Synchronization of chaotic oscillators due to common delay time modulation

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We have found a synchronization behavior between two identical chaotic systems when their delay times are modulated by a common irregular signal. This phenomenon is demonstrated both in two identical chaotic maps whose delay times are driven by a common chaotic or random signal and in two identical chaotic oscillators whose delay times are driven by a signal of another chaotic oscillator. We analyze the phenomenon by using the Lyapunov exponents and discuss it in relation to generalized synchronization.

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Synchronization in chaotic oscillators [1–4], which is characterized by the loss of exponential instability or neutrality in the transverse direction due to the interaction, has given rise to much attention for its application to diverse disciplines of science such as biology [5], chemistry [6], and physics [1–4]. Extensive investigations have been undertaken to understand its underlying mechanism [7–10]. Synchronization can be classified depending on the characteristics of coupled systems. Complete synchronization (CS) [4] is observed in identical systems while phase synchronization [7] and lag synchronization [8] occur in slightly detuned systems.

Recently a more general type of synchronization has been described in coupled systems with different dynamics, which is called generalized synchronization (GS) [9,10]. GS is characterized by the appearance of a functional relationship between the master, $\dot{\mathbf{y}}=\mathbf{G}(\mathbf{y})$, and the slave systems, $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x},\mathbf{h}(\mathbf{y}))$, where $\mathbf{h}(\mathbf{y})$ is the function that describes the coupling between the master and the slave. Equivalently, the existence of a functional relationship implies that CS has been established between the slave and its replica $\dot{\mathbf{x}}'=\mathbf{F}(\mathbf{x}',\mathbf{h}(\mathbf{y}))$ such that $\|\mathbf{x}-\mathbf{x}'\|\to 0$ as $t\to\infty$ [9,10]. Generally, it is thought that this type of synchronization phenomenon is also established even if the driving signal is noisy [11].

In a real situation, time delay is inevitable, since the propagation speed of the information signal is finite [12–18]. Since Volterra's predator-prey model [19], the delay time has been considered in various forms to incorporate realistic effects, e.g., distributed, state-dependent, and time-dependent delay times. Up to now, the effects of these forms of delay in dynamical systems have been extensively studied in many fields of physics [20], biology [19], and economy [21]. Also it was reported that synchronous behavior is enhanced in neural systems by discrete time delays [22]. Recently, the effect of delay time modulation on the characteristics of the chaotic signal was reported [23]. In this report, the delayed system transits to a complex state and does not simplify the chaotic attractor into a low-dimensional manifold. In this regard, it is important to understand the effect of delay time modulation on the synchronization of chaotic oscillators, because it is one of the fundamental phenomena in dynamical systems.

In this paper, we report a synchronization phenomenon between two identical chaotic systems when their delay times are driven by a common irregular signal. We analyze the synchronous behaviors in two identical logistic maps by studying the conditional and maximal Lyapunov exponents, and demonstrate them in two Rössler oscillators whose delay times are driven by a Lorenz oscillator.

Our system can be described as follows:

$$\dot{\mathbf{y}}=\mathbf{f}(\mathbf{y}),\quad \text{master},$$

$$\dot{\mathbf{x}}=\mathbf{g}(\mathbf{x},\mathbf{x}(t-\tau(\mathbf{y}))),\quad \text{slave 1},$$

$$\dot{\mathbf{x}}'=\mathbf{g}(\mathbf{x}',\mathbf{x}'(t-\tau(\mathbf{y}))),\quad \text{slave 2},\tag{1}$$

where the signal $\mathbf{y}$ of the master system drives the two slaves. The model describes two identical chaotic systems that are influenced by a common delay time modulation (DTM) [23]. If the delay time is a constant, the two slaves become two independent time-delayed systems with fixed delay. To emphasize, when the delay times are modulated in time, what we observed is that the two slaves transit to the synchronization state above the threshold.

