Band theory of light localization in one-dimensional disordered systems

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A simple approach to the problem of light localization in one-dimensional system is presented. The role of the Bragg reflection in one-dimensional localization of light is discussed. Contrary to the existent viewpoint, we show that the origin of band gaps of regular crystals and the localization due to disorder have a common nature, that is, the Bragg reflection. We expand the concept of band structure to random systems of finite thickness L and relate the Anderson localization of light with the total band gap growth, which is observed in our computer simulation of disordered system, as L increases.

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In quantum mechanics (QM) it is well established [1–3] that in a one-dimensional case one-particle wave function is localized in any infinite disordered system. The similarity of the Schrodinger and Maxwell equations suggests that light should be also localized in a one-dimensional disordered system. This has been confirmed in computer simulations [4]. It is worth mentioning that often the concept of localization has different meaning in QM and in optics. In QM [3] when dealing with the eigenvalue problem localization is understood as the existence of a special solution to the stationary Schrodinger equation. This solution decays on average exponentially with distance from a certain bounded domain in space, which is a characteristic of the particular energy and system realization. Optics considers, as a rule, a scattering problem and understands by light localization the total reflection from semi-infinite space filled by disordered medium [1]. In the mathematical language the difference between these definitions lies in the application of different boundary conditions. The reason for employing the QM results in optics or even in a QM scattering problem is that the QM definition implies that the localized wave function weakly depends on boundary conditions [2].

However, the situation is fraught with a conflict. The consideration of a one-dimensional photonic crystal with a single defect [5] has shown that in the infinite photonic crystal inside the band gap there appears a defect mode with field distribution satisfying the QM definition of localization. Thus, at frequency of the defect mode the light is localized in the infinite system. On the other hand, any finite sample of the crystal with a defect in the middle is nothing more than a Fabry-Perot filter where two fragments of the photonic crystal form the dielectric mirrors. The system is transparent at the very frequency of the defect mode at any system size. Hence, in terms of the optical definition of localization, the light wave is delocalized in such a system.

Moreover, there are special boundary conditions under which the light localization does not take place in any one-dimensional disordered system at all. To illustrate this statement, we use the T-matrix language. A T matrix relates phasors of incident and outgoing waves on both sides of the layer. The T matrix of two layers is equal to the product of the T matrices of these layers. A set of all T matrices forms a group [3]. Hence for any given finite sample of any one-dimensional disordered system we can find a finite sample (this sample may belong to another ensemble of random systems) that has a T matrix, which at a given frequency is inverse to the T matrix of the initial sample. The system of these two samples is absolutely transparent. It is the boundary conditions realized between the samples that warrant the absence of localization in the initial sample.

So, we have to be careful about recruiting the QM results in optics. This adoption is aggravated by the formal character of the existing reasoning proving the fact of localization in QM [3]. The most rigorous proof [3] is based on Furstenberg's theorem [6] and may be reduced to the statement that all solutions (with probability 1) of the involved random equations have "the exponential growth." Unfortunately, the physical reasons of this exponential growth do not follow from Frustenberg's theorem. The abstract form of Ishii's constructions in Ref. [3] often hinders physicists from application of the results (see, e.g., Refs. [7,8] devoted to delocalization due to the correlated disorder where the results of Ref. [3] are ignored). As a clear physical pattern of wave localization is necessary, a search for new arguments and interpretation continuous (see, e.g., Refs. [5,9–12]).

In this respect it is necessary to mention the papers considering a transfer from regular photonic crystals to disordered systems [13–16]. Being based on the results of these works but contrary to their conclusion [16] that the band gap in crystals and localization in random media are phenomena of different nature [17], we state that it is the Bragg reflection that is responsible not only for the appearance of band gaps but also for the Anderson localization of light in the one-dimensional case. We use the T matrix language because it can describe both scattering and eigenvalue problem. So the results obtained in terms of T matrices language are of the universal character.

Any random system and, what is even more, its finite part are not translationally invariant. Thus, a direct, mathematically rigorous application of the band theory is impossible in these cases. To attribute a band structure to any finite system, we build up a periodic system that has this particular finite system as a primitive cell. A band structure of the periodic system is regarded as the associated band structure of the

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finite system. If there exists a band structure while the primitive cell size tends to infinity, such a band structure can be ascribed to the proper infinite system.

Below, we assume all elementary layers to be of an identical thickness d. This restriction is not a determinative one but it is taken to simplify our speculations and computer simulation because it permits us to deal with impermeable ingredients only. Indeed, the dispersion in the permittivity values guarantees dispersion in both optical paths and impedance values. We treat only the normal incidence of waves.

