Nonradiating sources with connections to the adjoint problem

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A general description of localized nonradiating (NR) sources whose generated fields are confined (nonzero only) within the source's support is developed that is applicable to any linear partial differential equation (PDE) including the usual PDEs of wave theory (e.g., the Helmholtz equation and the vector wave equation) as well as other PDEs arising in other disciplines. This description, which holds for both formally self-adjoint and non-self-adjoint linear partial differential operators (PDOs), is derived in the context of both the governing PDE and the corresponding adjoint PDE of the associated adjoint problem. It is shown that a necessary and sufficient condition for a source to be NR is that it obeys an orthogonality relation with respect to any solution in the source's support of the corresponding homogeneous adjoint PDE. For real linear PDOs, this description takes on a more relaxed form where, in addition to the previous necessary and sufficient condition, the obeying of a complementary orthogonality relation with respect to any solution in the source's support of the homogeneous form of the same governing PDE is also both necessary and sufficient for the source to be NR.

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I. INTRODUCTION

Nonradiating (NR) sources whose generated fields remain confined (are nonzero only) within the source's region of localization have been of interest for a long time in connection with physical and inverse theories (see [1,2] for recent overviews) and other applications (see [3] for a recent application to propagation control in lattice strings). Particular descriptions applicable to specific partial differential operators (PDOs), including the Helmholtz operator [4,6], the vector wave equation operator [5], and, in the time domain, the D'Alembertian operator [7], have appeared throughout the years. A more general description that is applicable to any real, formally self-adjoint linear PDO was presented in [8,9]. In this broad context, a NR source is a "source" to a partial differential equation (PDE) in a general (e.g., spatial or spatial-temporal) coordinate space of interest whose generated "field" remains confined within the source's support. This generalization enables treatment within common grounds of both the usual NR sources to the different wave equations as well as to "NR sources" in other areas such as magnetostatics, where field-confining sources (whose fields vanish identically everywhere outside the source) are of enormous importance in connection with Aharonov-Bohm experiments [10,11]. The present work generalizes the presentation in [8,9] to *any* linear PDO, including nonreal and formally non-self-adjoint PDOs. Examples of non-self-adjoint PDOs of interest include the PDO of the Schrödinger equation (see, e.g., [12], p. 438), the PDO of the diffusion equation (see, e.g., [12], p. 437), the PDO of the small-amplitude (linearized) version of the Korteweg–de Vries equation (see, e.g., [13], p. 4), and the PDO of the equation of motion of a mechanical oscillator with damping (see, e.g., [12], p. 851). Physical realizations of NR sources in the last three areas translate (correspondingly) into excitations that control zero external diffusion, or that generate no soliton propagation, or that drive finite-duration mechanical oscillations. In the framework of the Schrödinger equation, the relevance of sources in general is discussed in [14] in connection with Bohmian mechanics. Our motivation is twofold since, in addition to generalizing the scope of applicability of the theory in [8,9] to a broader class of PDOs (in particular, all linear PDOs), we also derive additional proofs of previously known results. In particular, the arguments employed in the present general description are stronger than those in [8,9] since (unlike in [8,9]) they do not rely on operator self-adjointness. Hence, they depict a more robust picture that complements the one developed in [8,9], facilitating also new proofs of other, closely related results, applicable to particular PDOs, due to Kim and Wolf [4] and Friendlander [15].

II. OUR MAIN RESULTS

We consider a general scalar, vector, or tensor source-field system (ρ,ψ) governed by a linear PDE

$$(L\psi)(x) = \rho(x), \tag{1}$$

where the vector $x=(x_1,x_2,\ldots,x_n)\in\Omega\subset\Re^n$ denotes the relevant space or space-time coordinates in a given observation domain Ω with boundary $\partial\Omega$, L is a linear PDO, and $\psi(x)$ is the scalar, vector, or tensor field produced by a scalar, vector, or tensor source $\rho(x)$ of support $D\subset\Omega$. For example, for time-harmonic electromagnetic fields, the relevant source $\rho(x)$ and field $\psi(x)$ can be the space-dependent part $i\mu\omega\mathbf{J}(\mathbf{r})$ and $\mathbf{E}(\mathbf{r})$ of the electric source (current density) and field vectors, respectively, where ω is the angular oscillation frequency and μ is the medium's permeability, while the relevant PDO L is the vector wave equation operator $(\nabla\times\nabla\times-k^2)$ where k is the wave number of the field (for more examples, see [8]).