For a simple example, we consider a system that consists of a logistic map for the master and two logistic maps for the slaves as follows:

$$y_{n+1}=4y_n(1-y_n),\quad \text{master},$$

$$x_{n+1}=\gamma\overline{x_n}(\tau)[1-\overline{x_n}(\tau)],\quad \text{slave 1},$$

$$x'_{n+1}=\gamma\overline{x'_n}(\tau)[1-\overline{x'_n}(\tau)],\quad \text{slave 2},$$

where $\overline{x_n}(\tau)=(1-\alpha)x_n+\alpha x_{n-\tau}$ and α is the coupling strength. Here we take $\tau=[\Lambda y_n]$ as the common delay time and Λ is a scaling parameter of the DTM. Here $[\Lambda y_n]$ is the largest integer less than Λy_n which is introduced to get an integer number for the iteration. Figure 1 shows the temporal behaviors of two coupled logistic maps by a common DTM. The modulated delay time as a function of time is presented in Fig. 1(a) and the temporal behavior of one of the slaves is presented in Figs. 1(b) and 1(c) at two different reference points [*A* and *B* in Fig. 3(a)], respectively. While the differ-

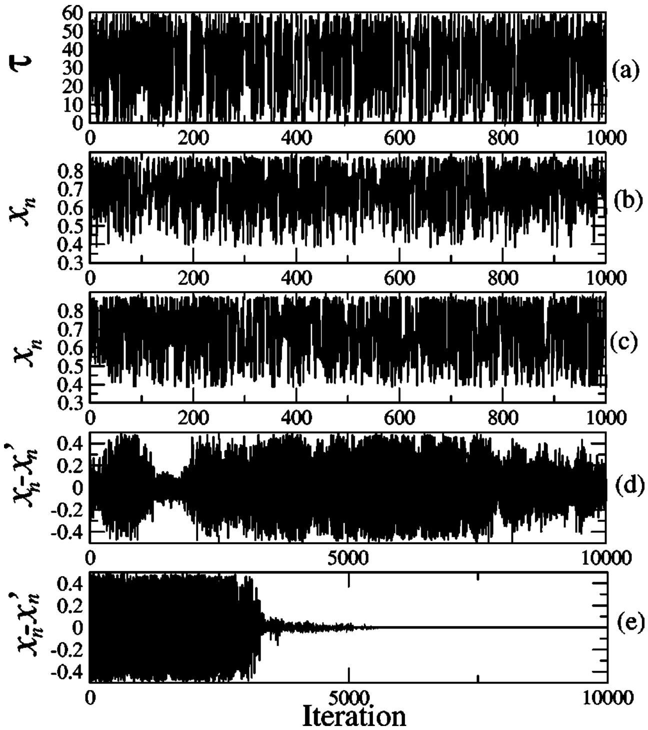

FIG. 1. Temporal behaviors of the slave logistic maps when $\gamma = 3.5$ and $\Lambda = 60$. (a) The modulated delay time τ as a function of time; (b) x_n and (d) $x_n - x_n'$ at $\alpha = 0.7$ [the reference point A in Fig. 3(a)]; (c) x_n and (e) $x_n - x_n'$ at $\alpha = 0.8$ [the reference point B in Fig. 3(a)].

ence of the two slave systems is chaotic as shown in Fig. 1(d) below the threshold [i.e., at the reference point A in Fig. 3(a)], surprisingly the slaves are synchronized above the threshold [i.e., at the reference point B in Fig. 3(b)] just by a common DTM as shown in Fig. 1(e) without any changing of the chaotic behaviors of the two slave oscillators. We emphasize that this phenomenon is purely originated from the common DTM. If the modulation is turned off, the two slaves become two independent systems with a fixed time delay and so they cannot be synchronized.

