Kelvin-Helmholtz instability for relativistic fluids

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We reexamine the stability of an interface separating two nonmagnetized relativistic fluids in relative motion, showing that, in an appropriate reference frame, it is possible to find analytic solutions to the dispersion relation. Moreover, we show that the critical value of the Mach number, introduced by compressibility, is unchanged from the nonrelativistic case if we redefine the Mach number as $\mathcal{M}=[\beta/(1-\beta^2)^{1/2}][\beta_s/(1-\beta_s^2)^{1/2}]^{-1}$, where β and β_s are, respectively, the speed of the fluid and the speed of sound (in units of the speed of light).

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I. INTRODUCTION

The stability of the interface between two fluids in relative motion is a classical problem of fluid dynamics dating to the end of nineteenth century, and to the work of [1,2] for incompressible fluids. In this limit, any relative tangential motion between two uniform fluids is found to be unstable. These classic studies were successively extended to include important additional physical ingredients [3]. For example, the inclusion of compressibility makes stable all modes whose effective Mach number is larger than a critical value, the effective Mach number being defined using the projection of the velocity on the wave number direction. The relativistic version of the instability has been studied extensively in the astrophysical context, most prominently in [4,5,6], where numerical solutions to the dispersion relation were found. In this note we reexamine the Kelvin-Helmholtz instability for relativistic flows, showing that, in the appropriate reference frame, the dispersion relation has a form that can be solved analytically. Moreover, we show that the critical Mach number, introduced by compressibility, is unchanged from the nonrelativistic case if we use the relativistic Mach number definition $\mathcal{M}=[\beta/(1-\beta^2)^{1/2}][\beta_s/(1-\beta_s^2)^{1/2}]^{-1}$, introduced in [7,8] in the context of steady solutions.

In Sec. II we present the relevant equations and the dispersion relation, while in Sec. III we analyze its properties. In Sec. IV we summarize our results.

II. EQUATIONS AND DISPERSION RELATION

We study the linear stability of a planar interface separating two fluids in relative motion. Without loss of generality, we assume the interface to be located in the $x-z$ plane, and describe the system in a frame of reference in which the two fluids move with equal and opposite velocities, that is,

$$\mathbf{u} = \begin{cases} (+U,0,0) & \text{for} \quad y > 0, \\ (-U,0,0) & \text{for} \quad y < 0, \end{cases} \tag{1}$$

where U is positive. Furthermore, we assume that the two fluids are initially in pressure equilibrium and that they have the same proper density ρ.

Our starting points are the equations of relativistic hydrodynamics for a relativistic perfect fluid in flat Minkowskian geometry [9,10]

$$\frac{\partial(\gamma\rho)}{\partial t} + \nabla \cdot (\gamma\rho\mathbf{u}) = 0, \tag{2}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{c^2}{\rho h\gamma^2}\left(\nabla p + \frac{\mathbf{u}}{c^2}\frac{\partial p}{\partial t}\right), \tag{3}$$

$$\gamma\frac{\partial p}{\partial t} + \gamma\mathbf{u}\cdot\nabla p = \frac{\partial(\gamma\rho h)}{\partial t} + \nabla\cdot(\gamma\rho h\mathbf{u}), \tag{4}$$

where $\mathbf{u}$ is the fluid velocity, c is the speed of light, ρ is the proper rest mass density, h is the specific enthalpy, and γ denotes the Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1-\mathbf{u}^2/c^2}}.$$

The system (2)–(4) is closed by an equation of state

$$h = h(p,\rho), \tag{5}$$

from which the speed of sound can be evaluated according to

$$c_s \equiv c\sqrt{\left.\frac{1}{h}\frac{\partial p}{\partial \rho}\right|_s}, \tag{6}$$

where the derivative has to be taken at constant entropy. In what follows we do not make any particular assumption on the choice of the equation of state (5); we do, however, recall that for a relativistic (nondegenerate) perfect gas the sound speed cannot be larger than the limiting value $c/\sqrt{3}$ [11–13].