The generated field is given by

$$\psi(x) = \int_D dx'\rho(x')G(x,x'), \tag{2}$$

where $G(x,x')$ is the scalar or tensor Green function associated with the PDO L that obeys suitable boundary conditions imposed by the problem description on $\partial\Omega$ [the Green function $G(x,x')$ is generally a tensor if the source $\rho(x)$ and field $\psi(x)$ are vectors or tensors].

Our first goal in this work is to derive the following result, applicable to any NR source $\rho_{\rm NR}(x)$ of support D to a general linear PDE (1), for which (by definition) the NR field

$$\psi_{\rm NR}(x) = \int_D dx' \rho_{\rm NR}(x') G(x,x') = 0 \quad \text{if } x \notin D. \tag{3}$$

Theorem 1. A necessary and sufficient condition for a source $\rho_{\rm NR}(x)$ of support D to a general linear PDE (1) to be NR is that it obeys the "orthogonality" relation

$$\int_D dx \rho_{\rm NR}(x) v^*(x) = 0, \tag{4}$$

where * denotes the complex conjugate, with respect to any function $v(x)$ that obeys everywhere in the source's support D (the boundary ∂D of D included) the homogeneous PDE

$$(\widetilde{L}v)(x) = 0 \quad \text{if } x \in D, \tag{5}$$

where $\widetilde{L}$ is the adjoint of the PDO L [as defined, e.g., in Chap. 9 of [12], in Chap. 7 of [16], and in connection with the generalized Green theorem in Eqs. (3.3), (3.4), and (3.5) of [17]].

Another interesting result, also to be established in this work, is the following corollary of Theorem 1, which holds for all real linear PDOs L, for which $(L\psi)^*(x) = (L\psi^*)(x)$. Examples of such real PDOs include the vector wave equation operator $(\nabla\times\nabla\times - k^2)$, the Helmholtz operator $(\nabla^2 + k^2)$, and the PDO of the diffusion equation. The PDO of the Schrödinger equation is not real.

Corollary 1. A necessary and sufficient condition for a source $\rho_{\rm NR}(x)$ of support D to a PDE (1) with real linear PDO L to be NR is that it obeys the orthogonality relation (4) with respect to any function $v(x)$ that obeys everywhere in the source's support D (the boundary ∂D of D included) the homogeneous form of the same governing PDE, in particular,

$$(Lv)(x) = 0 \text{ if } x \in D. \tag{6}$$

These two results—Theorem 1, which holds for *all* linear PDOs, and Corollary 1, which holds for all *real* linear PDOs—constitute a general description of a localized NR source to a linear PDE that is based on orthogonality or noninteractivity [8,9] of a NR source to free fields, corresponding to the adjoint problem (as in Theorem 1) or to the original problem (as in Corollary 1).