In order to understand the threshold behavior, we analyze the Lyapunov exponents of the two logistic maps [11]. For these we consider the difference dynamics as follows:

$$\Delta X_{n+1} = J_n \Delta X_n + K_n, \tag{2}$$

where

$$J_n = (1-\alpha)\gamma\{1 - [\overline{x_n}(\tau) + \overline{x_n'}(\tau)]\},$$

$$K_n = \alpha\gamma\{1 - [\overline{x_n}(\tau) + \overline{x_n'}(\tau)]\}(x_{n-\tau} - x'_{n-\tau}),$$

and $\Delta X_n = x_n - x_n'$. The above equation is nonautonomous and has the unusual term of K_n. Accordingly, we iterate the above equation with one master and two slave equations, altogether. Here ΔX_n is treated as an independent variable. From the iteration, we can evaluate the conditional Lyapunov exponent, which describes the synchronization behaviors between the two slave systems, such that $\lambda_c = \lim_{N\to\infty}(1/N)\log(|\Delta X_N/\Delta X_0|)$ [11]. In order to understand the dynamical property of the whole system, we calculate the maximal Lyapunov exponent λ_m, which describes the chaotic property of a system. In this case, we need one more replica of the master system with different initial conditions such that $\Delta X_{n+1} = \overline{J_n}\Delta X_n + \overline{K_n}$, where $\overline{J_n} = (1-\alpha)\gamma\{1 - [\overline{x_n}(\tau) + \overline{x_n'}(\tau')]\}$ and $\overline{K_n} = \alpha\gamma\{1 - [\overline{x_n}(\tau) + \overline{x_n'}(\tau')]\}(x_{n-\tau} - x'_{n-\tau'})$.

The conditional and maximal Lyapunov exponents are presented as functions of (α, Λ) in Fig. 2. The conditional Lyapunov exponent shows the synchronized regime [the gray region of Fig. 2(a)] for the two slaves in the (α, Λ) space. In that regime the transverse variable ΔX_n converges to zero. That is, the system becomes stable in the transverse direction ΔX_n due to DTM. The maximal Lyapunov exponent which describes the chaotic property of the system is positive except in the narrow periodic regime [the dark gray region on the (α, Λ) plane of Fig. 2(b)]. Therefore one see synchronization of the two slave logistic maps in the regime where the two slaves are chaotic.

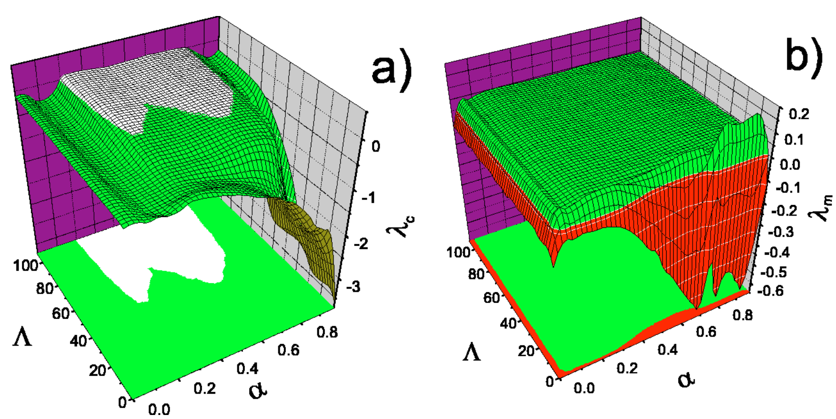

FIG. 2. (a) Conditional Lyapunov exponent λ_c as a function of (Λ, α). The gray (green) region indicates the regime in which the conditional Lyapunov exponent is negative (i.e, the synchronization regime) and the white region shows the regime in which the exponent is positive. (b) Maximal Lyapunov exponent λ_m as a function of (Λ, α). The gray (green) region shows the regime where the exponent is positive and the dark gray (red) region shows the regime in which the exponent is negative. We added a small noise of order 10^{-8} to one of the slaves in order to avoid abrupt synchronization due to round-off error in the simulation.