The relativistic character of Eqs. (2)–(4) enters in two distinct ways. The first effect is purely kinematical and becomes important when the relative bulk motion of the fluids is close to the speed of light c (i.e., $\gamma \gg 1$). On the other hand, we can have thermodynamically relativistic fluids where sufficiently high temperatures lead to microscopic relativistic velocities; in this case the specific enthalpy h can significantly exceed the rest mass energy of the fluid ($h \gg c^2$).

In order to obtain the dispersion relation we must first find the perturbative solutions of the linearized version of the relativistic equations (2)–(4). For this purpose, we start by observing that these solutions may be easily found in the reference frames in which the fluids are at rest: in these frames, in fact, we have sound wave solutions. Denoting a generic three-dimensional perturbation of the flow variables by δq (here q is one of ρ, $\mathbf{u}$, p) we have

$$\delta\tilde{q}_{\pm} \propto \exp\left[i\left(\tilde{k}_{\pm}\tilde{x} + \tilde{l}_{\pm}\tilde{y} + \tilde{m}_{\pm}\tilde{z} - \tilde{\omega}_{\pm}\tilde{t}\right)\right],$$

where the tilde denotes quantities in the rest frames and the + and − subscripts refer, respectively, to the fluid initially in the region $y>0$ (with positive x velocity) and $y<0$ (with negative x velocity). In the rest frames, the components of the spatial wave vector ($\tilde{k}_{\pm}$, $\tilde{l}_{\pm}$, and $\tilde{m}_{\pm}$) and the frequency $\tilde{\omega}_{\pm}$ are connected by the dispersion relation for sound waves

$$\tilde{\omega}_{\pm}^2 = (\tilde{k}_{\pm}^2 + \tilde{l}_{\pm}^2 + \tilde{m}_{\pm}^2)c_s^2. \tag{7}$$

In the laboratory frame, where the fluids have the initial configuration given by Eq. (1), the two solutions will still have the form

$$\delta q_{\pm} \propto \exp[i(kx + l_{\pm}y + mz - \omega t)],$$

this time with ω, k, and m equal on both sides of the interface. However, since k, $l_{\pm}$, m, and ω are, respectively, the spatial and temporal components of the wave four-vector $K_{\pm}^{\mu}=(k,l_{\pm},m,\omega)$, we can find their relationship to $\tilde{k}_{\pm}$, $\tilde{l}_{\pm}$, $\tilde{m}_{\pm}$, and $\tilde{\omega}_{\pm}$ by means of a Lorentz transformation. Using this result, we can write the dispersion relation (7) in the laboratory frame as

$$\gamma^2(\omega \mp kU)^2 = \left[\gamma^2\left(k \mp \omega\frac{U}{c^2}\right)^2 + l_{\pm}^2 + m^2\right]c_s^2. \tag{8}$$

The pressure has to be continuous at the interface between the two fluids, i.e., $\delta p_+|_{y=0} = \delta p_-|_{y=0} \equiv \delta p$; furthermore, the fluid displacements $\delta\xi_{\pm}(t,x)$ need to match at this interface. Since the Langrangian derivative of the latter, $d\delta\xi_{\pm}/dt$, is equal to the transverse velocity δv_y of the fluid element, matching the displacements is equivalent to

$$\frac{\delta v_{y+}}{\omega - kU} = \frac{\delta v_{y-}}{\omega + kU}, \tag{9}$$

where the tangential velocity δv_y can be expressed as

$$\delta v_{y\pm} = \frac{c^2 l_{\pm}}{(\omega \mp kU)\rho h \gamma^2}\delta p, \tag{10}$$

a result which follows upon properly linearizing the transverse component of Eq. (3).

