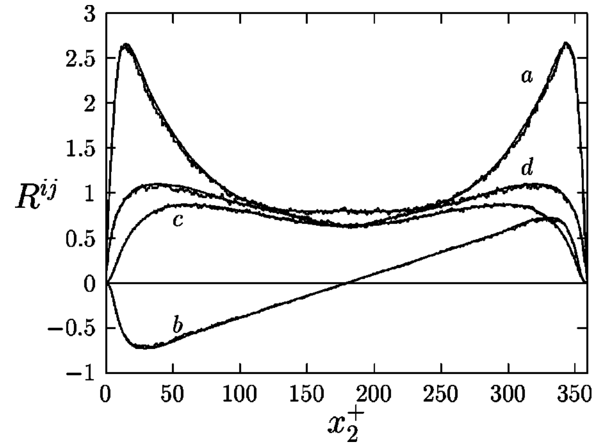

FIG. 5. Comparison between input statistics for the Reynolds tensor R^{ij}, from DNS [45], and simulated data from Monte Carlo. The data are almost undistinguishable. (a) R^{11}, (b) R^{12}, (c) R^{22}, (d) R^{33}.

For $x_2^+ < 140$, Eq. (105) coincides with the interpolation formula quoted in Ref. [37]. Thus, Eulerian time scales span from $\tau_E^{\max} \simeq 140$ in the channel center to $\tau_E^{\min} \simeq 14$ at the walls.

We have considered neither the effect of gravity, nor that of Brownian motion. The second may be important in the case of submicrometer particles. We have included, instead, the contribution from the Saffman lift [31]; indicating as usual with d the particle diameter and γ the particle to fluid density ratio,

$$v_G = 0.39 \frac{\tau_S}{\gamma d} \left| \nu \frac{\partial \overline{u_1}}{\partial x_2} \right|^{1/2} \text{sign} \left(\frac{\partial \overline{u_1}}{\partial x_2} \right) (u_1 - v_1),$$

which is known to contribute to the solid particle dynamics in the range $\tau_S \gtrsim 10$ [37].

As input data for the model, we have utilized the one-point statistics from the DNS by Kim, Moin, and Moser [45]. The channel width L_c is nearly 360 wall units and the Reynolds number is of order 3300, based on the maximum mean velocity and the channel half-width.

We have carried on Monte Carlo simulations with $N_{\rm tot} = 10\,000$ particles, uniformly distributed at the initial time. In order to obtain, after a suitable time, a stationary concentration profile, we have introduced a source of particles at the channel center to balance the deposition flux at the walls. As in Ref. [44], for these conditions and for quite all Stokes times, it seems that a simulation time $T_{\rm sim}$ of about 700 is sufficient to achieve a stationary distribution for the solid particles.

As a validation, Fig. 5 shows that the model reproduces, in the fluid particle case, the input statistics. Furthermore, the well-mixed condition has been verified: despite the possible numerical complications arising from the presence of a multiplicative noise, a uniform passive tracer concentration profile is preserved in time and no tracer deposition on the walls takes place.

In Fig. 6 we give the profile of the fluctuation amplitude for the normal velocity of a particle with $\tau_S = 60$. The Monte

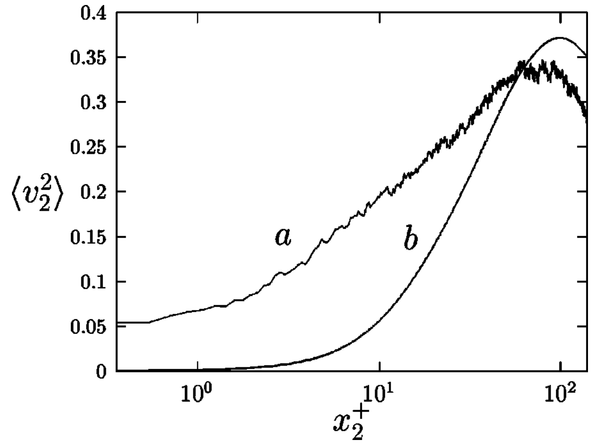

FIG. 6. Comparison between the Monte Carlo simulation results for $\langle v_2^2 \rangle$ profile (a), and the homogeneous turbulence estimate for the same quantity (b).

