Poincaré Husimi representation of eigenstates in quantum billiards

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(Received 25 July 2003; revised manuscript received 2 December 2003; published 14 September 2004)

For the representation of eigenstates on a Poincaré section at the boundary of a billiard different variants have been proposed. We compare these Poincaré Husimi functions, discuss their properties, and based on this select one particularly suited definition. For the mean behavior of these Poincaré Husimi functions an asymptotic expression is derived, including a uniform approximation. We establish the relation between the Poincaré Husimi functions and the Husimi function in phase space from which a direct physical interpretation follows. Using this, a quantum ergodicity theorem for the Poincaré Husimi functions in the case of ergodic systems is shown.

DOI: 10.1103/PhysRevE.70.036204 PACS number(s): 05.45.Mt, 03.65.Sq, 03.65.Ge

I. INTRODUCTION

The study of eigenfunctions of quantum systems, in particular, their dependence on the classical dynamics, has attracted a lot of attention. A prominent class of examples is provided by two-dimensional billiard systems, which are classically given by the free motion of a particle inside some domain with elastic reflections at the boundary. The corresponding quantum system is described by the Helmholtz equation inside a compact domain $\Omega \subset \mathbb{R}^2$ (in units $\hbar = 1 = 2m$),

$$\Delta\psi_n(\boldsymbol{x}) + k_n^2\psi_n(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega, \tag{1}$$

with (for example) Dirichlet boundary conditions

$$\psi_n(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega, \tag{2}$$

where the eigenfunctions $\psi_n(\boldsymbol{x})$ are in $L^2(\Omega)$. Assuming that the eigenvalues k_n^2 are ordered with increasing value, the semiclassical limit corresponds to $n \to \infty$. A detailed knowledge of the behavior of the eigenvalues k_n^2 and the structure of eigenstates is relevant for applications, for example, microwave cavities or mesoscopic systems (see, e.g., Ref. [1], and references therein).

For the description of the phase space structure of quantum systems usually the Wigner function [2] or Husimi function [3] is used. However, for a system with d degrees of freedom these are $2d$-dimensional functions, which are difficult to visualize for $d > 1$. Therefore, one usually considers the position representation, or the momentum representation [4], or sections through the Wigner or Husimi function, see, e.g., Ref. [5].

Another approach is the use of representations on the billiard boundary, acting as a global Poincaré section. In the literature one can find several variants for these representations, see, e.g., Refs. [6–8]. The reason is, as emphasized in Ref. [7], that there is no natural definition of a scalar product for functions on the billiard boundary. This raises the question whether one of these definitions has advantages over the others, which will be addressed in the following.

The representation of eigenstates on the Poincaré section plays an important role in several applications. For example, it is used to define scar measures [8,9], or to study conductance fluctuations, see Ref. [10], and references therein. Furthermore, these representations are used to determine the coupling of leads in open systems [11]. Another important application is the detection of regions where eigenstates localize, see, e.g., Refs. [12,13,11] (for an alternative approach based on the scattering approach see Refs. [14,15]). Representations of eigenstates on the Poincaré section have also been useful to understand the behavior of optical microresonators, see, e.g., Ref. [16], and references therein. More generally, the approach is not just applicable for billiard systems but it is also useful for Poincaré sections arising from Bogomolny's transfer operator approach [17].

In this paper we first compare two different definitions for the Poincaré Husimi representation, discuss their properties (Sec. II), and based on this we select one particular definition for the following. In Sec. III we derive the behavior of these Poincaré Husimi functions when averaged over several energies. In Sec. IV we establish a relation between the well-known Husimi function in phase space and the Poincaré Husimi function on the billiard boundary. This allows for a direct physical interpretation of the Poincaré Husimi functions. Moreover, for ergodic systems a quantum ergodicity theorem for the Poincaré Husimi functions is shown.

II. HUSIMI REPRESENTATION ON THE BOUNDARY

Let us first recall the definition and some properties of Husimi functions in phase space. For a solution ψ_n of the Helmholtz equation (1) with energy $E = k_n^2$ the Husimi function $H_n^B(\boldsymbol{p},\boldsymbol{q})$ is given by its projection onto a coherent state, i.e.,

$$H_n^B(\boldsymbol{p},\boldsymbol{q}) := \left(\frac{k_n}{2\pi}\right)^2 |\langle \psi^B_{(\boldsymbol{p},\boldsymbol{q}),k_n}, \psi_n \rangle_\Omega|^2. \tag{3}$$

Here

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$$\langle \psi_1, \psi_2 \rangle_\Omega := \int_\Omega \overline{\psi_1}(\boldsymbol{q}) \psi_2(\boldsymbol{q}) d^2 q \tag{4}$$

is the scalar product in Ω, and $\overline{\psi_1}$ denotes the complex conjugate of ψ_1.

The coherent states are defined as

$$\psi^B_{(\boldsymbol{p},\boldsymbol{q}),k}(\boldsymbol{x}) := \left(\frac{k}{\pi}\right)^{1/2} (\det \operatorname{Im} B)^{1/4} e^{ik[\langle \boldsymbol{p}, \boldsymbol{x}-\boldsymbol{q}\rangle + (1/2)\langle \boldsymbol{x}-\boldsymbol{q}, B(\boldsymbol{x}-\boldsymbol{q})\rangle]}, \tag{5}$$

where $(\boldsymbol{p},\boldsymbol{q}) \in \mathbb{R}^2 \times \mathbb{R}^2$ denotes the point in phase space around which the coherent state is localized, and B is a symmetric complex 2×2 matrix which determines the shape of the coherent state. For the conventional coherent states one has $B = i\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and in general one has the condition $\operatorname{Im} B > 0$, i.e., $\langle v, \operatorname{Im} B\, v\rangle > 0$ for all $v \in \mathbb{R}^2 \setminus \{0\}$. Notice that because the variance of the coherent states is proportional to k, all Husimi functions are concentrated around the energy shell $|\boldsymbol{p}|^2 = 1$ (and not around $|\boldsymbol{p}|^2 = k^2$). By this it is possible to compare Husimi functions with different energies k_n^2, and, for example, consider their mean, see Eq. (7) below.

Such Husimi functions can be interpreted as probability distributions on phase space, because they satisfy the relation

$$\langle \psi_n, \boldsymbol{A}\psi_n \rangle_\Omega = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} a(\boldsymbol{p},\boldsymbol{q}) H_n^B(\boldsymbol{p},\boldsymbol{q}) d^2 p\, d^2 q + O(k_n^{-1}), \tag{6}$$

where $a(\boldsymbol{p},\boldsymbol{q})$ is a function on phase space and $\boldsymbol{A}$ its quantization. This relation also shows that the choice of the matrix B in the definition of the coherent states does not affect the leading order behavior of H_n^B as a probability density, since the left hand side of Eq. (6) does not depend on B.

The average of all Husimi functions $H_n^B(\boldsymbol{p},\boldsymbol{q})$ up to some energy $k^2 = E$ converges for $k \to \infty$ to the normalized invariant measure on the energy shell,

$$\lim_{k\to\infty} \frac{1}{N(k)} \sum_{k_n \leqslant k} H_n^B(\boldsymbol{p},\boldsymbol{q}) = \frac{1}{\pi A} \chi_\Omega(\boldsymbol{q}) \delta(1 - |\boldsymbol{p}|^2). \tag{7}$$

Here $N(k)$ denotes the spectral staircase function, $N(k) := \#\{k_n \leqslant k\}$, χ_Ω is the characteristic function on Ω, and A is the area of Ω. The mean behavior (7) is similar to the mean behavior of the spectral staircase function, which is given by the Weyl formula, i.e., for $k \to \infty$ one has $N(k) \sim (A/4\pi)k^2$. A similar asymptotic behavior can be derived for the mean of normal derivative functions, see Ref. [19] for a detailed discussion.