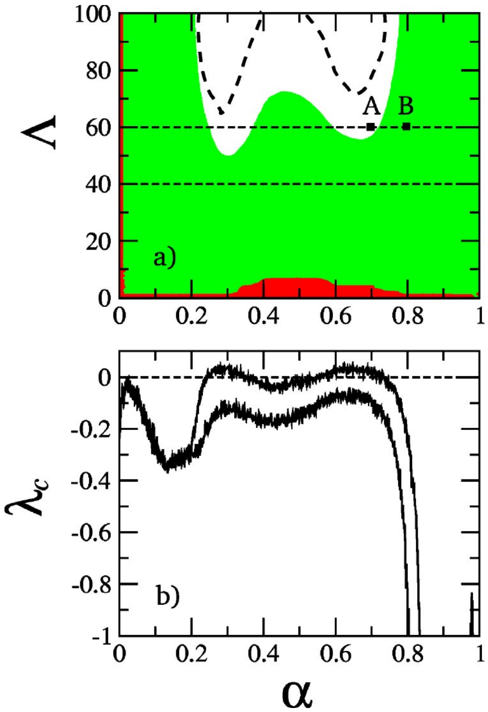

FIG. 3. (a) Synchronization regime determined by the time series. The gray (green) region shows the synchronization regime which coincides with that of the contour plot of Fig. 2(a). *A* and *B* indicate the reference points used to present the time series in Fig. 1. The dashed lines are the border of synchronization when the driving signal is replaced by a random signal ξ_n. One sees that the synchronization regime is extended in this case. (b) The conditional Lyapunov exponents on the two reference lines of (a) as a function of α. The upper line is at $\Lambda=60$ and the lower at $\Lambda=40$.

The periodic regime corresponds to the imprint of the periodic behavior of slave systems when the delayed feedbacks are absent. However, when the delayed feedback is turned on, the system becomes chaotic. The slaves return to the independent logistic maps when $\alpha=0$ or $\Lambda<1$. (The results of Figs. 1–3 show the chaotic output of the slave systems and a wide synchronization regime depending on the modulation amplitude and the coupling strength, even though we took $\gamma=3.5$. We also performed the same studies with other parameters $\gamma=3.8$, 3.9, and 4.0 and observed a synchronization regime.)

By tracing the time series in the (α,Λ) space, we obtain the synchronization regime which we show in Fig. 3(a). One can see that the synchronization regime in Fig. 3(a) coincides with that of Fig. 2(a). To confirm the numerical results of Fig. 3(a), we redraw the conditional Lypunov exponent as shown in Fig. 3(b) when $\Lambda=60$ and $\Lambda=40$. Even when we replace the master system by a random signal ξ_n, we can observe a similar synchronization regime whose border is presented by a dashed line in Fig. 3(a). One sees that the synchronization is enhanced when the delay time modulation is noisy. This fact leads us to understand the synchronization phenomenon in the framework of the synchronization by a common signal which can be chaotic or noisy [11]. Specifically, in our system the driving common signal is fed into the delay time implicitly, while in previous systems the driving signal is explicitly introduced [11].

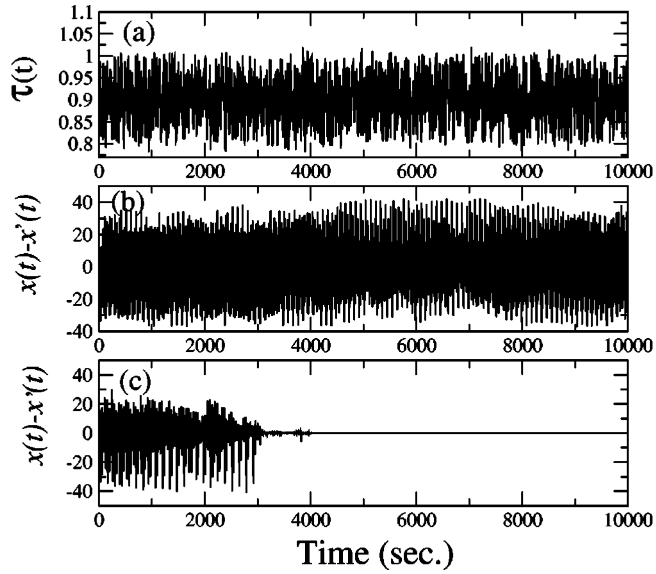

FIG. 4. Temporal behaviors of the Rössler oscillators of Eqs. (3) and (4). (a) $\tau(t)$ as a function of time when $\epsilon=0.012$, $\beta=0.0067$, and $\tau_0=0.9$; and the difference motion between two slaves when (b) the coupling strength $\alpha=0.238$ and (c) 0.240.