Carlo data strongly differ from the profile obtained from the homogeneous isotropic turbulence estimate provided by Eq. (104) and illustrate the difficulty in the *a priori* determination of a reference PDF $\rho_L(\mathbf{v},\mathbf{x},t)$ in one-fluid Lagrangian models (see discussion at the end of Sec. VI).

In Fig. 7 we give account of the particle concentration build-up in the near wall regions. The peak height appears to increase with τ_S up to $\tau_S \simeq 10$ and to decrease afterwards; the same decrease was observed in Ref. [46]. In agreement with both Refs. [44,46], and in contrast with the one-fluid model in Ref. [39], we observe that the concentration maximum occurs in the viscous sublayer at $x_2^+ \lesssim 1$. Conversely, numerical data on the peak height present in literature show a definite scatter; anyway, our data are closer to those in Ref. [44] than in Ref. [46], with an overestimation of the order of 50% with respect to the first.

As regards particle deposition, we have studied the dependence on τ_S of the deposition flux,

$$J_w = \frac{L_c N_d}{T_{\rm sim} N_{\rm tot}},$$

being L_c the channel width, N_d the number of deposited particles in the simulation time $T_{\rm sim}$, and $N_{\rm tot}$ the total number of

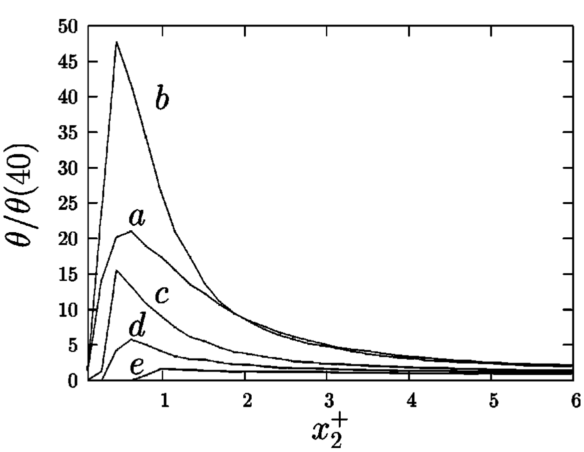

FIG. 7. Concentration profile θ vs x_2^+. (a) τ_S=3, (b) τ_S=10, (c) τ_S=20, (d) τ_S=30, (e) τ_S=60.

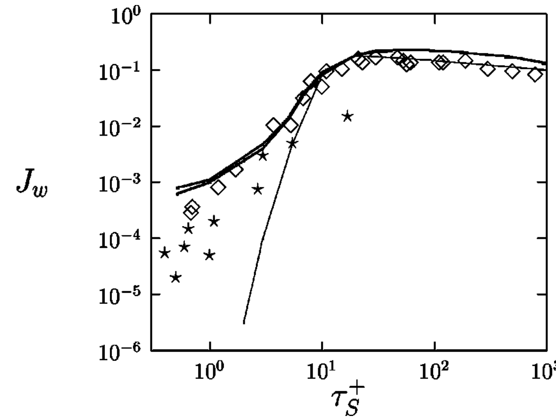

FIG. 8. Comparison of experimental data on particle deposition by Liu and Agarwal (◇, Ref. [48]) and by Wells and Chamberlin (⋆, Ref. [47]), with the results of our Monte Carlo simulations (thick lines). The lower line corresponds to simulations without taking into account the effect of the Saffman lift. The thin line is the result of simulations from a one-fluid model (Ref. [39]).

particle simultaneously present in the channel. In our simulations, we consider a particle deposited, when its distance from a wall is smaller than its radius $d/2$. Assuming that air is the suspending medium, we fix for the density ratio the value γ=1000, and for the viscosity ν=0.15 cm^2/s; from relation $\tau_S = 1/18\gamma d^2/\nu$, the particle diameter will then be, in wall units,

$$d \simeq 0.134\tau_S^{1/2}.$$

Our results are shown in Fig. 8 and compared with experimental data by Refs. [47,48], and with an example of one-fluid Lagrangian model [39]. The agreement is good with respect to the data in Ref. [48], apart of a slight overestimation in the range $\tau_S > 10$. On the contrary, our model performs much better than the one-fluid model in the range $\tau_S < 10$. As expected, the effect of the Saffman lift is felt only in the range $\tau_S < 10$; in any case, the contribution to both deposition and particle accumulation appears to be small.