Let us now introduce the dimensionless phase velocity $\phi=\omega/(c_3 k)$, the classical Mach number $M=U/c_s$, and $\beta=U/c$, so that Eq. (8) can be solved in order to express $l_{\pm}$ in terms of ϕ, k, m, M, and β:

$$l_{\pm} = \gamma k\sqrt{(\phi \mp M)^2 - \left(1 \mp \frac{\phi\beta^2}{M}\right)^2 - \frac{m^2}{\gamma^2 k^2}}. \tag{11}$$

Notice that $l_{\pm}$ will be, in general, complex numbers; therefore in order to satisfy appropriate boundary conditions at infinity, $\mathrm{Im}(l_+)$ must have positive sign for $y>0$, while $\mathrm{Im}(l_-)$ must be negative for $y<0$. Furthermore, perturbations must be carried by outgoing waves as $y \to \pm\infty$: this is known as the Sommerfeld radiation condition and has to be applied in the frame in which the fluid is at rest [3].

The dispersion relation can now be obtained by substituting Eq. (10) into Eq. (9), with $l_{\pm}$ given by Eq. (11); after a bit of algebra we find the following equation for ϕ:

$$\frac{\sqrt{(\phi+M)^2 - (1+\phi\beta^2/M)^2 - \alpha^2(1-\beta^2)}}{(\phi+M)^2} = \frac{\sqrt{(\phi-M)^2 - (1-\phi\beta^2/M)^2 - \alpha^2(1-\beta^2)}}{(\phi-M)^2}, \tag{12}$$

where $\alpha=m/k$.

Equation (12) represents the desired dispersion relation. Following [7,8] we introduce the relativistic Mach number, defined as $\mathcal{M}=\gamma M/\gamma_s$, with $\gamma_s=(1-c_s^2/c^2)^{-1/2}$; the dispersion relation (12) can then be squared to obtain the fifth-order polynomial

$$\phi\left[\left(\frac{\phi}{M}\right)^4(\mathcal{M}^2+2\beta^2) - 2\left(\frac{\phi}{M}\right)^2(\mathcal{M}^2+1+\alpha^2-\beta^2) + (\mathcal{M}^2-2-2\alpha^2)\right] = 0. \tag{13}$$

Notice that α is related to the angle θ between the fluid velocity and the wave number projection in the $x-z$ plane by

$$\cos\theta = \frac{1}{\sqrt{1+\alpha^2}},$$

which allows us to write the solutions to Eq. (13) as

$$\phi = 0, \tag{14}$$

$$\frac{\phi^2}{M^2} = \frac{\mathcal{M}_e^2 + 1 - \beta_e^2 \pm \sqrt{4\mathcal{M}_e^2(1-\beta_e^2) + (1+\beta_e^2)^2}}{\mathcal{M}_e^2 + 2\beta_e^2}, \tag{15}$$

where the effective relativistic Mach number $\mathcal{M}_e$ and effective fluid velocity (in units of c) β_e are defined as

$$\mathcal{M}_e = \mathcal{M}\cos\theta, \quad \beta_e = \beta\cos\theta.$$

III. DISCUSSION

The solutions of Eq. (13) form a two-parameter family, which we will describe in terms of the effective Mach number $\mathcal{M}_e$ and the effective β_e. Since $c_s/c=\beta/M$, an equivalent choice may be given by any combination of two parameters among $\mathcal{M}$ (or M), β and c_s/c; when the Mach number (either classic or relativistic) is used, the restriction $c_s/c<1/\sqrt{3}$ corresponds to the requirement that only the region $M>\sqrt{3}\beta$