For billiards an extremely useful approach for describing the dynamics is the use of a Poincaré section $\mathcal{P}$ together with the corresponding Poincaré mapping P. Usually the section $\mathcal{P} := \{(q,p) \mid q \in [0,L], p \in [-1,1]\}$ is parametrized by the arclength coordinate q along the boundary $\partial\Omega$ of the billiard and the projection p of the (unit) momentum $\hat{\boldsymbol{p}}$ after the reflection on the tangent $\hat{\boldsymbol{t}}(q)$, i.e., $p = \langle \hat{\boldsymbol{p}}, \hat{\boldsymbol{t}}(q)\rangle$. By this the billiard flow induces an area-preserving map $P: \mathcal{P} \to \mathcal{P}$, where the invariant measure is given by $d\mu = dq\, dp$.

In order to have the advantages of such a reduced representation in quantum mechanics as well, one is interested in a Husimi representation $h_n(q,p)$ on the Poincaré section $\mathcal{P}$ which is associated with an eigenstate ψ_n. Such a *Poincaré Husimi function* should have similar properties as the ones expressed by relations (6) and (7) for the Husimi functions in phase space, and our aim is to study to what extent this is possible. More precisely, one would like that for the Husimi function on the billiard boundary a spectral average,

$$\mathcal{H}_k(q,p) := \frac{1}{N(k)} \sum_{k_n \leqslant k} h_n(q,p), \tag{8}$$

tends to the invariant measure on $\mathcal{P}$ as $k \to \infty$, in the same way as in Eq. (7).

The Husimi representation on the billiard boundary is usually defined using the normal derivative of the eigenfunction (hereafter called the boundary function),

$$u_n(s) := \langle \hat{\boldsymbol{n}}(s), \boldsymbol{\nabla}\, \psi_n(\boldsymbol{x}(s))\rangle, \tag{9}$$

where $\boldsymbol{x}(s)$ is a point on the boundary $\partial\Omega$, parametrized by the arclength s, and $\hat{\boldsymbol{n}}(s)$ denotes the outer normal unit vector to $\partial\Omega$ at $\boldsymbol{x}(s)$. The boundary functions are a natural starting point for defining a Husimi representation because they determine the eigenfunctions uniquely, see Eq. (30). Thus the boundary functions form a reduced representation of the system. If an eigenfunction ψ_n is normalized, then the corresponding boundary function u_n fulfils the normalization condition [21]

$$\frac{1}{2} \int_{\partial\Omega} |u_n(s)|^2 \langle \hat{\boldsymbol{n}}(s), \boldsymbol{x}(s)\rangle ds = k_n^2. \tag{10}$$

For alternative derivations of Eq. (10) and more general boundary conditions see Refs. [22,23]. Notice that while the integrand depends on the chosen origin for the vector $\boldsymbol{x}(s)$, the integral is independent of this choice.

Starting from the boundary function a Husimi function on the Poincaré section can be defined by a projection onto a coherent state. There are different possibilities to define coherent states on the boundary of a billiard. A natural choice is the periodization of the usual one-dimensional coherent states,

$$c^b_{(q,p),k}(s) := \left(\frac{k}{\pi}\right)^{1/4} (\operatorname{Im} b)^{1/4} \sum_{m\in\mathbb{Z}} e^{ik[p(s-q+mL) + (b/2)(s-q+mL)^2]}, \tag{11}$$

where $(q,p) \in \partial\Omega \times \mathbb{R}$, and L denotes the length of the boundary. The parameter $b \in \mathbb{C}$, $\operatorname{Im} b > 0$, determines the shape of the coherent state. Then for an eigenstate ψ_n with boundary function u_n a Husimi function on the Poincaré section $\mathcal{P}$ (or more precisely, on the cylindric phase space $\partial\Omega \times \mathbb{R}$) can be defined as [6,7]

$$h_n(q,p) = \frac{1}{2\pi k_n}\left| \int_{\partial\Omega} \overline{c^b_{(q,p),k_n}}(s) u_n(s) ds \right|^2. \tag{12}$$

The completeness relation for the coherent states gives

$$\int_{\partial\Omega}\int_{\mathbb{R}} h_n(q,p) dp\ dq = \frac{1}{k_n^2}\int_{\partial\Omega} |u_n(s)|^2 ds, \tag{13}$$

so in view of relation (10) the Poincaré Husimi function $h_n(q,p)$ will in general not be normalized. This can be fixed by dividing $h_n(q,p)$ by the factor $(1/k_n^2)\int |u_n(s)|^2 ds$, as was done, for instance, in Refs. [12,13]. But later on we will see that it is more natural to work with the non-normalized Husimi functions (12).

A different Poincaré representation has been proposed in Ref. [8],

$$\widetilde{h}_n(q,p) = \frac{1}{2k_n^2}\frac{\left| \int_{\partial\Omega} \overline{c^b_{(q,p),k_n}}(s) u_n(s) \langle \hat{\boldsymbol{n}}(s), \boldsymbol{x}(s)\rangle ds \right|^2}{\int_{\partial\Omega} \overline{c^b_{(q,p),k_n}}(s) c^b_{(q,p),k_n}(s) \langle \hat{\boldsymbol{n}}(s), \boldsymbol{x}(s)\rangle ds}, \tag{14}$$

where the inclusion of the factor $\langle \hat{\boldsymbol{n}}(s), \boldsymbol{x}(s)\rangle$ is motivated by its appearance in the normalization condition (10). In order to compare the two definitions, we use the fact that for large k the coherent state becomes more and more concentrated around $s=q$ and so $\langle \hat{\boldsymbol{n}}(s), \boldsymbol{x}(s)\rangle\ \overline{c^b_{(q,p),k_n}}(s) \sim \langle \hat{\boldsymbol{n}}(q), \boldsymbol{x}(q)\rangle \overline{c^b_{(q,p),k_n}}(s)$. This leads to the relation

$$\widetilde{h}_n(q,p) \sim \langle \hat{\boldsymbol{n}}(q), \boldsymbol{x}(q)\rangle h_n(q,p), \tag{15}$$

between the two definitions for Husimi functions.

Let us first illustrate the behavior of the Husimi representation given by Eq. (12). As a concrete example we consider a member of the family of limaçon billiards introduced by Robnik [24,25], whose boundary is given in polar coordinates by $\rho(\varphi)=1+\varepsilon \cos(\varphi)$, where $\varepsilon \in [0,1]$ is the family parameter. At $\varepsilon=0.3$ the billiard has a mixed phase space (see Fig. 1 in Ref. [12]) and at $\varepsilon=1$ it turns into the fully chaotic (i.e., ergodic, mixing, …) cardioid billiard. Because of the symmetry of the billiard we consider the half-limaçon billiard with Dirichlet boundary conditions everywhere. The eigenvalues have been computed using the conformal mapping technique [25,26] and then the boundary element method has been used to compute the eigenfunctions (see Ref. [27], and references therein). Figure 1 shows a comparison of eigenstates $\psi_n(\boldsymbol{q})$ with their Husimi representations $h_n(q,p)$ as gray-scale plots with black corresponding to large values. For the computations $b:=i\sigma^{-1}=i$ was chosen. In (a) an eigenstate which is localized around a stable periodic orbit with period three is shown which is clearly reflected in its Poincaré Husimi function to the right. The symmetry $h_n(q,p)=h_n(q,-p)$ is due to the time-reversal symmetry of the system and the symmetry $h_n(q,p)=h_n(L-q,p)$ stems from the reflection symmetry of the system. The plots in