XI. CONCLUSIONS

We have studied the statistical properties of trajectories extracted from a random velocity field with nonzero correlation time, analyzing the conditions for a Markovian approximation of the Lagrangian velocity. Our result is that a generalized form of the Thomson-87 well-mixed condition [4] can be derived also in the case of random fields, provided their velocity structure functions scale linearly at small space-time separations. In the incompressible case, the Markovian approximation for the Lagrangian velocity defines a Lagrangian model obeying the Thomson-87 well-mixed condition, with uniform concentration PDF given by the Eulerian one-point PDF for the random field.

Depending on the circumstances, a random field based approach to Lagrangian modeling may be advantageous. In the compressible case, knowledge of the one-point Eulerian PDF for the random field is sufficient to determine the coefficients of the associated Lagrangian model. The Thomson-87 approach, instead, requires knowledge of the particular Lagrangian PDF (indicated in Ref. [4] with g_a) which originates from an initial concentration profile equal to the instantaneous local fluid density, and which does not necessarily coincide with ρ_E. Solid particle transport is an example in which implementation of the Thomson-87 approach is not straightforward, unless ad-hoc hypotheses are made on the Lagrangian statistics.

A second advantage of this approach concerns the nonuniqueness problem: knowledge of the two-point Eulerian correlations completely fixes the form of the Lagrangian model, which is of interest for turbulent flows in complex geometry, where it is not clear which model satisfying the well-mixed condition, should be chosen. The relation of some of the nonunique terms with helicity [26] and rotation [6] is confirmed, and additional terms associated with strain have been identified. Similar approaches, in which DNS informations on the two-point Eulerian correlations are used to determine the form of Lagrangian models, have been recently adopted in Ref. [27]. Alternative formulations for the treatment of the nonuniqueness exist, in which the Lagrangian acceleration is modeled by a Langevin equation, on the same footing of the Lagrangian velocity (second order models [9]). In these models, however, the nonuniqueness problem is only displaced to the higher order acceleration.

A third advantage of the random field approach concerns situations in which it is difficult to characterize an inertial range, and in which concepts like the constant C_0 cease to be meaningful (e.g., in the buffer region of a turbulent boundary layer). Comparing our approach with the Thomson-87 technique, the main difference is, to lowest order in the SO(3) expansion, the form of the noise and the parameter a taking the place of C_0. Both noise expressions require knowledge of quantities estimated from large scale features of the flow: the viscous dissipation $\bar{\epsilon}$ and the Eulerian correlation time τ_E, whose relative dependence (as the one between a and C_0) is not an intrinsic characteristic of the models. In our approach, however, a precise relation can be obtained between the parameter a and the ratio of the Lagrangian to the Eulerian correlation time τ_L/τ_E, which is valid also when the Reynolds number is low. Using for this ratio the value obtained in Ref. [40], we obtain $a \simeq 0.65$.

We have tested our model, with isotropic noise and Gaussian statistics, to study solid particle transport both in homogenous isotropic turbulence and in channel flow geometry.

In homogeneous isotropic turbulence, the correct renormalization for the correlation time for the fluid velocity along the solid particle trajectories have been obtained without resorting to *ad hoc* parametrizations. The form of Eq. (101) descends directly from the random field and Markovianization along trajectories [see Eqs. (8) and (65)]. This illustrates the importance of the parameter a in providing the most simple characterization of space correlations in the turbulent flow. It is important to stress that, had we not taken its contribution into account, Eq. (101) would have been unable to reproduce the anisotropy of the time correlations.

An interesting aspect we have observed is satisfaction of the ergodic property in solid particle transport by homogeneous turbulence, this, despite compressibility of the solid particle flow. It is not clear whether this is an artifact of the model; in any case, it is a nontrivial effect since the ergodic property can be shown to be violated by very simple compressible flows [see Eqs. (71)–(73) and discussion therein].