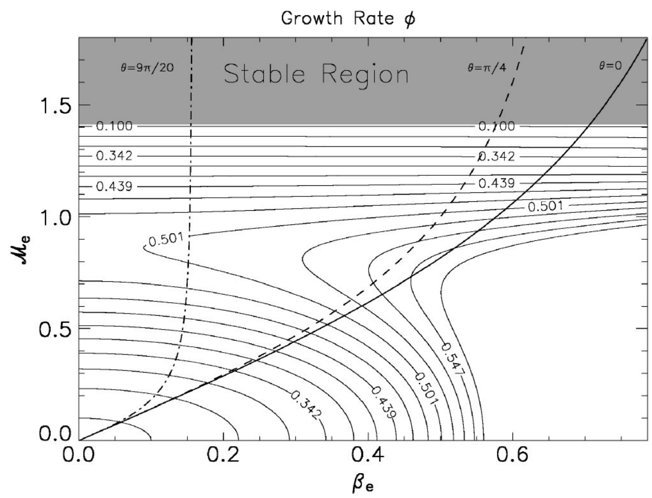

FIG. 1. Growth rate contours for the unstable mode discussed in the text as a function of the relativistic effective Mach number $\mathcal{M}_e$ and β_e. The allowed range for $\mathcal{M}_e$ and β_e is restricted by the unphysical region, where the sound speed exceeds the limiting value $c/\sqrt{3}$. The three curves represent the upper boundaries of such unphysical regions for three different angles ($\theta=0$, solid line; $\theta=\pi/4$, dashed line; $\theta=9\pi/20$, dash-dotted line). For $\mathcal{M}_e>\sqrt{2}$ this mode becomes stable.

(or, equivalently, $\mathcal{M}>\sqrt{2}\gamma\beta$) be physically accessible.

Returning to the solutions (14) and (15), we thus find that the first root $\phi=0$, which corresponds to a neutrally stable mode, is valid only for $\mathcal{M}_e\geqslant 1$. This neutral solution becomes important when one considers the stability properties of a smooth shear layer (rather than a vortex sheet), where it has been shown (see [14] for the classical case) that this mode is destabilized.

Of the remaining roots (15), only the one with the minus sign is valid and gives an unstable mode in the range $\mathcal{M}_e<\sqrt{2}$ as ϕ (and therefore ω) becomes purely imaginary; for $\mathcal{M}_e>\sqrt{2}$ the solution is stable. In addition, since the physically allowed region has to satisfy $\mathcal{M}>\sqrt{2}\gamma\beta$, perturbations with $\beta>(1+\cos^2\theta)^{-1/2}$ are always linearly stable, regardless of the value of the Mach number (and therefore of the sound speed). The growth rate for this mode is shown in Fig. 1 as a function of $\mathcal{M}_e$ and β_e. In the same figure the three curves represent the boundaries of the physically allowed regions for three different angles (i.e., the condition $\mathcal{M}>\sqrt{2}\gamma\beta$ is satisfied only above these curves).

We note that with this definition of the Mach number the stability conditions for the two modes in this reference frame are the same ones found in the classical case. This can be understood by recalling that [7,8] have shown that the relativistic steady equations for an ideal fluid can be transformed into an equivalent Newtonian form by a suitable set of transformations, one of which is the substitution of the relativistic Mach number for the classical one. In our case, when we consider the neutral mode or the unstable mode at cutoff, we have $\phi=0$ and therefore we are dealing with steady solutions. Thus all the relations that are valid for them in the Newtonian case still hold in the relativistic case, provided we make the suitable transformations and, in particular, the critical values of the Mach number remain the same when we substitute the classical definition with the relativistic definition.

The previous results can be easily reformulated in terms of the classical Mach number $M=U/c_s$, by simply recalling that, according to its definition, we have

$$M=\sqrt{\mathcal{M}^2(1-\beta^2)+\beta^2}. \tag{16}$$

We remind the reader that both the classical and relativistic Mach numbers as previously introduced refer to either one layer or the other as seen in the laboratory frame. To find the relative Mach numbers between the two layers, we have to apply the correct relativistic velocity composition, that is,

$$\tilde{U}_{\pm}=\pm\frac{2U}{1+\beta^2}, \tag{17}$$

where, again, the tilde denotes quantities measured in their rest frame. In other words, $\tilde{U}_+$ is the velocity of the upper $(y>0)$ layer as measured from the rest frame of the lower $(y<0)$, and similarly for $\tilde{U}_-$. Since thermodynamic quantities such as c_s and γ_s are relativistically invariant by definition, the classical Mach number transforms in the same way as U, while the relativistic Mach number becomes $\tilde{\mathcal{M}}=2\gamma\mathcal{M}$.