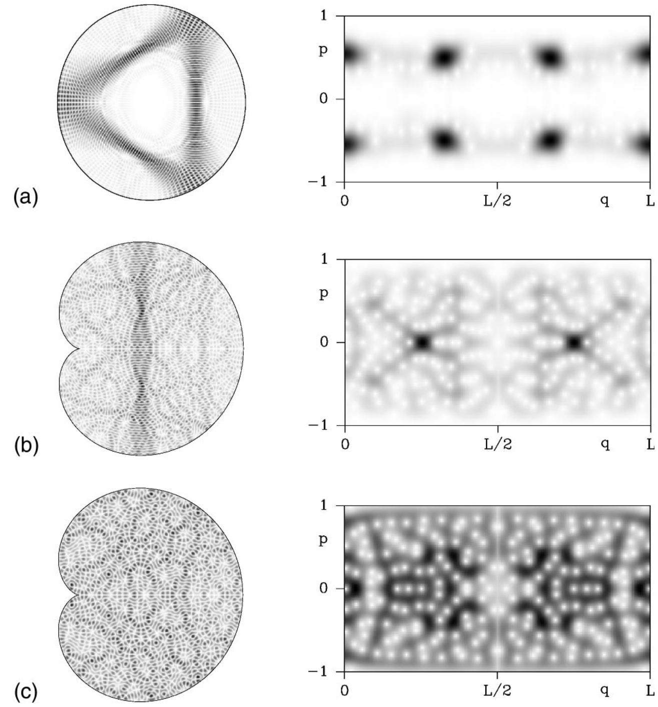

FIG. 1. Examples of eigenstates $\psi_n(\boldsymbol{q})$, shown to the left, and to the right their Poincaré Husimi functions $h_n(q,p)$. In (a) an eigenstate ($n=1952$) localizing around a regular orbit for the limaçon billiard at $\varepsilon=0.3$ is shown. In (b) and (c) two eigenstates for the cardioid billiard are shown ($n=1817$ and $n=1277$).

Figs. 1(b) and 1(c) are at $\varepsilon=1.0$, i.e., for the cardioid billiard. The eigenstate shown in (b) is localized around an unstable periodic orbit of period two which is also nicely seen in the prominent peaks for the corresponding Poincaré Husimi function. In (c) an irregular state in the cardioid billiard is displayed which is spread out over the full billiard and also $h_n(q,p)$ does not show any prominent localization.

Now we turn to a comparison of the two Poincaré Husimi representations given by Eqs. (12) and (14). In Fig. 2 a plot of $\mathcal{H}_k(q,p)$ is shown where $k=125.27\ldots$ is chosen such that the first 2000 states are taken into account. Both definitions, Eqs. (12) and (14), lead to a similar nonuniform behavior of $\mathcal{H}_k(q,p)$ in p direction. We will discuss this behavior in more detail in the following section. In addition we observe that $\mathcal{H}_k(q,p)$ has a minimum at $(q,p)=(0,0)$ and $(q,p)=(\pm\mathcal{L}/2,0)$, which is due to the desymmetrization. Figure 2(b) shows a plot of $\widetilde{H}_k(q,p)$ which is defined as $\mathcal{H}_k(q,p)$, but instead of $h_n(q,p)$ the functions $\widetilde{h}_n(q,p)$ are used, see definition (14). In this case we observe in addition a clear variation in q. The reason for this is the factor $\langle \hat{\boldsymbol{n}}(q), \boldsymbol{x}(q)\rangle$ as explained by relation (15). Another important point is that the definition (14) depends on the chosen origin as the factor $\langle \hat{\boldsymbol{n}}(q), \boldsymbol{x}(q)\rangle$ does, and therefore the integrals in Eq. (14) are not invariant under a shift of the origin. Because of the variation of $\widetilde{h}_n(q,p)$ in q and the dependence on the origin we prefer the definition (12) and will use this exclusively in the following.

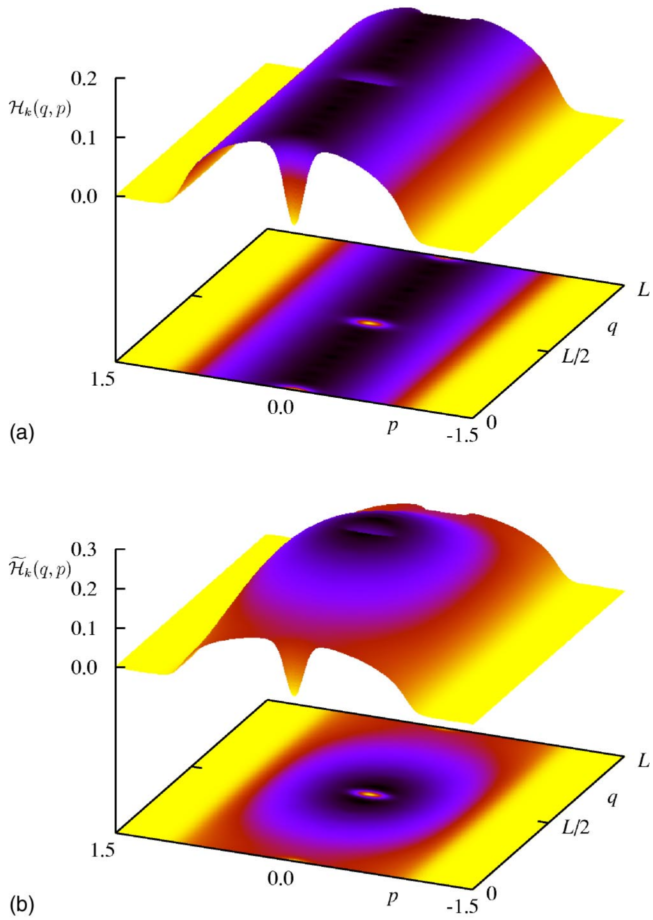

FIG. 2. Plot of $\mathcal{H}_k(q,p)$ for $k=125$ using the first 2000 eigenstates in the limaçon billiard of odd symmetry at $\varepsilon=0.3$. In (a) the result for $\mathcal{H}_k(q,p)$ using definition (12) for $h_n(q,p)$ is shown and in (b) a corresponding $\widetilde{\mathcal{H}}_k(q,p)$ using definition (14) is displayed. In addition to the symmetry related dips at $(q,p)=(0,0)$ and $(L/2,0)$ one clearly sees the variation in p direction in both cases and in (b) we, moreover, observe a variation in q.

III. MEAN BEHAVIOR OF BOUNDARY HUSIMI FUNCTIONS

In this section we determine the asymptotic behavior of the mean $\mathcal{H}_k(q,p)$ of the boundary Husimi functions for large energies. To this end we will use the methods from our previous work [19]. Let us introduce

$$g^{\rho}(k,s,s') := \sum_{n\in\mathbb{N}} \frac{u_n(s)\overline{u_n}(s')}{k_n^2}\rho(k-k_n), \tag{16}$$

where ρ is a smooth function whose Fourier transform $\hat{\rho}$ is supported in a neighborhood $[-\eta,\eta]$, with η smaller than the length of the shortest periodic orbit of the billiard flow, and satisfies in addition $\hat{\rho}(0)=1$. The function $g^{\rho}(k,s,s')$ was studied in Ref. [19] and an asymptotic expansion was derived. Its leading term reads

$$g^{\rho}(k,s,s') = \frac{k}{2\pi^2}\int_0^{2\pi} \langle \hat{\boldsymbol{n}}(s),\hat{\boldsymbol{e}}(\varphi)\rangle \times \langle \hat{\boldsymbol{n}}(s'),\hat{\boldsymbol{e}}(\varphi)\rangle e^{ik\langle \boldsymbol{x}(s)-\boldsymbol{x}(s'),\hat{\boldsymbol{e}}(\varphi)\rangle} d\varphi[1+O(k^{-1})], \tag{17}$$

where $\boldsymbol{x}(s)$ denotes the position vector on the boundary at point s, $\hat{\boldsymbol{n}}(s)$ denotes the outer unit normal vector to the boundary at s, and $\hat{\boldsymbol{e}}(\varphi)=(\cos\varphi,\sin\varphi)$ is the unit vector in direction φ. In general the right hand side of Eq. (17) is a sum of oscillating terms corresponding to reflected orbits, the condition on the support of the Fourier transform of ρ is necessary in order that only one term contributes.