In channel flow geometry, we have found good agreement with experimental data on particle deposition [47,48], and partial agreement with numerical data on near wall accumulation [44,46]. We stress that these results have been obtained without any parameter fitting, apart from the choice $a = 0.65$ inferred from Ref. [40].

Clearly, a Gaussian model with isotropic noise cannot account for the effect of coherent structures and intermittency, which are an important feature of turbulence in channel flows. Imposing the appropriate form for the one-point PDF and going to higher orders in the SO(3) expansion allows consideration of these effects. Preliminary analysis suggests that inclusion in the model of non-Gaussianity, noise anisotropy, and nonunique terms, strongly affects particle deposition and transport in wall turbulence. Modeling the structure of the turbulent correlations, based on empirical considerations, appears to lead to models that perform worse, compared to the data, than the simple isotropic Gaussian model, a situation similar to that observed in Ref. [49]. This suggests that careful consideration of the structure of the turbulent correlation, based on DNS data, may be necessary; this will be part of a different paper.

Some issues remain to be clarified as regards the definition of a random field purely in terms of its local properties. The global extension provided by Eq. (15), in which the random field is assumed "Markovian" along rectilinear cuts in space-time is only one of the possibilities. This choice produces effects on the form of the trajectories, which can be accounted for only in the non-Markovian approach described in Sec. VIII. (Markovianization along trajectories corresponds to considering only local properties of the random field.) An open question remains which global structure of a random field would lead, for fixed local structure, to transport properties which are approximated best by a Markovian Lagrangian model. This, beside understanding whether the assumption in Eq. (15), which leads in the Gaussian case to exponential scaling of the random field correlations, fits the turbulent structure in an appropriate way. For the choice provided by Eq. (15), and for homogeneous isotropic conditions and Gaussian statistics, the error in the ratio τ_L/τ_E corresponding to $a \simeq 0.65$ appears to be of the order of 15% in defect.

Related to this issue is the fact that consideration of long-lived coherent structures, corresponding to small values of τ_L/τ_E and to large errors in the Markovian approximation, is probably out of the range of applicability of our model. An alternative strategy, which would allow taking into account long-lived coherent structures, is the non-Markovian approach described in Sec. VIII. The noise and drift terms, however, must to be rederived including the condition at the emission point following Eqs. (97) and (100).

Extension of the present approach beyond one-point statistics is possible in principle, but is limited by the unphysical scaling of the random field structure function at small separations. Only concentration fluctuations on the scale of a correlation length could then be taken into account.

ACKNOWLEDGMENT

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APPENDIX A: NON-GAUSSIAN CASE

A symmetric one-dimensional distribution with unitary variance and kurtosis larger than three can be modeled by means of a bi-Gaussian,

$$P(x) = \frac{\alpha}{\pi^{1/2}}\exp(-x^2) + \frac{(1-\alpha)}{(2\pi\beta)^{1/2}}\exp\left(-\frac{x^2}{2\beta}\right), \quad \text{(A1)}$$

where

$$\alpha = \frac{4\langle x^4\rangle - 12}{4\langle x^4\rangle - 9} \quad \text{and} \quad \beta = \frac{2}{3}\langle x^4\rangle - 1 \quad \text{(A2)}$$

parametrize the strength of the kurtosis $\langle x^4\rangle$. From here, we can obtain the expression for an isotropic non-Gaussian velocity distribution,

$$\rho_E = \frac{1}{(\pi u_T^2)^{3/2}}\left[\alpha \exp\left(-\frac{u^2}{u_T^2}\right) + \frac{1-\alpha}{(2\beta)^{3/2}}\exp\left(-\frac{u^2}{2\beta u_T^2}\right)\right] \quad \text{(A3)}$$

and for an anisotropic distribution, in which one of the velocity components, in the diagonal reference frame for the Reynolds tensor, is non-Gaussian,

$$\rho_E = \alpha\rho_1 + (1-\alpha)\rho_2 = \rho_x[\alpha\rho_{y1} + (1-\alpha)\rho_{y2}]\rho_z, \quad \text{(A4)}$$

where

$$\rho_x = \frac{1}{(2\pi\hat{R}^{11})^{1/2}}\exp\left(-\frac{\hat{u}_2^1}{2\hat{R}^{11}}\right),$$