The field in each layer is a sum of the left-going and right-going waves. Employing continuity of the electrical and magnetic fields at the interface surfaces the amplitudes A_{j+1}, B_{j+1}, and A_{j-1}, B_{j-1} of such waves in the layers adjacent to the jth layer can be linked with a T matrix. Let us introduce an auxiliary vacuum layer of the zero thickness between any adjacent layers. As the T matrix of this layer is equal to the identity matrix, this layer is of no significance. Now, the T matrix of the jth layer depends on its own (and vacuum) properties only. In the general case, after extracting the Jordan form the T matrix of a primitive cell can be written as

$$T_{cell} = S_{cell} J S_{cell}^{-1} = S_{cell} \begin{Vmatrix} e^{ik_{eff}L_{cell}} & 0 \\ 0 & e^{-ik_{eff}L_{cell}} \end{Vmatrix} S_{cell}^{-1}, \qquad (1)$$

where the S_{cell} matrix depends on the cell structure. The T matrix of a system containing M of identical primitive cells has the form (1) with L_{cell} substituted by the system size $L = ML_{cell}$. So, k_{eff} could be regarded as the sought wave number. As $\mathrm{Tr}(SJS^{-1}) = \mathrm{Tr}(J)$ [19] the dispersion equation can be written employing the trace of T_{cell} [20]:

$$\mathrm{Tr}\,(T_{cell}) = \mathrm{Tr}\,(J) = 2\cos\,(k_{eff}L_{cell}). \qquad (2)$$

For the simplest primitive cell built up of two elementary layers with different values of permittivity, Eq. (2) yields the well-known result [21]:

$$\begin{aligned} 2\cos\,(k_{eff}2d) &= \mathrm{Tr}\,(T_{cell}) \\ &= 2\cos\,(k_0\sqrt{\varepsilon_1}d)\cos\,(k_0\sqrt{\varepsilon_2}d) - (\sqrt{\varepsilon_1/\varepsilon_2} \\ &\quad + \sqrt{\varepsilon_2/\varepsilon_1})\sin\,(k_0\sqrt{\varepsilon_1}d)\sin\,(k_0\sqrt{\varepsilon_2}d), \end{aligned} \qquad (3)$$

with $k_0 = \omega/c$. Equation (3) predicts the existence of band gaps where $|\mathrm{Tr}(T_{cell})| > 2$ and k_{eff} is a pure imaginary quantity. The imaginary part of k_{eff} is usually referred to as the Lyapunov exponent $\gamma^{\mathrm{Tr}} = \mathrm{Im}(k_{eff}) = \mathrm{Im}\{\arccos \times [\mathrm{Tr}(T_{cell})/2]\}/L$.

Let us trace the development of a band structure as the construction of a primitive cell becomes more complicated. The complication of a primitive cell implies an increase in the number of its elementary layers. The complication can be achieved by various methods. We can introduce new types of elementary layers having different values of permittivity or simply combine several neighboring cells and intermix the available elementary layers. Now, for comparing systems with different L_{cell} we follow the second way. In fact, a simple joining of the neighboring cells of a regular system cannot lead to any new physical consequences. However, after mixing the order of the elementary layers inside a new supercell, additional frequency gaps appear where $\mathrm{Tr}(T_{cell})$ exceeds 2 (see Fig. 1). Unfortunately, the increase of the number of band gaps is accompanied with a decrease in the gap width. Thus, we cannot directly identify this increase with the rise of the degree of localization in the system. We have to look at the total width of all the gaps.

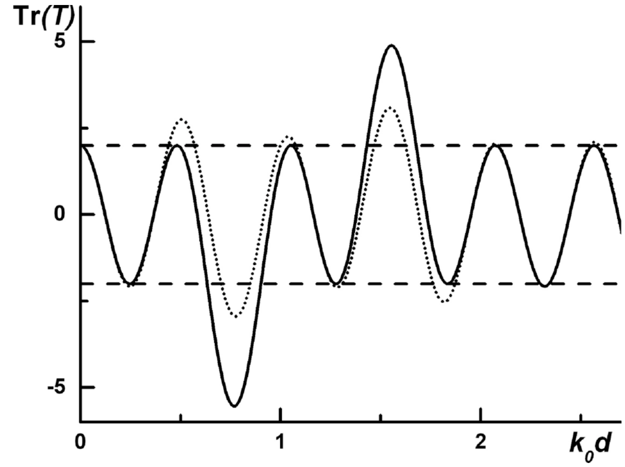

FIG. 1. The dependence of Tr(T) of six-layer supercells on frequency. In the first cell (solid line) ε takes the values of 2,7,2,7,2,7, in the second cell (dot line) ε takes values 2,7,7,2,7,2.