III. CONNECTION TO PREVIOUS RESULTS

The statements in Theorem 1 (valid for all linear PDOs) and Corollary 1 (valid for all real linear PDOs) encompass analogous results on NR sources for particular PDOs encountered in the literature. In particular, for the real, self-adjoint Helmholtz operator, the substitution $\widetilde{L} = \nabla^2 + k^2$ in Eq. (5) (due to self-adjointness) or $L = \nabla^2 + k^2$ in Eq. (6) (due to reality) yields, in connection with Theorem 1 or Corollary 1, respectively, the particular results, applicable to this operator, derived in [4], where it was shown that any NR source $\rho_{\rm NR}(\mathbf{r})$ of support D to the Helmholtz equation $(\nabla^2 + k^2)\psi(\mathbf{r}) = \rho(\mathbf{r})$ must be orthogonal [essentially as in Eq. (4), with $x = \mathbf{r}$] to all solutions $v(\mathbf{r})$ of the homogeneous Helmholtz equation $(\nabla^2 + k^2)v(\mathbf{r}) = 0$ in D. This condition was shown to be also sufficient for a source $\rho_{\rm NR}(\mathbf{r})$ of support D to be NR. Even though the analysis in [4] assumes continuous NR sources, the general results can be shown to hold in a more general, distributional sense. In fact, in our statement of Theorem 1 and of Corollary 1 we have made no assumption about the well-behavedness of the sources and fields since the stated results can be shown to hold in the sense of distributions. Finally, for the vector wave equation operator we arrive, by a similar substitution, in particular the substitution $L = \widetilde{L} = (\nabla\times\nabla\times - k^2)$ in Eqs. (5) and (6), at the particular version of our results related to Eqs. (4)–(6) corresponding to this operator derived in the appendix of [9].

IV. FORMULATION

In this section, we prove Theorem 1 for any linear PDO L. We also establish the accompanying Corollary 1 for real linear PDOs L.

Our starting point is the following result, concerning the necessary part of Theorem 1.

Lemma 1. If $\rho_{\rm NR}(x)$ is a NR source of support D to a general linear PDO L, then, necessarily, the orthogonality relation (4) must hold for any function $v(x)$ obeying the homogeneous form of the adjoint PDE, in particular Eq. (5), everywhere in the source's support D.

The real PDO version of Lemma 1 [where $(L\psi)^*(x) = L\psi^*(x)$] was established in [8,9]. To keep our presentation self-contained, we establish next the proof of Lemma 1 in the most general case (not necessarily real PDOs).

It is not hard to deduce from Eq. (1) that the most general NR source $\rho_{\rm NR}(x)$ of support D whose localized field is $\psi_{\rm NR}(x)$ in D [and is, according to Eq. (3), zero elsewhere] must be of the form $\rho_{\rm NR}(x) = (L\psi_{\rm NR})(x)$ (see also [5]). By using this representation of a localized NR source and the generalized Green theorem in its general form applicable to any linear PDO (regardless of its scalar, dyadic, or other particular nature) as encountered in treatments of functional analysis (see, e.g., Chap. 7 of [18]), operator theory [see, e.g., Eqs. (3.3), (3.4), and (3.5) and related discussion in [17]], and wave theory (see, e.g., [16], pp. 870–883), we obtain

$$\int_D dx \rho_{\rm NR}(x) v^*(x) = \int_D (L\psi_{\rm NR})(x) v^*(x) = \int_D \psi_{\rm NR}(x) (\widetilde{L}v)^*(x), \tag{7}$$

where $v(x)$ is an arbitrary function of x and where the (missing) boundary terms in Eq. (7) vanish due to the vanishing of the NR field $\psi_{\rm NR}(x)$ for $x \notin D$. If $(\widetilde{L}v)(x) = 0$ in the source's support D, then the integral in Eq. (7) vanishes, which establishes Lemma 1 corresponding to the necessary part of Theorem 1. Let us establish next the sufficient part of the same theorem.