$$\rho_z = \frac{1}{(2\pi\hat{R}^{33})^{1/2}}\exp\left(-\frac{\hat{u}_2^1}{2\hat{R}^{33}}\right),$$

$$\rho_{yi} = \frac{1}{(2\pi\hat{R}_i^{22})^{1/2}}\exp\left(-\frac{\hat{u}_2^2}{2\hat{R}_i^{22}}\right), \quad i = 1,2 \quad \text{(A5)}$$

with $\hat{R}_1^{22} = \hat{R}^{22}/2$, $\hat{R}_2^{22} = \beta\hat{R}^{22}$, and the hat indicating the diagonal reference frame.

Let us calculate explicitly the drift terms in the case of Eqs. (A4) and (A5). Substituting into Eq. (19), we find immediately that $\overline{A}$ is given by the superposition of the contributions from each of the Gaussians ρ_1 and ρ_2,

$$\overline{A}_\mu{}^i \mathrm{d}x^\mu = -\frac{1}{2\rho}\langle \mathrm{d}w^i \mathrm{d}w^j\rangle[\alpha\rho_1 S_{jk}^1 + (1-\alpha)\rho_2 S_{jk}^2]u^k \quad \text{(A6)}$$

(notice the absence of the hats; it is not necessary here to work in the diagonal reference frame). The contribution from the Φ and Ψ terms has a more complicated form. Let us take the laboratory frame with the inhomogeneity direction along x^2 (the usual channel flow geometry in which x^1 is the mean flow direction). We use the ansatz,

$$\Phi_\mu{}^i = \alpha\Phi_{\mu 1}{}^i + (1-\alpha)\Phi_{\mu 2}{}^i + \Delta\Phi_\mu{}^i, \quad \Delta\hat{\Phi}_\mu{}^i = \hat{F}_\mu\delta_2^i, \quad \text{(A7)}$$

where Φ_1 and Φ_2 give the form of Φ in the case $\rho_E = \rho_1$ and $\rho_E = \rho_2$. Substituting into Eq. (20) and using Eqs. (A4) and (A5), we obtain

$$\partial_{\hat{u}^2}\hat{F}_\mu = -\delta_\mu^2\hat{\partial}_2\alpha(\rho_1 - \rho_2) \quad \text{(A8)}$$

leading to the result in the laboratory reference frame

$$\Delta\Phi_\mu{}^i = -\delta_\mu^2\Omega_2^i\frac{\rho_x\rho_y}{(2\pi)^{1/2}}\partial_2\alpha\int_{\hat{u}_2/\sqrt{R_2^{22}}}^{\hat{u}_2/\sqrt{R_1^{22}}}\mathrm{d}\hat{u}\,\mathrm{e}^{-\hat{u}^2/2}, \quad \text{(A9)}$$

where Ω_2^i is the rotation matrix defined through $u^i = \Omega_j^i\hat{u}^j$, i.e., $\Omega_j^i = \mathbf{e}^i\cdot\hat{\mathbf{e}}_j$.

Analogous procedure is followed to obtain the Ψ term. From Eq. (21), in analogy with Eq. (26), write

$$\psi^j = \alpha\psi^j + (1-\alpha)\psi^j + \Delta\psi^j, \quad \Delta\hat{\psi}^j = \delta_2^j G, \quad \text{(A10)}$$

where $2\partial_{u^i}\psi^j = -\Delta\Phi_i^i = -\hat{F}_2$, and decompose $\Psi_j{}^i$ in an analogous way. From Eq. (A9), we obtain immediately,

$$G = \frac{\rho_x\rho_y}{2(2\pi)^{1/2}}\Omega_2^2\partial_2\alpha\int_{-\infty}^{\hat{u}_2}\mathrm{d}\hat{u}\int_{\hat{u}/\sqrt{R_2^{22}}}^{\hat{u}/\sqrt{R_1^{22}}}\mathrm{d}x\;\mathrm{e}^{-x^2/2}, \quad \text{(A11)}$$

and substituting again into Eq. (26) [in real space: $\Psi_j{}^i = \delta_j^i\partial_{u^k}\psi^k - \partial_{u^j}\psi^i$], we find

$$\Delta\Psi_j{}^i = \delta_j^i\partial_{\hat{u}^2}G - \Omega_2^i\Omega_j^l\partial_{\hat{u}^l}G. \quad \text{(A12)}$$