The stability conditions for the first and second modes [given, respectively, by Eqs. (14) and (15)], as seen in these rest frames, are obtained simply by using Eq. (17) together with its inverse relation at cutoff. This yields

$$\tilde{\mathcal{M}}\cos\tilde{\theta}>2\mathcal{M}_c\tilde{\chi}, \tag{18}$$

where

$$\tilde{\chi}=\frac{\tilde{\gamma}+1}{2}\sqrt{1+\frac{1-\tilde{\gamma}}{1+\tilde{\gamma}}\cos^2\tilde{\theta}}, \tag{19}$$

with $\mathcal{M}_c=1$, $\mathcal{M}_c=\sqrt{2}$ for the two modes, respectively. In Eq. (19) we have introduced the relative Lorentz factor between the two layers, $\tilde{\gamma}=(1-\tilde{\beta}^2)^{-1/2}$, and the angle $\tilde{\theta}=\tan^{-1}\tilde{m}/\tilde{k}$.

The relativistic critical Mach number is now a monotonically increasing function of the relative velocity $\tilde{\beta}$ between the two fluids, reaching its minimum value in the limit of vanishing $\tilde{\beta}$, where Eq. (18) reduces to the well known classical stability conditions, i.e., $\tilde{\mathcal{M}}\cos\tilde{\theta}>2$ for the neutrally stable mode and $\tilde{\mathcal{M}}\cos\tilde{\theta}>\sqrt{8}$ for the unstable mode. The critical classical Mach number, however, decreases with

TABLE I. Stability conditions for the two modes described in the text in terms of the effective relativistic and classical Mach numbers $\mathcal{M}_e$ and $M_e \equiv M \cos\theta$. For clarity of exposition we set $\tilde{\eta} = \{2 + [(1-\tilde{\gamma})/(1+\tilde{\gamma})]\cos^2\tilde{\theta}\}^{1/2}$, while $\tilde{\mathcal{M}}_e = \tilde{\mathcal{M}}\cos\tilde{\theta}$ and $\tilde{M}_e = \tilde{M}\cos\tilde{\theta}$ are the effective relativistic and classical Mach numbers in the rest frame. The Newtonian limit is recovered by letting γ, $\tilde{\gamma} \to 1$, $\beta_e \to 0$, and $\tilde{\chi} \to 1$.

Mode	Laboratory frame	Rest frame
$\phi = 0$	$\mathcal{M}_e > 1$, $M_e > \sqrt{1/\gamma^2 + \beta_e^2}$	$\tilde{\mathcal{M}}_e > 2\tilde{\chi}$, $\tilde{M}_e > \gamma + 1/\gamma$
$\phi = \phi_-$	$\mathcal{M}_e > \sqrt{2}$, $M_e > \sqrt{2/\gamma^2 + \beta_e^2}$	$\tilde{\mathcal{M}}_e > \sqrt{8}\tilde{\chi}$, $\tilde{M}_e > [(\tilde{\gamma}+1)/\tilde{\gamma}]\tilde{\eta}$

increasing $\tilde{\beta}$. The stability criteria for the two modes in terms of M and $\mathcal{M}$ are summarized in Table I.

IV. SUMMARY

Our results can be briefly summarized as follows.

(1) We have derived the dispersion relation for the Kelvin-Helmholtz instability for relativistic flows, showing that it can be solved analytically.

(2) Using the definition of relativistic Mach number given by [7,8] we have shown that in the laboratory frame the stability criteria are the same as those found in the classical case.

(3) We find that, for a given perturbation whose wave number makes an angle θ with the flow direction, there exists a critical velocity above which the fluid is always stable, regardless of the value of the Mach number. In the laboratory frame, this value is conveniently expressed in terms of the Lorentz factor as $\gamma = \sqrt{1 + 1/\cos^2\theta}$, while in the rest frame we have $\tilde{\gamma} = 1 + 2/\cos^2\tilde{\theta}$.

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