Multiplying Eq. (17) with $\overline{c^b_{(q,p),k}}(s)$ and $c^b_{(q,p),k}(s')$ and integrating over s and s' leads to

$$\sum_{n\in\mathbb{N}} \rho(k-k_n)h_n(q,p) = \frac{k^2}{4\pi^3}\int_0^{2\pi} \left| \int_{\partial\Omega} \langle \hat{\boldsymbol{n}}(s),\hat{\boldsymbol{e}}(\varphi)\rangle\, e^{ik\langle \boldsymbol{x}(s),\hat{\boldsymbol{e}}(\varphi)\rangle}\overline{c^b_{(q,p),k}}(s)ds \right|^2 d\varphi \times[1+O(k^{-1})], \tag{18}$$

where we have used $\|c^b_{(q,p),k}-c^b_{(q,p),k_n}\|^2 \leqslant C(k-k_n)^2/(k+k_n)^2$ in order to obtain the left hand side. The s integral can be computed by the method of stationary phase,

$$\begin{aligned}\int_{\partial\Omega} &\langle \hat{\boldsymbol{n}}(s),\hat{\boldsymbol{e}}(\varphi)\rangle e^{ik\langle \boldsymbol{x}(s),\hat{\boldsymbol{e}}(\varphi)\rangle}\overline{c^b_{(q,p),k}}(s)ds \\ &= \left(\frac{k}{\pi}\right)^{1/4}(\mathrm{Im}\, b)^{1/4}\int_{-\infty}^{\infty}\langle \hat{\boldsymbol{n}}(s),\hat{\boldsymbol{e}}(\varphi)\rangle \times e^{ik[\langle \boldsymbol{x}(s),\hat{\boldsymbol{e}}(\varphi)\rangle - p(s-q)-(\bar{b}/2)(s-q)^2]}ds \\ &= \left(\frac{4\pi}{k}\right)^{1/4}\frac{(\mathrm{Im}\, b)^{1/4}}{[i\tilde{b}]^{1/2}}\langle \hat{\boldsymbol{n}}(q),\hat{\boldsymbol{e}}(\varphi)\rangle \times e^{ik[\langle \boldsymbol{x}(q),\hat{\boldsymbol{e}}(\varphi)\rangle + (1/2\tilde{b})\{p-\langle \hat{\boldsymbol{t}}(q),\hat{\boldsymbol{e}}(\varphi)\rangle\}^2]}[1+O(k^{-1/2})],\end{aligned} \tag{19}$$

with

$$\tilde{b} = \bar{b} + \kappa(q)\langle \hat{\boldsymbol{n}}(q),\hat{\boldsymbol{e}}(\varphi)\rangle, \tag{20}$$

where $\kappa(q)$ is the curvature of the boundary at q. Inserting this result we obtain

$$\sum_{n\in\mathbb{N}} \rho(k-k_n)h_n(q,p) = \frac{2k^2}{(2\pi)^3}\left(\frac{4\pi}{k}\right)^{1/2}\int_0^{2\pi}\frac{(\mathrm{Im}\, b)^{1/2}}{|\tilde{b}|}|\langle \hat{\boldsymbol{n}}(q),\hat{\boldsymbol{e}}(\varphi)\rangle|^2 \times e^{-k(\mathrm{Im}\, b/|\tilde{b}|^2)[p-\langle \hat{\boldsymbol{t}}(q),\hat{\boldsymbol{e}}(\varphi)\rangle]^2}\, d\varphi[1+O(k^{-1/2})], \tag{21}$$

and for $|p|<1$ the φ integral can again be solved by the method of stationary phase (notice that there are two stationary points) which yields

$$\sum_{n\in\mathbb{N}}\rho(k-k_n)h_n(q,p)=\frac{k}{\pi^2}\sqrt{1-p^2}[1+O(k^{-1/2})]. \quad (22)$$

By integrating this equation, and using a Tauberian Lemma as in proofs of the Weyl formula (see, e.g., Ref. [28], Lemma 17.5.6), we finally obtain

$$\mathcal{H}_k(q,p)\equiv\frac{1}{N(k)}\sum_{k_n\leqslant k}h_n(q,p)=\frac{2}{A\pi}\sqrt{1-p^2}+O(k^{-1/2}). \quad (23)$$

In the derivation of Eq. (22) from Eq. (21) we have assumed that $|p|<1$ because then the stationary points are nondegenerate. For $|p|>1$ the stationary points become complex and the integral is exponentially decreasing for $k\to\infty$.

Previously, such a $\sqrt{1-p^2}$ behavior appeared in the context of Fredholm methods for Poincaré Husimi functions [30] and was also obtained in connection with the inverse participation ratio [9].

Next we want to derive a uniform approximation which describes the mean behavior of the Husimi functions near $|p|=1$ and the crossover from the regime $|p|<1$ to the exponential decrease for $|p|>1$. We will study the case $p\approx 1$, the case $p\approx -1$ is completely analogous. Let φ_0 be the angle corresponding to the direction of $\hat{\boldsymbol{t}}(q)$ and expanding the amplitude and phase function in Eq. (21) around φ_0 leads to

$$\begin{aligned}\sum_{n\in\mathbb{N}}\rho(k-k_n)h_n(q,p)&=\frac{4k^2}{(2\pi)^3}\left(\frac{4\pi}{k}\right)^{1/2}\int_0^\infty\frac{(\operatorname{Im}b)^{1/2}}{|\tilde{b}|}\varphi^2e^{-k(\operatorname{Im}b/|\tilde{b}|^2)(p-1+\varphi^2)^2}d\varphi\,[1+O(k^{-1/2})]\\&=\frac{4k^2}{(2\pi)^3}\left(\frac{4\pi}{k}\right)^{1/2}\int_0^\infty\frac{(\operatorname{Im}b)^{1/2}}{|\tilde{b}|}x^{1/2}e^{-k(\operatorname{Im}b/|\tilde{b}|^2)(p-1+x)^2}dx[1+O(k^{-1/2})]\\&=e^{-k(\operatorname{Im}b/|\tilde{b}|^2)(p-1)^2}\frac{(2k)^{3/4}}{2\pi^{5/2}}\left(\frac{|\tilde{b}|^2}{\operatorname{Im}b}\right)^{1/4}\int_0^\infty x^{1/2}e^{-\{(2k\operatorname{Im}b)^{1/2}/|\tilde{b}|\}(p-1)x-x^2/2}dx[1+O(k^{-1/2})]\\&=\frac{(2k)^{3/4}}{(2\pi)^2}e^{-k(\operatorname{Im}b/2|\tilde{b}|^2)(1-p)^2}\left(\frac{|\tilde{b}|^2}{\operatorname{Im}b}\right)^{1/4}D_{-3/2}\left(\frac{(2k\operatorname{Im}b)^{1/2}}{|\tilde{b}|}(p-1)\right)[1+O(k^{-1/2})], \quad (24)\end{aligned}$$

where $D_{-3/2}(x)$ denotes the parabolic cylinder function and we have used one of the standard integral representations, see, e.g., Ref. [31].