Assuming that the statistics of d**w** be independent of the velocity, as we have done in Sec. II, has the consequence that all of the non-Gaussianity is contained in the drift. The small scale structure of the correlations, associated with d**w** remain therefore Gaussian. This breaks, for large values of the kurtosis, the direct relationship between the drift coefficients and the correlation lengths [25]. It is easy to see what happens looking at Eqs. (A4) and (A6). Whenever the value of u goes above u_T, the slowly decaying ρ_2 becomes dominant in Eq. (A6) and leads to a reduced decay rate for the fluctuation that can thus slowly grow to produce a burst. We thus have a hierarchy of time scales (focus for simplicity on variations along time),

$$\tau_E \to \text{Time scale for background } (u \sim u_T),$$

$$\beta\tau_E \to \text{Time scale for bursts } (u \sim \beta^{1/2}u_T),$$

$$\beta^2\tau_E \to \text{Spacing between bursts},$$

This difference between the time scale for bursts and background fluctuations is not physically meaningful in general. This has the effect of overshooting the burst contribution to turbulent dispersion, which is estimated by the product of the probability β^{-1} of a burst, its time scale and the square of its velocity scale,

$$\beta^{-1} \times \beta\tau_E \times \beta u_T^2 = \beta u_T^2 \tau_E. \tag{A13}$$

(In comparison, the background contribution is $u_T^2\tau_E$, and, if the time scales for burst and background were the same, the background and burst contributions would be identical.)

APPENDIX B: SPHERICAL TENSORS AND THE SO(3) TECHNIQUE

A symmetric two-index tensor function can be decomposed in spherical tensors in the form

$$\delta^{ij}Y_J(\mathbf{x}), \quad \partial^i\partial^j Y_J(\mathbf{x}), \quad x^i x^j Y_J(\mathbf{x}), \quad (x^i\partial^j + x^j\partial^i)Y_J(\mathbf{x}),$$

$$x_n(\epsilon^{jnm}x^i + \epsilon^{inm}x^j)\partial_m Y_J(\mathbf{x}) \quad \text{and} \quad x_n(\epsilon^{jnm}\partial^i + \epsilon^{inm}\partial^j)\partial_m Y_J(\mathbf{x}), \tag{B1}$$

where $Y_J(\mathbf{x})$ is a J-order polynomial in the components x_i,

$$Y_J(\mathbf{x}) = y^{i_1 i_2 \ldots i_J} x_{i_1} x_{i_2} \ldots x_{i_J}, \tag{B2}$$

and $y^{i_1 i_2 \ldots i_J}$ is traceless with respect to any pair of indices [19]. In consequence of this, the spherical tensors in Eq. (B1) will be polynomials of order $L=J$, $J-2$, $J+2$, J, $J+1$, and $J-1$, respectively. In the case of the noise tensor $\langle \Delta w^i \Delta w^j \rangle$, we have the additional symmetry with respect to spatial inversion, which imposes the condition that L be even. This implies J even for the first four spherical tensors and J odd for the last two. Limiting the analysis to $J \leqslant 2$, we notice immediately that the last spherical tensor in Eq. (B1) disappears. Similarly the $J=1$ contribution from $x_n(\epsilon^{jnm}x^i + \epsilon^{inm}x^j)\partial_m Y_J(\mathbf{x})$ is absent due to incompressibility; writing $Y_1(\mathbf{x}) = y_m x^m$,

$$\partial_i x_n(\epsilon^{jnm}x^i + \epsilon^{inm}x^j)\partial_m Y_1(\mathbf{x}) = 5\epsilon^{jnm}x_n y_m = 0$$

which imposes $y_m=0$. We are thus left only with the $J=0$ and $J=2$ contributions.