To show the universal feature of this type of synchronization, we consider the Lorenz oscillator as master and the Rössler oscillators as slaves [1] as follows:

$$\epsilon\dot{p} = \sigma(q-p),$$

$$\epsilon\dot{q} = -pr + ap - q,$$

$$\epsilon\dot{r} = pq - br, \quad \text{master}, \tag{3}$$

$$\dot{x} = y - z,$$

$$\dot{y} = \overline{x} + cy,$$

$$\dot{z} = d + z + \overline{x}, \quad \text{slave}, \tag{4}$$

where $\sigma=10$, $a=28$, $b=8/3$, $c=0.15$, $d=0.2$, and $\overline{x}=(1-\alpha)x+\alpha x[t-\tau(p)]$. Here ϵ is the time scaling parameter to control the average oscillation frequency of the driving system. We take the delay time in the form of $\tau(p)=\beta p(t)+\tau_0$, where β describes the modulation amplitude and τ_0 is the center of the delay time. In this model, the Lorenz oscillator plays the role of a driving system for a common DTM. Figure 4 shows the temporal behaviors of the two slave chaotic systems near the synchronization threshold when $\epsilon=0.012$, $\beta=0.0067$, and $\tau_0=0.9$. Above the threshold, the two slaves are synchronized as shown in Fig. 4(c), while each oscillator is in a chaotic state. As we analyzed the logistic maps, we can easily understand the synchronization phenomenon in these systems.

It is worth discussing the relationship between this phenomenon and GS. GS is characterized by CS between the slave oscillator $\mathbf{\dot{x}}=\mathbf{f}(\mathbf{x},\mathbf{h}(\mathbf{y}))$ and its replica $\mathbf{\dot{x}}'=\mathbf{f}(\mathbf{x}',\mathbf{h}(\mathbf{y}))$. The master signal $\mathbf{y}$ is directly fed into the slaves in the form of the explicitly defined function $\mathbf{h}$. On the contrary, since the delay times of our slave systems are modulated by the master signal, the functional dependency between the master and the slave is not explicitly revealed. That is to say, the effective forces acting on two slaves are quite different from the case of GS until the two slaves are converged into a synchronization state, because the feedback signal is proportional to the value of its own state vector, not a common feeding signal as in GS [i.e., feedback signals are not common since $\mathbf{x}(t-\tau)$ is for slave 1 and $\mathbf{x}'(t-\tau)$ is for slave 2]. If one introduces a multiplicative coupling such that $x(t)y(t)$, the feedback forces can be different in the two slaves. However, the force is proportional to the master signal as well as the slave one in this case, while the feedback force is proportional to the slave signal only in our systems, because $x(t-\tau)$ is just a previous trajectory of the slave systems. In this respect, the observed synchronization could be classified into an extended type of GS.

Regarding an experimental realization of our method, we can consider a laser system with optical feedback, where the delay time can be modulated by a vibrating feedback mirror using a piezoelectric or electromagnetic cell (see the second reference of Ref. [22]). Also in an electronic circuit, the delay time modulation can be implemented by using a digital delay line or computer interface.

In conclusion, we have investigated the synchronization behavior between two chaotic systems whose delay time is modulated by a common irregular signal. We have demonstrated that synchronization can be achieved by a common DTM in chaotic maps and flows. And we have clarified that a common DTM alters the stability of two chaotic oscillators and leads the systems to be synchronized. We have confirmed the observed phenomenon through analysis of the conditional and maximal Lyapunov exponents. We expect that the observed phenomenon will extend the concept of GS, and that the introduced systems will be useful for understanding synchronization phenomena with time delay in such various fields as neurology [22,24] and population dynamics [21].

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