If $\widetilde{G}(x,x')$ is a Green's function of the adjoint PDO $\widetilde{L}$ (obeying arbitrary adjoint boundary conditions on the boundary $\partial\Omega$ of the observation domain Ω), then, clearly,

$$(\widetilde{L}\widetilde{G})(x',x) = 0 \tag{8}$$

if $x' \in D$ and $x \notin D$. It follows that for $x' \in D$ and $x \notin D$, we can put $v(x') = \widetilde{G}(x',x)$ [which obeys, as read in Eq. (8), the required Eq. (5)] in the orthogonality condition (4) so that

$$\int_D dx' \rho_{\mathrm{NR}}(x') \widetilde{G}^*(x',x) = 0 \quad \text{if } x \notin D. \tag{9}$$

It is not hard to show with appropriate reference to Eq. (7) (see, e.g., [16], pp. 882–883) that for any Green's function $\widetilde{G}(x,x')$ of the adjoint PDO $\widetilde{L}$ there is associated a Green's function $G(x,x')$ of the PDO L such that the reciprocity condition $\widetilde{G}^*(x',x) = G(x,x')$ holds. Then Eq. (9) with the substitution $\widetilde{G}^*(x',x) = G(x,x')$ is seen to reduce to the nonradiation requirement Eq. (3), which establishes sufficiency. Furthermore, since the adjoint Green function in Eqs. (8) and (9) is arbitrary, then this nonradiation requirement can be shown to hold not only for the particular Green function implicit in Eq. (3), but also for any other Green's function. We conclude that nonradiation under a certain boundary condition implies nonradiation under any boundary condition or, alternatively, that nonradiation to a given Green's function implies nonradiation to any Green's function of the governing PDE. The latter result provides a general explanation of a result derived in [15] (see also [1] for other connections) for the particular context of the scalar wave equation. In particular, it was shown there that if a source is NR to the usual retarded Green function, then it must also be NR to the corresponding advanced Green function (where the differentiating boundary conditions are *boundary conditions in time*). From our more general standpoint applicable to any linear PDE, this result can be interpreted as arising from Theorem 1 and, in particular, from the fact that nonradiation to a given Green's function of any governing linear PDO L (under specific boundary conditions) implies, in turn, nonradiation to any other Green's function of L (under other suitable boundary conditions).

Having established Theorem 1, we are now in a position to address Corollary 1, which applies to the important class of real linear PDOs. For a real PDO L (having a corresponding real adjoint PDO $\widetilde{L}$),

$$(\widetilde{L}\widetilde{G}^*)(x,x') = 0 \tag{10}$$

if $x' \in D$ and $x \notin D$. Then by lines analogous to those used in connection with our derivation of Eq. (9), we find that according to Eq. (10) and Lemma 1, necessarily,

$$\int_D dx' \rho_{\mathrm{NR}}(x') \widetilde{G}(x,x') = 0 \quad \text{if } x \notin D. \tag{11}$$

The adjoint Green function $\widetilde{G}(x,x')$ in Eq. (11) is arbitrary. The result in Eq. (11) indicates that, from the point of view of the adjoint problem, the source $\rho_{\mathrm{NR}}(x)$, when perceived as a source to the adjoint PDO $\widetilde{L}$, is a NR source *also* to the adjoint Green function $\widetilde{G}(x,x')$, i.e., its generated adjoint field as given by $\int_D dx' \rho_{\mathrm{NR}}(x') \widetilde{G}(x,x')$ vanishes everywhere outside the source's support D. Furthermore, since this adjoint Green function $\widetilde{G}(x,x')$ in Eq. (11) is arbitrary, we have then arrived at the fundamental finding that if a source $\rho_{\mathrm{NR}}(x)$ to a real PDO L is NR to a Green's function $G(x,x')$ of the PDO L [as required in Eq. (3)], then, necessarily, this source must also be NR [as indicated in Eq. (11)] to any Green's function $\widetilde{G}(x,x')$ of the adjoint PDO $\widetilde{L}$. Furthermore, it is not hard to show by going, next, backwards in the same argument, in particular from nonradiation to a Green's function of the adjoint PDO $\widetilde{L}$ to nonradiation to any Green's function of the original PDO L [where $\widetilde{L}$ and L play, respectively, the (reversed) roles of the "governing" and the "adjoint" PDOs here], that, in general, if a source ρ_{NR} is NR with respect to a given Green's function of either L or $\widetilde{L}$, then it must automatically be NR with respect to both any Green's function of L and any Green's function of $\widetilde{L}$. This result essentially establishes that for real PDOs, nonradiation to a PDO L implies nonradiation to the associated adjoint PDO $\widetilde{L}$ and vice versa.