From Eq. (39), the $J=0$ and $J=2$ contributions to B^{ij} will have the form

$$u_T B_0^{ij}(\mathbf{x}) = a|x|\delta^{ij} + \hat{a}\frac{x^i x^j}{|\mathbf{x}|}, \tag{B3}$$

$$u_t B_2^{ij}(\mathbf{x}) = \frac{4b^{lm}}{|\mathbf{x}|}x_l x_m \delta^{ij} + 2c^{lm}|\mathbf{x}|\partial_i\partial_j x_l x_m + \frac{d^{lm}}{|\mathbf{x}|^3}x_l x_m x^i x^j + \frac{e^{lm}}{2|\mathbf{x}}(x^j\partial^i + x^i\partial^j)x_l x_m. \tag{B4}$$

Applying the incompressibility condition $\partial_i B^{ij}=0$ leads to the equations

$$\hat{a} = -\frac{a}{3},$$

$$8b^{lm} + 4c^{lm} + 8e^{lm} = 0,$$

$$3d^{lm} - 4b^{lm} - e^{lm} = 0. \tag{B5}$$

Substituting the solution to Eq. (B5) into Eqs. (B3) and (B4) leads to Eq. (40).

We give next the expressions for the spherical tensors contributing to an antisymmetric two-index tensor,

$$\epsilon^{ijk}x_k Y_J, \quad \epsilon^{ijk}\partial_k Y_J, \quad \text{and} \quad (x^i\partial^j - x^j\partial^i)Y_J. \tag{B6}$$

In the case of the antisymmetric part of the correlation C_A, we have the additional property of antisymmetry with respect to spatial inversion, which implies that J be even for the first two and odd for the last.

In the case of a vector field, an analogous decomposition can be obtained in terms of spherical vectors in the form

$$x^i Y_J(\mathbf{x}), \quad \partial^i Y_J(\mathbf{x}), \quad \text{and} \quad \epsilon^{ijk}x_j\partial_k Y_J(\mathbf{x}). \tag{B7}$$

If the vector field does not have an axial component, only the first two spherical vectors can contribute. If we have axial symmetry, identified by a direction $\mathbf{u}$, the tensors $y^{i_1 i_2 \ldots i_l}$ entering Eq. (B2) will be zero trace symmetrized products of components u^i and of the identity matrix δ^{ij}. The first spherical vectors $x^i Y_J(\mathbf{x})$ and $\partial^i Y_J(\mathbf{x})$ are, respectively,

$$\mathbf{x}, \quad (\mathbf{u}\cdot\mathbf{x})\mathbf{x}, \quad \left((\mathbf{u}\cdot\mathbf{x})^2 - \tfrac{1}{3}|\mathbf{u}|^2|\mathbf{x}|^2\right)\mathbf{x} \tag{B8}$$

and

$$0, \quad \mathbf{u}, \quad (\mathbf{u}\cdot\mathbf{x})\mathbf{u} - \tfrac{1}{3}|\mathbf{u}|^2\mathbf{x}.$$

APPENDIX C: CONDITIONAL RANDOM FIELD STATISTICS

We want to calculate conditional velocity moments in the form

$$\langle u^i(\mathbf{x}_0,t_0)u^j(\mathbf{x}_0,t_0)\ldots|\mathbf{u}(\mathbf{x}_1,t_1),\mathbf{u}(\mathbf{x}_2,t_2),\ldots\rangle. \tag{C1}$$

Let us indicate, for $l=0,1,\ldots,n$, $\mathbf{U}_l = \mathbf{u}(\mathbf{x}_l,t_l)$, and $\mathbf{C}_{lm} = \langle \mathbf{U}_l\mathbf{U}_m\rangle$, assuming for simplicity a symmetric correlation tensor. For a Gaussian random field, the velocity correlations at points $\{t_l,\mathbf{x}_l\}$ are obtained from the generating function

$$\tilde{\rho}(\{\eta_l\}) = \exp\left(-\frac{1}{2}\sum_{l,m=0}^{n}\eta_l\cdot\mathbf{C}_{lm}\cdot\eta_m\right). \tag{C2}$$