This result was derived under the assumption $p\approx 1$ such that $(p^2-1)\approx 2(p-1)$. Substituting $(p-1)$ by $(p^2-1)/2$ allows us to combine the results for the different p regions in one formula,

$$\sum_{n\in\mathbb{N}}\rho(k-k_n)h_n(q,p)=\frac{k}{\pi^2}F_k(p)[1+O(k^{-1/2})], \quad (25)$$

where

$$\begin{aligned}F_k(p)&=\frac{1}{2(2k)^{1/4}}e^{-k(\operatorname{Im}b/8|\tilde{b}|^2)(1-p^2)^2}\left(\frac{|\tilde{b}|^2}{\operatorname{Im}b}\right)^{1/4}\\&\times D_{-3/2}\left(\frac{(k\operatorname{Im}b)^{1/2}}{2^{1/2}|\tilde{b}|}(p^2-1)\right). \quad (26)\end{aligned}$$

For $|p|<1$ one has $F_k(p)=\sqrt{1-p^2}+O(k^{-1})$, since $D_{-3/2}(x)\sim 2^{3/2}|x|^{1/2}e^{x^2/4}$ for $x\to-\infty$. Recall that $\tilde{b}$ is defined in Eq. (20). In Fig. 3 we compare the expression (26) with $|\tilde{b}|^2/\operatorname{Im}b=1$ for different values of k. It is clearly visible that the asymptotic result is reached slowly with increasing k.

Integrating Eq. (26), analogous to the transition from Eq. (22) to Eq. (23), one can compare the uniformized mean behavior with the numerical result. In Fig. 4 a section of $\mathcal{H}_k(q,p)$ at $q=3.0$ is shown for $k=125$, compare with Fig. 2(a). The remaining differences are due to higher order corrections.

In the derivation of the results (22) and (25) we have implicitly assumed that the boundary of Ω is sufficiently smooth, because only then we can use the stationary phase formula. But it is easy to extend the results to the case that the boundary is only piecewise smooth. Since we multiply in Eq. (18) by a coherent state centered in q, all the following computations remain valid if q is in the smooth part of the boundary, since the contributions from the singular points are exponentially suppressed then. So it could only happen that some additional mass sits at the singular points of the boundary, i.e., we have

$$\lim_{k\to\infty}\frac{1}{N(k)}\sum_{k_n\leqslant k}h_n(q,p)=\frac{2}{A\pi}\sqrt{1-p^2}+\mu_S(p,q), \quad (27)$$

where $\mu_S(p,q)dp\,dq$ is a measure supported on the singular part of the boundary (i.e., if $(p,q)\in\operatorname{supp}\mu_S$ then q is in the singular part of the boundary). We want to show that $\mu_S=0$ if the billiard is star shaped. We first show that $\mu_S\geqslant 0$, let $\mu_S=\mu_S^+ + \mu_S^-$ be the decomposition into its positive and negative parts, and let $S^\pm$ be the support of $\mu_S^\pm$. We define for any $\varepsilon>0$ $S_\varepsilon^-:=\{z;\inf_{s\in S^-}|z-s|\}$, and with $\lim_{\varepsilon\to 0}\int_{S_\varepsilon^-}\mu^+dpdq=0$ and $\lim_{\varepsilon\to 0}\int_{S_\varepsilon^-}\sqrt{1-p^2}dpdq=0$ we get

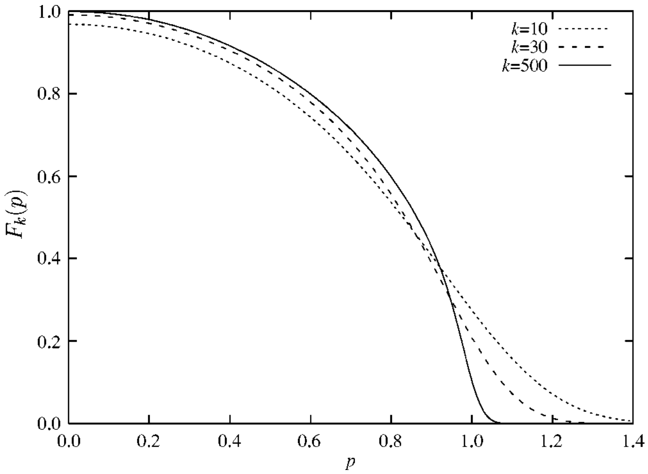

FIG. 3. Comparison of the uniformized asymptotic behavior $F_k(p)$, see Eq. (26), with $|\tilde{b}|^2/\mathrm{Im}\, b=1$ and for $k=10,30,500$. The asymptotic semicircle behavior is reached slowly.

$$\lim_{\varepsilon\to 0}\lim_{k\to\infty}\frac{1}{N(k)}\sum_{k_n\leqslant k}\int_{S_\varepsilon^-} h_n(q,p)dpdq = \int_{S^-} \mu_S^-. \qquad (28)$$

But the right hand side is negative, whereas the left hand side is positive, and therefore $\mu_S^-=0$ and $\mu_S\geqslant 0$. Now the completeness relation for the coherent states and the normalization (10) gives $\lim_{k_n\to\infty}\frac{1}{2}\int\int\langle\hat{\boldsymbol{n}}(q),\boldsymbol{x}(q)\rangle h_n(q,p)dpdq=1$, and together with the relation $1/2\int\int\langle\hat{\boldsymbol{n}}(q),\boldsymbol{x}(q)\rangle \times(2/A\pi)\sqrt{1-p^2}dpdq=1$ this yields

$$\int_{-1}^{1}\int_{\partial\Omega}\langle\hat{\boldsymbol{n}}(q),\boldsymbol{x}(q)\rangle\mu_S(p,q)dq\; dp = 0. \qquad (29)$$

But for a star-shaped billiard one can choose the origin of the coordinate system such that $\langle\hat{\boldsymbol{n}}(q),\boldsymbol{x}(q)\rangle>0$ for all $q\in\partial\Omega$, and so $\mu_S=0$. Therefore Eqs. (22) and (25) remain true for star-shaped billiards with piecewise smooth boundary with the only possible modification that the error term might decay more slowly at the singular points of the boundary.

IV. FROM HUSIMI FUNCTIONS IN PHASE SPACE TO HUSIMI FUNCTIONS ON THE BOUNDARY

In this section we derive a direct relation between the Husimi function in phase space and the one on the Poincaré section, as given by Eq. (12). By this we obtain a physical interpretation of the Poincaré Husimi representation. For the calculations in this section we have to assume that the billiard domain Ω is convex. Let ψ be a solution of the Helmholtz equation (1) in Ω which satisfies Dirichlet boundary condition on $\partial\Omega$. Any such function can be represented as

$$\psi(\boldsymbol{x}) = -\int_{\partial\Omega} G_k(\boldsymbol{x}-\boldsymbol{x}(s))u(s)ds, \qquad (30)$$

where $G_k(\boldsymbol{x}-\boldsymbol{y})$ is a free Greens function and $u(s)$ is the normal derivative of ψ on the boundary. Notice that the right hand side of Eq. (30) gives an extension of $\psi(\boldsymbol{x})$ to $\mathbb{R}^2$ with $\psi(\boldsymbol{x})=0$ for $\boldsymbol{x}\in\mathbb{R}\setminus\overline{\Omega}$ (this follows from Green's formula).