The necessary part of Corollary 1 can be verified as follows. First, we have found in Eq. (11) that if a source $\rho_{\mathrm{NR}}(x)$ is NR to a Green's function of the real linear PDO L, then it must also be NR to any Green's function of the adjoint PDO $\widetilde{L}$ (then, it is NR also to the adjoint PDO $\widetilde{L}$). Second, we can employ the following, adjoint version of Lemma 1: If $\rho_{\mathrm{NR}}(x)$ is a NR source of support D to the adjoint PDO $\widetilde{L}$, then, necessarily, the orthogonality relation [4] must hold for any function $v(x)$ obeying Eq. (6) everywhere in the source's support D. Finally, the above two statements imply that if a source $\rho_{\mathrm{NR}}(x)$ of support D is NR to a real linear PDO L, then it must necessarily obey the orthogonality relation (4) with respect to any function $v(x)$ that obeys the homogeneous form of the governing PDE, Eq. (6), which completes the proof.

The sufficient part of Corollary 1 follows at once by noting that for real PDOs L,

$$(LG^*)(x,x') = 0 \tag{12}$$

if $x' \in D$ and $x \notin D$. By putting $v(x') = G^*(x,x')$, which obeys according to Eq. (12) condition (6), in the orthogonality relation (4), we arrive at the nonradiation condition Eq. (3), which establishes the proof.

V. CONCLUSIONS

The results presented in this work provide a general framework for a number of results on NR sources published earlier for particular PDEs, the most interesting of which are summarized in [1,4,9]. Our analysis also generalized both in scope and in methodology a previous presentation in [8,9], which did not preserve sufficient generality by assuming from the start reality and self-adjointness of the relevant PDOs. Both our proofs and our final results are of a more general form applicable to rather general linear PDOs that may help clarify other questions in this intriguing field. Importantly, our characterization may open new applications of NR sources by enabling treatment, within common principles, of field-confining sources in other disciplines (e.g., Aharonov-Bohm magnetostatics, controlled diffusion and other transport phenomena, fluid dynamics in general, nonpropagating soliton excitations, field-confining sources in quantum field theory [19], among other areas) in addition to the usual wave disciplines where such NR sources have been of interest since the early days of electromagnetic theory. The present characterization of NR sources, i.e., sources in the null space of the propagators (Green functions) for general linear PDEs, is expected to be also of interest to workers on inverse problems where such NR sources and their corresponding null spaces play a central role (see, e.g., [2,6,7,9]). Physically, the general orthogonality relations established in this work, as defined in connection with Eqs. (4)–(6) and related discussion, can be shown to be related to noninteractivity of NR sources to (incident) fields or waves of the relevant source-field system: NR sources are noninteractors. In particular, by applying the integral defining the power or energy interaction between sources and fields in source-field systems of interest, e.g., the electromagnetic field [19], one finds that this integral must vanish for NR sources in light of the orthogonality relations (4)–(6). Then, NR sources do not interact, energetically, with free fields, neither in transmission nor in reception (for more details, see [9]). Finally, we wish to note that even though we have focused on linear PDEs, the possibility remains open of generalizing some of our results to the nonlinear regime by using nonlinear generalizations of Green's theorem involving the so-called dual PDOs that are nonlinear generalizations of the usual adjoint PDOs of linear theory (see, e.g., [20]). The corresponding nonlinear dual operators are expected to play in the more general nonlinear theory a role analogous to that of the adjoint operators of the present linear theory. Work in this direction may be of interest to cosmologists and other theorists in connection with NR gravitational objects, described by the nonlinear Einstein equations, and accounting, perhaps, along the unbounded lines of speculation, for "invisible" NR energy in the universe.

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