Let us introduce the marginal PDF,

$$\rho(\{\mathbf{U}_l, i=1,\ldots,n\}) = \mathcal{N}\exp\left(-\frac{1}{2}\sum_{l,m=1}^{n}\mathbf{U}_l\cdot\mathbf{D}_{lm}\cdot\mathbf{U}_m\right), \tag{C3}$$

where $\mathcal{N}$ is the normalization and $\mathbf{D}_{lm}$ is the inverse of the restriction of $\mathbf{C}_{lm}$ to $l,m=1,\ldots,n$. We shall indicate in the following this restriction with a prime:

$$\sum_{lm}{}' \equiv \sum_{l,m=1}^{n}, \quad \{\mathbf{U}_l'\} \equiv \{\mathbf{U}_l, l=1,\ldots,n\}, \quad \text{etc.} \tag{C4}$$

The generating function for $\mathbf{U}_0$ conditioned to $\mathbf{U}_l$, $l=1,\ldots,n$ is obtained by inverse Fourier transforming $\tilde{\rho}(\{\eta_l\})$ with respect to $\{\eta_l'\}$ in $\{\mathbf{U}_l'\}$ and dividing by the marginal PDF $\rho(\{\mathbf{U}_l'\})$,

$$\tilde{\rho}(\eta_0|\{\mathbf{U}_l'\}) = \frac{1}{\rho(\{\mathbf{U}_l'\})} \int \prod_l{}' \mathrm{d}^3\eta_l \exp\Bigg(-\mathrm{i}\sum_l{}' \eta_l \cdot \mathbf{U}_l - \frac{1}{2}\sum_{lm}{}' \eta_l \cdot \mathbf{C}_{lm} \cdot \eta_m - \sum_l{}' \eta_0 \cdot \mathbf{C}_{0l} \cdot \eta_l - \frac{1}{2}\eta_0 \cdot \mathbf{C}_{00} \cdot \eta_0 \Bigg). \tag{C5}$$

From here we can calculate the conditional moments in Eq. (C1). We calculate first the mean velocity in $\{t_0,\mathbf{x}_0\}$ given velocities $\mathbf{U}_l$ in $\{t_l,\mathbf{x}_l\}$ $l=1,\ldots,n$,

$$\langle \mathbf{U}_0|\{\mathbf{U}_l'\}\rangle = \frac{1}{\tilde{\rho}(0|\{\mathbf{U}_l'\})} \frac{\partial \tilde{\rho}(\eta_0|\{\mathbf{U}_l'\})}{\partial\, i\,\eta_0}\Bigg|_{\eta_0=0} = \frac{\sum_r{}' \mathbf{C}_{0r}}{\rho(\{\mathbf{U}_l'\})\tilde{\rho}(0||\{\mathbf{U}_l'\})} \cdot \int \prod_l{}' \mathrm{d}^3\eta_l\, \boldsymbol{\eta}_r \times \exp\Bigg(-\frac{1}{2}\sum_{lm}{}' \boldsymbol{\eta}_l \cdot \mathbf{C}_{lm} \cdot \boldsymbol{\eta}_m - \mathrm{i}\sum_l{}' \boldsymbol{\eta}_l \cdot \mathbf{U}_l \Bigg). \tag{C6}$$

Carrying out the Gaussian integrals, we obtain the result

$$\langle \mathbf{U}_0|\{\mathbf{U}_l'\}\rangle = \sum_{lm}{}' \mathbf{C}_{0l} \cdot \mathbf{D}_{lm} \cdot \mathbf{U}_m. \tag{C7}$$

The calculation of the second conditional moment is analogous and the result is

$$\langle \mathbf{U}_0\mathbf{U}_0|\{\mathbf{U}_l'\}\rangle = -\frac{1}{\tilde{\rho}(\boldsymbol{\eta}_0|\{\mathbf{U}_l'\})} \frac{\partial^2 \tilde{\rho}(\eta_0|\{\mathbf{U}_l'\})}{\partial\, \boldsymbol{\eta}_0\, \partial\, \boldsymbol{\eta}_0}\Bigg|_{\eta_0=0} = \mathbf{C}_{00} - \sum_{lm}{}' \mathbf{C}_{0l} \cdot \mathbf{D}_{lm} \cdot \mathbf{C}_{m0} + \langle \mathbf{U}_0|\{\mathbf{U}_l'\}\rangle \times \langle \mathbf{U}_0|\{\mathbf{U}_l'\}\rangle. \tag{C8}$$

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