In our computer simulations (Fig. 2), we calculated the fraction (measure) of frequencies at which the Lyapunov exponent is equal to zero

$$\tau = \lim_{K\to\infty}\left\{\frac{1}{K}\int_0^K [1 - \mathrm{sign}\{\gamma^{\mathrm{Tr}}(k_0)\}]dk_0\right\}.$$

The measure of the band gaps is equal to $1-\tau$. As we can see from Fig. 2, the measure decreases with an increase in the thickness L of a random system. For fairly thick systems almost all frequencies lie in the frequency gaps. It is reasonable to regard this fact as localization, identifying $1/\gamma^{\mathrm{Tr}}(k_0)$

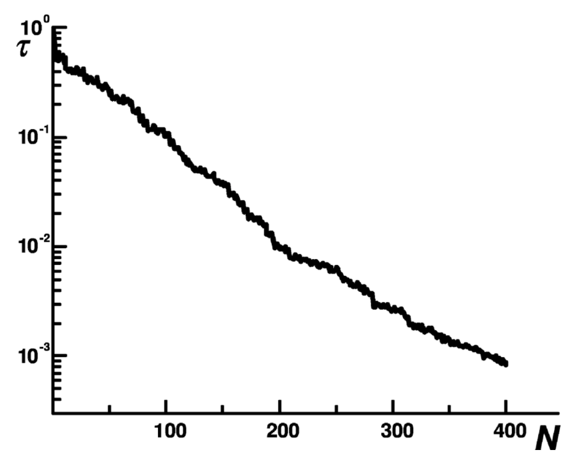

FIG. 2. The dependence of the measure of bands of transparency on the thickness L of the primitive cell. The permittivity ε equiprobably takes values from the interval $[2,3]$.

with localization length. This definition of the localization length coincides with the common one [1]

$$L_{loc} = \gamma_{loc}^{-1} = -\lim_{L\to\infty} \frac{L}{\langle \ln|t| \rangle}, \tag{4}$$

where t is the transmission coefficient, and the brackets indicate ensemble averaging. Indeed, the T matrix of any sample can be expressed in terms of the reflection coefficients r_R and r_L corresponding to the right-incident and left-incident waves, as well as through the transmission coefficient t (because $\det T=1$, the value of t is independent of the direction of incidence [22]):

$$T = \left\| \begin{matrix} \left(t - \dfrac{r_R r_L}{t}\right) & \dfrac{r_R}{t} \\ -\dfrac{r_L}{t} & \dfrac{1}{t} \end{matrix} \right\|$$

and $\mathrm{Tr}(T)=t+(1-r_R r_L)/t$. Taking into account that $\gamma^{\mathrm{Tr}} = \mathrm{Im}\{\arccos[\mathrm{Tr}(T_{cell})/2]\}/L$ gives

$$\gamma^{\mathrm{Tr}} = \left\{ \ln\left(\frac{1}{t}\right) + \ln\left[\frac{1 - r_R r_L + t^2}{2} + \sqrt{\left(\frac{1 - r_R r_L + t^2}{2}\right)^2 - t^2} \right] \right\} \Big/ L. \tag{5}$$

In a realization where the localization occurs at $L\to\infty$ we have $|t| \ll |r_R| \sim |r_L| \sim 1$ but $|r_R r_L - 1| \sim 1$ [23]. Comparison of Eqs. (4) and (5) reveals that $\gamma^{\mathrm{Tr}} \to \gamma_{loc}$.

In our approach the frequencies at which the system is transparent lie in the bands of the associated band structure. Though the measure of the bands tends to zero their number N_{bands} tends to infinity [24]. To understand what happens to the waves at these frequencies we evaluate the maximum value of the group velocity inside the bands.