Let $\psi_{\boldsymbol{z}}$ be a coherent state (5) centered at $\boldsymbol{z}=(\boldsymbol{p},\boldsymbol{q})\in T^*\mathbb{R}^2$, for reasons of simplicity we restrict ourselves to the case of a nonsqueezed symmetrical state, i.e., $B=i\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, and omit the index B in the following. We want to compute the overlap $\langle\psi,\psi_{\boldsymbol{z}}\rangle$ given by

$$\langle\psi,\psi_{\boldsymbol{z}}\rangle_\Omega = \langle\psi,\psi_{\boldsymbol{z}}\rangle_{\mathbb{R}^2} = -\int_{\partial\Omega}\langle G_k(\cdot-\boldsymbol{x}(s)),\psi_{\boldsymbol{z}}\rangle_{\mathbb{R}^2}\overline{u(s)}ds, \qquad (31)$$

where we have used the aforementioned extension of $\psi(\boldsymbol{x})$ to $\mathbb{R}^2$ given by Eq. (30). We now observe that

$$\langle G_k(\cdot-\boldsymbol{x}(s)),\psi_{\boldsymbol{z}}\rangle_{\mathbb{R}^2} = \mathbf{G}_k^\dagger\psi_{\boldsymbol{z}}(\boldsymbol{x}(s)), \qquad (32)$$

where

$$\mathbf{G}_k = \lim_{\varepsilon\to 0}\frac{-1}{\Delta+k^2+i\varepsilon} \qquad (33)$$

is the resolvent operator, whose kernel is the Greens function. From Eq. (32) we see that the function $\mathbf{G}_k^\dagger\psi_{\boldsymbol{z}}$ is restricted to the billiard boundary. For the resolvent operator we use the integral representation

$$\mathbf{G}_k^\dagger = \frac{i}{k}\int_{-\infty}^{0} e^{ikt}\,\mathbf{U}(t)dt, \qquad (34)$$

where $\mathbf{U}(t)=e^{(i/k)t\Delta}$ is the free time evolution operator with $1/k$ playing the role of $\hbar$, and inserting Eq. (34) into Eq. (32) we obtain

$$\langle G_k(\cdot-\boldsymbol{x}(s)),\psi_{\boldsymbol{z}}\rangle_{\mathbb{R}^2} = \frac{i}{k}\int_{-\infty}^{0} e^{ikt}\,\mathbf{U}(t)\psi_{\boldsymbol{z}}(\boldsymbol{x}(s))dt. \qquad (35)$$

But the free time evolution of a coherent state centered in $\boldsymbol{z}$ is well known (see, e.g., Refs. [32,33]) to give again a coherent state, centered around the image of $\boldsymbol{z}$ under the classical flow and with transformed variance,

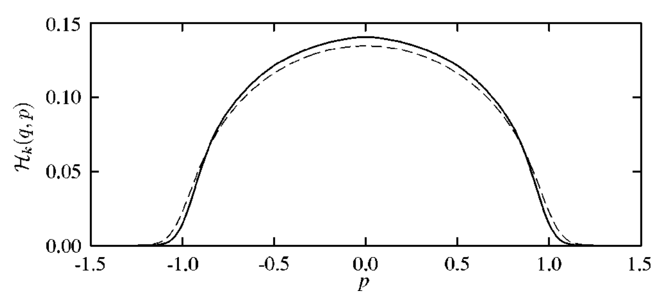

FIG. 4. The full curve shows a section of $\mathcal{H}_k(q,p)$ at $q=3.0$ with $k=125$ for the desymmetrized limaçon billiard, see Fig. 2(a), and the second line is the uniformized mean behavior. The remaining deviations are caused by higher order corrections.

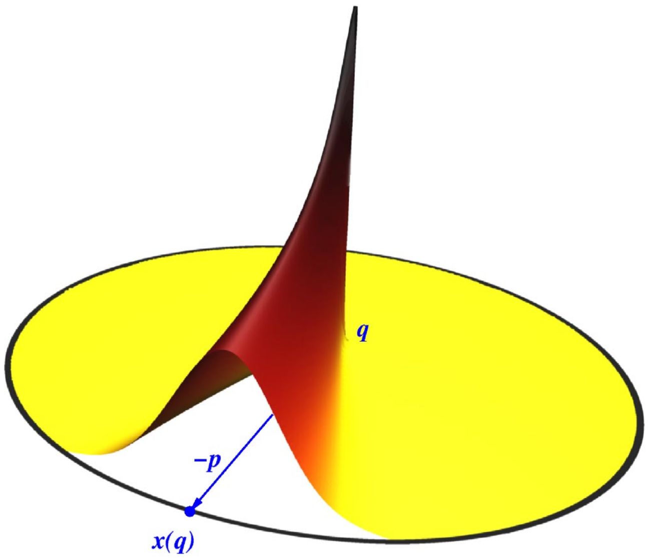

FIG. 5. Illustration of a Gaussian beam as given by Eq. (38) inside the limaçon billiard at $\varepsilon=0.3$.

$$\mathbf{U}(t)\psi_{\boldsymbol{z}}(\boldsymbol{x}) = e^{ik|\boldsymbol{p}|^2 t}\left(\frac{k}{\pi}\right)^{1/2}\frac{1}{1+2it}e^{ik[\langle \boldsymbol{p},\boldsymbol{x}-\boldsymbol{q}(t)\rangle+[i/\{2(1+2it)\}](\boldsymbol{x}-\boldsymbol{q}(t))^2]}, \tag{36}$$

with $\boldsymbol{q}(t)=\boldsymbol{q}+2t\boldsymbol{p}$. Therefore, $\mathbf{G}_k^\dagger\psi_{\boldsymbol{z}}(\boldsymbol{x})$ has the structure of a Gaussian beam emanating from the point $\boldsymbol{q}$ in direction $\boldsymbol{p}$ backwards in time. If we introduce a new coordinate system $\boldsymbol{x}=(x_\parallel,x_\perp)$ centered at $\boldsymbol{q}$ with $x_\parallel$ parallel to $\boldsymbol{p}$ and $x_\perp$ perpendicular to $\boldsymbol{p}$, we obtain by a stationary phase approximation that for $x_\perp$ and $1-|\boldsymbol{p}|$ small (i.e., near the energy shell)

$$\mathbf{G}_k^\dagger\psi_{\boldsymbol{z}}(\boldsymbol{x}) = \frac{i}{\sqrt{2}k(1+ix_\parallel)^{1/2}}e^{ik[x_\parallel+\{i/(2(1+ix_\parallel))\}x_\perp^2+(i/2)(1-|\boldsymbol{p}|)^2]} \times[1+O(k^{-1/2})] \tag{37}$$

holds, where we have assumed that $x_\parallel<0$. For $x_\parallel\approx 0$ and $x_\parallel>0$ the integral leads to an error function which describes the transition from the exponentially decaying regime with $x_\parallel>0$ to the regime $x_\parallel<0$ in Eq. (37). For $|\boldsymbol{p}|=1$ the result reads

$$\mathbf{G}_k^\dagger\psi_{\boldsymbol{z}}(\boldsymbol{x})|_{|\boldsymbol{p}|=1} = \frac{i}{\sqrt{2}k(1+ix_\parallel)^{1/2}}e^{ik[x_\parallel+\{i/(2(1+ix_\parallel))\}x_\perp^2]} \times\frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{k}{2}}\frac{x_\parallel}{(1+ix_\parallel)^{1/2}}\right)[1+O(k^{-1/2})], \tag{38}$$

where $\mathrm{erfc}(z)$ denotes the complementary error function, and the absolute value of this expression is shown in Fig. 5.