For a fixed frequency domain $\Delta\omega$ the N_{bands} is $O(L)$ and the distance between neighboring bands is about $\Delta\omega/(L/d)$ [24]. Coming back to Fig. 1 we can evaluate the derivative of $\mathrm{Tr}(T)$ inside a band as the ratio of the maximum value of the $\mathrm{Tr}(T)$ in the adjacent band gaps to the frequency distance between the bands. Taking into account that $\max_{gap}[\mathrm{Tr}(T)] \sim \exp(\gamma^{Tr} L)$, where we can use the value of $\langle \gamma^{Tr}(k_0) \rangle$ averaged over ensemble, we can evaluate the derivative as

$$\begin{aligned} \frac{1}{d}\frac{d}{dk_0}\mathrm{Tr}\,[T(k_0 L)] &= \frac{1}{d}\frac{d}{dk_0} 2\cos(k_{eff}L) \\ &= -\frac{dk_{eff}}{dk_0}\frac{L}{d}\sin(k_{eff}L) \\ &\sim \frac{L}{d}\exp(\langle \gamma^{Tr}(k_0)\rangle L). \end{aligned}$$

Ultimately $v_{gr}=c(dk_0/dk_{eff}) \sim \sin(k_{eff}L)\exp(-\langle \gamma^{Tr}(k_0)\rangle L)$ that vanishes as $L\to\infty$. Thus, we see that in spite of $\gamma^{Tr}(k_0)=0$ the wave cannot transfer energy. In this sense light is localized at any frequency.

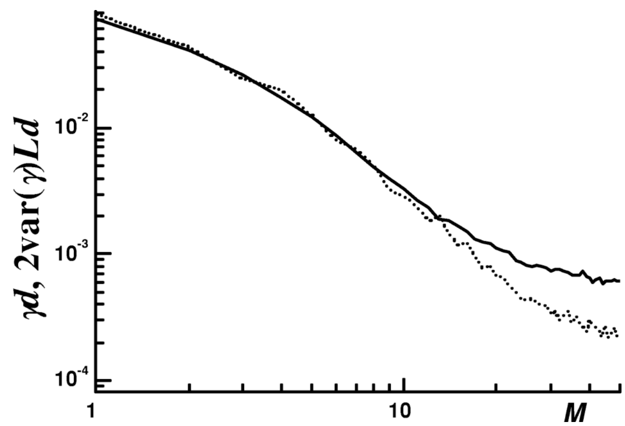

FIG. 3. The dependence of the Lyapunov exponent (solid line) and its variance (dot line) on the size M of the Bragg reflectors which have been randomly transformed to segments without Bragg reflectors.

The proposed band theory partly agrees with conclusions of work [25] that the Fourier harmonic of refractive index that satisfies the condition of the Bragg reflection plays a special role in the wave localization. Namely, confining to this harmonic the authors of Ref. [25] obtain an evaluation of the Lyapunov exponent, which coincides with the classical perturbative result [26]. It may seem that it is an infinitely expanded structure underlying the distribution of inhomogeneities (the Bragg harmonic) that determines whether the light is localized [25] or not [27].

Our scrupulous examination of the changes in light amplitude related to depth in the disordered system shows that the amplitude attenuation happens in accidentally shaped short segments having high value of the Lyapunov exponent (the Bragg reflectors). The rest of the layers plays no role in light localization [28]. Therefore, there is no need in a very long array of layers to attenuate the wave because a single Bragg reflector successfully does this.

To verify our hypothesis we have considered a realization of the length $L=20\,000$ and test one layer after another to see whether this layer together with $M-1$ foregoing layers build up an M-layer Bragg reflector. If it happens we randomly change the permittivity of the layer until the Lyapunov exponent of any j-layer cell ($j \leq M$) ending by the layer under consideration becomes equal to zero. Thus, after considering all layers we obtain a system that has the Bragg reflectors of a length greater than M only. We watch a decrease in the Lyapunov exponent of the whole resulting system as we successively perform the procedure increasing M from 2 to 50 (see Fig. 3). The dependence $\gamma(M)$ given in Fig. 3 is a result of averaging over 200 random realizations. We can see that not only $\gamma(M)$ but also the variance of $\gamma(M)$ tends to zero. This fact permits us to say that removing the Bragg reflectors makes the system transparent. In so doing we get a random system which may have long-range correlations (at least of size M). In the literature there are many examples of systems with correlated randomness [11] where the waves are delocalized. A simple analysis shows that in these systems there are no Bragg reflectors too. We see that the existence of Bragg reflectors is a necessary condition for localization.

We relate the effect of wave localization with the effect of the total growth of band gaps in the associated band structure. This makes evident why the majority of the solutions have the exponential growth. These band gaps are real band gaps with the zero density of state. The localized states come about from bands of transparency as the system size comes to infinity. At the same time the frequency width of these bands as well as the group velocity of the states come to zero. The latter property permits us to consider these states as localized ones.

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