Next we want to evaluate this expression on the boundary. To this end, let $\boldsymbol{x}(q)$ be the point of intersection between the boundary and the line from $\boldsymbol{q}$ in direction $-\boldsymbol{p}$. (Here we need the assumption that the billiard domain Ω is convex, in order that there is only one such point.) Then we obtain with $\boldsymbol{x}(s)=\boldsymbol{x}(q)+\hat{\boldsymbol{t}}(q)(s-q)-\kappa(q)/2\hat{\boldsymbol{n}}(q)(s-q)^2+O((s-q)^3)$ that

$$x_\parallel = |\boldsymbol{q}-\boldsymbol{x}(q)|+p(s-q)-\frac{\kappa(q)}{2}(1-p^2)^{1/2}(s-q)^2 + O((s-q)^3), \tag{39}$$

$$x_\perp = (1-p^2)^{1/2}(s-q)+O((s-q)^2), \tag{40}$$

where $p:=\langle\hat{\boldsymbol{p}},\hat{\boldsymbol{t}}\rangle\in[-1,1]$. Inserting these expressions in Eq. (37) gives

$$\langle G_k(\cdot-\boldsymbol{x}(s)),\psi_{\boldsymbol{z}}\rangle_{\mathbb{R}^2} = \frac{i\pi^{1/4}}{\sqrt{2}k^{5/4}}\frac{1}{(1-p^2)^{1/4}}e^{ik|\boldsymbol{q}-\boldsymbol{x}(q)|+i\theta}e^{-(k/2)(1-|\boldsymbol{p}|)^2}c^b_{(q,p),k}(s) \times[1+O(k^{-1/2})], \tag{41}$$

where $c^b_{(q,p),k}(s)$ is a coherent state on the boundary, as defined in Eq. (11), with variance $b=i(1-p^2)/[1+i|\boldsymbol{q}-\boldsymbol{x}(q)|]-\kappa(q)(1-p^2)^{1/2}$ and $e^{i\theta}=[|\boldsymbol{q}-\boldsymbol{x}(q)|+i]^{1/2}/[|\boldsymbol{q}-\boldsymbol{x}(q)|^2+1]^{1/4}$. Notice that although we started with a symmetric coherent state in the interior, the projected coherent state on the boundary is no longer symmetric and has a nontrivial squeezing parameter b which depends on the position of the original state, the angle of intersection of the ray in direction $-\boldsymbol{p}$ with the boundary, and the curvature of the boundary.

If we insert the expression (41) into Eq. (31) we obtain a semiclassical relation between the projection of an eigenstate onto a coherent state in the interior and the projection of the normal derivative on the boundary onto a coherent state on the boundary,

$$\langle\psi_n,\psi_{\boldsymbol{z}}\rangle_\Omega = -\frac{i\pi^{1/4}}{\sqrt{2}\,k_n^{5/4}}\frac{1}{(1-p^2)^{1/4}} \times e^{ik_n|\boldsymbol{q}-\boldsymbol{x}(q)|+i\theta}e^{-(k_n/2)(1-|\boldsymbol{p}|)^2} \times\langle u_n,c^b_{(q,p),k_n}\rangle_{\partial\Omega}[1+O(k_n^{-1/2})]. \tag{42}$$

In turn from this we obtain the central result of this section, a direct relation between the corresponding Husimi functions

$$H_n(\boldsymbol{p},\boldsymbol{q}) = \delta_{k_n}(1-|\boldsymbol{p}|)\frac{1}{4}\frac{h_n(q,p)}{\sqrt{1-p^2}}[1+O(k_n^{-1/2})], \tag{43}$$

with

$$\delta_{k_n}(1-|\boldsymbol{p}|):=\left(\frac{k_n}{\pi}\right)^{1/2}e^{-k_n(1-|\boldsymbol{p}|)^2}. \tag{44}$$

Let us first discuss the meaning of the individual terms on the right hand side of Eq. (43). The function $\delta_{k_n}(1-|\boldsymbol{p}|)$ is a delta sequence for $k_n\to\infty$, and describes the localization of $H_n(\boldsymbol{p},\boldsymbol{q})$ around the energy shell. The factor $1/\sqrt{1-p^2}$ comes from the projection of the Gaussian beam to the plane tangent to the boundary, see Fig. 5. The right hand side of Eq.(43) has still a dependence on the phase space point $(\boldsymbol{p},\boldsymbol{q})$ on the left hand side through the parameter b in the coherent state in Eq. (41). But as we will discuss after Eq. (45) below (and in more detail in the Appendix), when integrating the

Husimi function against an observable the result does not depend on b in leading order.

As in the preceding section we have assumed that the boundary is smooth. But by the localization of the coherent states the results can be again extended to the case that the boundary is piecewise smooth, then Eq. (43) remains valid if q is not a singular point of the boundary.

The direct connection between the Husimi function in the interior and the one on the boundary, given by Eq. (43), allows us to derive interesting relations between the two Husimi functions and can be used to give a direct physical interpretation of the Husimi function on the boundary. From Eq. (6) together with relation (43) we obtain

$$\langle \psi_n, \boldsymbol{A}\psi_n\rangle_\Omega = \int_{-1}^{1}\int_{\partial\Omega} \frac{h_n(q,p)}{4\sqrt{1-p^2}}\langle a\rangle(q,p) l(q,p)dq\, dp + O(k_n^{-1/2}), \tag{45}$$

where $l(q,p)$ denotes the length of a ray emanating from $\boldsymbol{q}(q) \in \partial\Omega$ in the direction determined by p until it hits the boundary again. Furthermore,

$$\langle a\rangle(q,p) := \frac{1}{l(q,p)}\int_0^{l(q,p)} a(\boldsymbol{q}(q) + t\hat{\boldsymbol{e}}(q,p), \hat{\boldsymbol{e}}(q,p))dt \tag{46}$$

is the mean value of the classical observable between two bounces, where $\hat{\boldsymbol{e}}(q,p)$ denotes the unit vector at $\boldsymbol{q}(q)$ in direction p. A relation of the same type as Eq. (45) has been obtained recently by different methods in Ref. [34] for certain localized functions on the boundary. Equation (45) now shows that the dependence on the parameter b in the coherent states used to define h_n can be discarded in leading order, see the Appendix for a detailed discussion. This means that if we move from the pointwise relation (43) to the integral relation (45), we gain the freedom to define the Husimi functions on the boundary with an arbitrary parameter b.

We conclude from relation (45) that

$$\mathfrak{h}_n(q,p) := \frac{1}{4}\frac{h_n(q,p)}{\sqrt{1-p^2}} \tag{47}$$

is a reduction of the probability density defined by the Husimi function on the whole phase space to the boundary. So if one wants a proper representation of eigenfunctions on the Poincaré section which is an approximate probability density, and whose general properties are independent of the billiard shape, then Eq. (47) seems to be the best choice. Of course a drawback of the function (47) is the singularity of $1/\sqrt{1-p^2}$ at $p=\pm 1$ which is relevant at any finite energy. So for numerical computations the definition (12) is more suitable and the importance of Eq. (47) lies in the physical interpretation.

In particular, relation (45) implies an asymptotic normalization condition on $\mathfrak{h}_n(q,p)$,

$$\int_{-1}^{1}\int_{\partial\Omega} \mathfrak{h}_n(q,p) l(q,p) dq\, dp = 1 + O(k_n^{-1/2}). \tag{48}$$

Since $l(q,p)dq\,dp$ is the phase space volume in the energy shell corresponding to the volume element $dq\,dp$ of the Poincaré section, the factor $l(q,p)$ can be viewed as a normalization which makes $\mathfrak{h}_n(q,p)$ independent of the billiard shape, i.e., for any $\mathcal{D}\subset \partial\Omega\times[-1,1]$, we get that $\int_{\mathcal{D}} \mathfrak{h}_n(q,p)l(q,p)dq\,dp$ is the probability for the particle in the state ψ_n to be found in the region $\hat{D} := \Pi^{-1}\mathcal{D}$ on the energy shell, where the map Π describes the projection of the domain $\hat{D}$ to the boundary.

We would like to close this section with some remarks on the implications of quantum ergodicity to the behavior of the Poincaré Husimi functions. If the classical billiard flow in Ω is ergodic, then the quantum ergodicity theorem [35,36] (see Ref. [20] for an introduction) tells us that almost all Husimi functions $H_n(\boldsymbol{p},\boldsymbol{q})$ tend weakly to $1/2\pi A$. Our result (43) then immediately implies that in the semiclassical limit almost all Poincaré Husimi functions $h_n(q,p)$ tend to $[2/\pi A]\sqrt{1-p^2}$ in the weak sense. So this proves a quantum ergodicity theorem for the boundary Husimi functions. Recently related results have been obtained establishing quantum ergodicity for observables on the Poincaré section [35,37,38]. Notice that the $\sqrt{1-p^2}$ behavior is also visible in the plot of $h_n(q,p)$ for the irregular state shown in Fig. 1(c) for the ergodic cardioid billiard.

V. SUMMARY

Poincaré representations of eigenstates play an important role in several areas. However, *a priori* there is no unique way for their definition. In this paper we single out the definition given by Eq. (12) and show that the asymptotic mean behavior of these Husimi functions is proportional to $\sqrt{1-p^2}$. For this asymptotic semicircle behavior we in addition derive a uniform asymptotic formula. Furthermore we establish a direct relation between the Husimi function in phase space and the Poincaré Husimi function (12) on the billiard boundary. By this a physically meaningful interpretation, see Eq. (43), of the previously *ad hoc* chosen definition for the Poincaré Husimi function is obtained. Namely, the Poincaré Husimi function $\mathfrak{h}_n(q,p)$ can be viewed as a probability density on the Poincaré section. For ergodic systems our result implies a quantum ergodicity theorem for the Poincaré Husimi functions, i.e., almost all Poincaré Husimi functions become equidistributed with respect to the appropriate measure.

ACKNOWLEDGMENTS

A.B. and R.S. would like to thank the Mathematical Sciences Research Institute, Berkeley, USA, for financial support and hospitality where part of this work was done. R.S. was supported by the European Commission under the Research Training Network (Mathematical Aspects of Quantum Chaos) Grant No. HPRN-CT-2000-00103 of the IHP Programme.

APPENDIX: HUSIMI FUNCTIONS AND EXPECTATION VALUES

In this appendix we recall some facts about Husimi functions, see, e.g., Ref. [18] and the contribution by Helffer in the same volume. With this information we discuss the dependence of the Husimi functions on the parameter b, as given in the definition (11) of the coherent states. In the following we will use the notation $z=(p,q)$. A Husimi function is a smoothed version of the Wigner function,

$$\frac{k}{2\pi}|\langle c_{z,k}^{b},u\rangle|^2 = \int W[c_{z,k}^{b}](z')W[u](z')dz', \qquad \text{(A1)}$$

where $W[u](z')$ denotes the Wigner function of u. The Wigner function of the coherent state $c_{z,k}^{b}$ is given by $W[c_{z,k}^{b}](z')=(k/\pi)e^{-k\langle z'-z,g(z-z')\rangle}+O(e^{-c/k})$, where

$$g = \begin{pmatrix} 1/\mathrm{Im}\ b & -\mathrm{Re}\ b/\mathrm{Im}\ b \\ -\mathrm{Re}\ b/\mathrm{Im}\ b & \mathrm{Im}\ b + (\mathrm{Re}\ b)^2/\mathrm{Im}\ b \end{pmatrix}. \qquad \text{(A2)}$$

Relation (A1) holds as well if b depends on z.

We will now use the fact that if $\boldsymbol{A}$ is the Weyl quantization of a function $a(z)$, see, e.g., Ref. [29], then $\langle u,\boldsymbol{A}u\rangle =\int a(z)W[u](z)dz$. Using this and Eq. (A1) one obtains

$$\int a(z)\frac{k}{2\pi}|\langle c_{z,k}^{b},u\rangle|^2 dz = \int\int a(z)W[c_{z,k}^{b}](z')W[u](z')dzdz' = \langle u,\tilde{\boldsymbol{A}}u\rangle, \qquad \text{(A3)}$$

where $\tilde{\boldsymbol{A}}$ is the Weyl quantization of the function

$$\tilde{a}(z) = \int a(z')W[c_{z',k}^{b}](z)dz'. \qquad \text{(A4)}$$

If we assume that the matrix g is either constant, or satisfies $\|\partial_z^{\alpha}g(z)\|\leqslant C_{\alpha}$ for all $\alpha\in\mathbb{N}^2$ and $z\in\mathrm{supp}\ a$, which is equivalent to the requirement that $b(z)$ is smooth and $\mathrm{Im}\ b(z)>0$ for $z\in\mathrm{supp}\ a$, then the method of stationary phase gives

$$\tilde{a}(z) = a(z) + \frac{1}{k}R(k,z), \qquad \text{(A5)}$$

where $R(k,z)$ is a smooth bounded functions with bounded derivatives. Hence the Weyl quantization of $R(k,z)$ is bounded by the Calderon-Vallaincourt theorem (see Ref. [29]), so $\|\boldsymbol{A}-\tilde{\boldsymbol{A}}\|\leqslant C/k$ and therefore

$$\left|\int a(z)\frac{k}{2\pi}|\langle c_{z,k}^{b},u\rangle|^2\ dz - \langle u,\boldsymbol{A}u\rangle\right| \leqslant C/k. \qquad \text{(A6)}$$

Since $\langle u,\boldsymbol{A}u\rangle$ is independent of b we have for any smooth $b(z),\tilde{b}(z)$ with $\mathrm{Im}\ b(z)>0$, $\mathrm{Im}\tilde{b}(z)>0$ for $z\in\mathrm{supp}\ a$ the estimate

$$\left|\int a(z)\frac{k}{2\pi}|\langle c_{z,k}^{b},u\rangle|^2\ dz - \int a(z)\frac{k}{2\pi}|\langle c_{z,k}^{\tilde{b}},u\rangle|^2 dz\right| \leqslant C/k. \qquad \text{(A7)}$$

This shows that in the definition of the family of coherent states we can choose any nondegenerate, possibly z-dependent, parameter b and still get in leading order the same probability distribution defined by the corresponding Husimi densities. In this sense the dependence of the Husimi functions on b is weak.

Let us now look at relations (43) and (45) from the perspective of the preceding discussion. In the Husimi function appearing on the right hand side of Eq. (43) the parameter b is given by $b=i(1-p^2)/[1+i|\boldsymbol{q}-\boldsymbol{x}(q)|]-\kappa(q)(1-p^2)^{1/2}$, so it depends on $z=(p,q)$ and additionally on $\boldsymbol{q}$, and it degenerates for $p\rightarrow\pm1$. If the classical observable a in relation (45) has support in the interior of Ω, then $\langle a\rangle$ is supported away from $p=\pm1$ and we can replace b by any nondegenerate $\tilde{b}$. If the support of a includes the boundary $\partial\Omega$, then $\langle a\rangle$ is not necessarily zero at $p=\pm1$ and we can only replace b by one which has the same type of behavior for $p\rightarrow\pm1$, such as, e.g., $b^{(0)}(p,q)=i(1-p^2)-\kappa(q)(1-p^2)^{1/2}$.

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